Most groups are hyperbolic, or ... most groups are trivial ?

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Outline

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Gromov's claim

Claim (Gromov '87)

Most finite presentations of groups, present an hyperbolic infinite group.

- Stated in his influential paper on hyperbolic groups: "Essays in group theory", 75-263, Springer, 1987,
- no proof, only the idea,
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Presentations of groups

Notation

- $A = \{a_1, \ldots, a_k\}$ *is a finite alphabet (n letters).*
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_k, a_k^{-1}\}$.
- *Usually,* $A = \{a, b, c\}$.
- $(A^{\pm 1})^*$ *the free monoid on* $A^{\pm 1}$ *(words on* $A^{\pm 1}$*).*
- $F_A = (A^{\pm 1})^* / \sim$ *is the free group on A (words on* $A^{\pm 1}$ *modulo reduction).*
- *Every* $w \in A^*$ *has a unique reduced form,*
- 1 *denotes the empty word, and* $|\cdot|$ *the (shortest) length in* F_A*:*
  $|1| = 0, \quad |aba^{-1}| = |abbb^{-1}a^{-1}| = 3, \quad |uv| \leqslant |u| + |v|$.
- *The free group* F_A *is usually denoted by:*

$$F_A = \langle a_1, \ldots, a_r \mid - \rangle.$$

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Presentations of groups

Theorem

Every finitely generated group G is a quotient of F_A (for some r), i.e.

$$G \simeq F_A/N = \langle a_1, \ldots, a_r \mid w_1, w_2, \ldots \rangle,$$

where N is the normal closure of $w_1, w_2, \ldots \in F_A$ in F_A.

- If G admits a presentation with finitely many w_i's (*relations*) we say it is *finitely presented*.
- Very different presentations can give isomorphic groups:

$$\langle a \mid a \rangle = 1 = \langle a, b \mid a^{-1}ba = b^2,\ b^{-1}ab = a^2 \rangle$$

- Deciding whether a finite presentation presents the trivial group is algorithmically unsolvable.

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Hyperbolicity

Let G be a group, $S \subseteq G$, and $\chi(G,S)$ the Cayley graph of G w.r.t. S.

- $\chi(G,S)$ is connected if and only if S generates G.
- $\chi(G,S)$ has non-trivial closed paths if and only if S satisfy non-trivial relations.
- $\chi(G,S)$ is a tree if and only if G is free with basis S.

Definition

A group G is δ-hyperbolic if every geodesic triangle in $\chi(G,S)$ is δ-thin. (Free groups are 0-thin with respect to bases).

So, intuitively, hyperbolic groups are "close" to free groups (in a geometric sense).

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The meaning of "most"

Let X be an infinite set. What is the meaning of sentences like "most elements in X have property $\mathcal{P}$" ?

- Define a notion of size, $|\cdot| \colon X \to \mathbb{N}$, with finite preimages.
- Define the balls: $B(n) = \{x \in X \mid |x| \leqslant n\}$ (which are finite).
- Count the proportion $\rho_n = \frac{|\{x \in X | x \text{ satisfies } \mathcal{P}\}|}{|B(n)|} = \frac{|\mathcal{P} \cap B(n)|}{|B(n)|}$.
- Define the density of X as $\rho = \lim_{n\to\infty} \rho_n$ ($\in [0, 1]$ if it exists).
- $\mathcal{P}$ is generic (or generically many elements satisfy $\mathcal{P}$) if $\rho = 1$.
- $\mathcal{P}$ is negligible if $\rho = 0$.

Of course, everything depends on the chosen size function, i.e. on the direction to infinity inside X.

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Classical example: visible points

Definition

A point $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ *is visible if* $\gcd(x_1, \ldots, x_k) = 1$.

Theorem (Mertens, 1874 (case $k = 2$))

The density of visible points in $\mathbb{Z}^k$ *is* $1/\zeta(k)$, *where* $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$ *is the Riemann zeta-function (with respect to* $||\cdot||_1$*).*

In particular, visible points in the plane have density $\frac{6}{\pi^2}$.

With artificial definitions of size, one can force it to be any $\alpha \in [0, 1]$.

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Arzhantseva-Ol'shanskii's proof

- Fix $r \geqslant 2$ and $k \geqslant 1$.
- Consider the free group $F_A = \langle a_1, \ldots, a_r \mid - \rangle$.
- In F_A we have the natural notion of size and balls.
- For $w_1, \ldots, w_k \in F_A$, let $G_{w_1,\ldots,w_k} = \langle a_1, \ldots, a_r \mid w_1, \ldots, w_k \rangle$.

Theorem (Arzhantseva-Ol'shanskii, '96)

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- Hence, generically many presentations present an infinite hyperbolic group.
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- Fix $r \geqslant 2$ and $k \geqslant 1$.
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Outline

A new point of view

Observation

Let $N = \langle w_1, \ldots, w_k \rangle \leqslant F_A$. Then,

$$\langle a_1, \ldots, a_r \mid w_1, \ldots, w_k \rangle \simeq \langle a_1, \ldots, a_r \mid N \rangle.$$

and let us count f.g. subgroups N of F_A, instead of counting k-tuples of words.

Advantages:

- r still fixed, but not k.
- less redundancy.
- it will be an equally natural way of counting.

... but with very different results... this is a very different direction to infinity.

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Stallings automata

Definition

A Stallings automaton *is a finite A-labeled oriented graph with a distinguished vertex,* (X, v)*, such that:*

1- X *is connected,*

2- *no vertex of degree 1 except possibly* v *(*X *is a* core-graph*),*

3- *no two edges with the same label go out of (or in to) the same vertex.*

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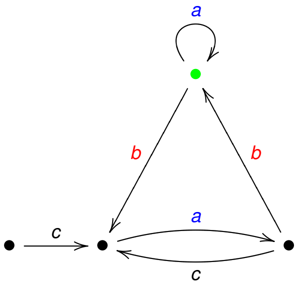

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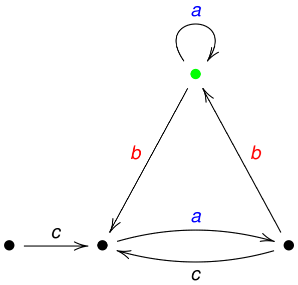

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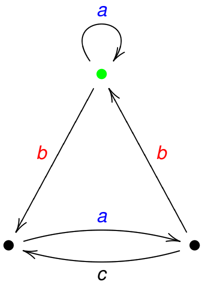

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Stallings (building on previous works) gave a bijection between finitely generated subgroups of F_A and Stallings automata:

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To any given (Stallings) automaton (X, v), we associate its fundamental group:

$$\pi(X, v) = \{ \text{labels of closed paths at } v\} \leqslant F_A,$$

clearly, a subgroup of F_A.

$$\pi(X, \bullet) = \{1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \ldots\}$$

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Membership problem in $\pi(X, \bullet)$ is solvable.

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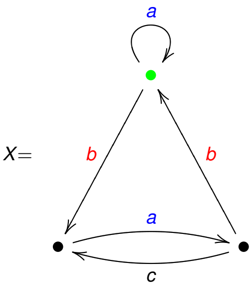

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A basis for $\pi(X, v)$

Proposition

For every Stallings automaton (X, v), the group $\pi(X, v)$ is free of rank $rk(\pi(X, v)) = 1 - |VX| + |EX|$.

Proof:

- Take a maximal tree T in X.
- Write $T[p, q]$ for the geodesic (i.e. the unique reduced path) in T from p to q.
- For every $e \in EX - ET$, $x_e = label(T[v, \iota e] \cdot e \cdot T[\tau e, v])$ belongs to $\pi(X, v)$.
- Not difficult to see that $\{x_e \mid e \in EX - ET\}$ is a basis for $\pi(X, v)$.
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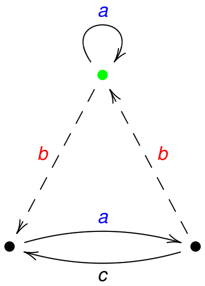

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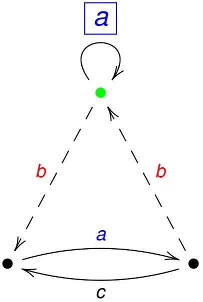

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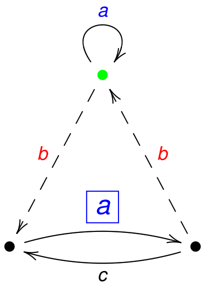

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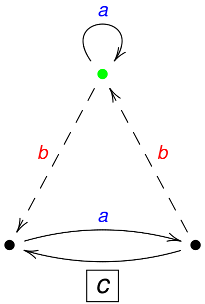

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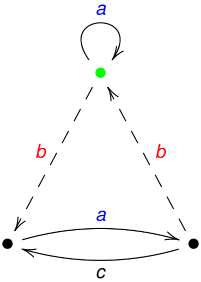

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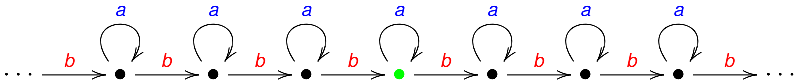
$rk(H) = 1 - 3 + 5 = 3.$

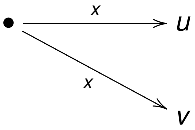
Example-2



$$F_{\aleph_0} \simeq H = \langle \ldots, b^{-2}ab^2, b^{-1}ab, a, bab^{-1}, b^2ab^{-2}, \ldots \rangle \leqslant F_2.$$

Constructing the automata from the subgroup

In any automaton containing the following situation, for $x \in A^{\pm 1}$,



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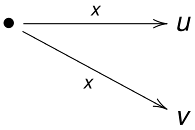

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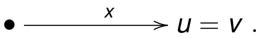

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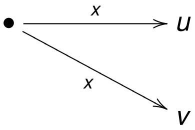

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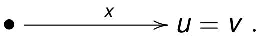

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If $(X, v) \rightsquigarrow (X', v')$ is a Stallings folding then $\pi(X, v) = \pi(X', v')$.

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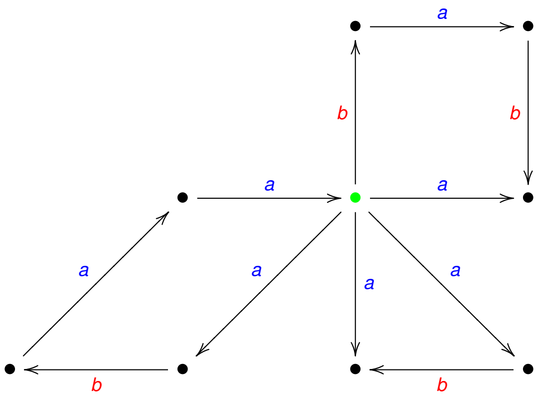

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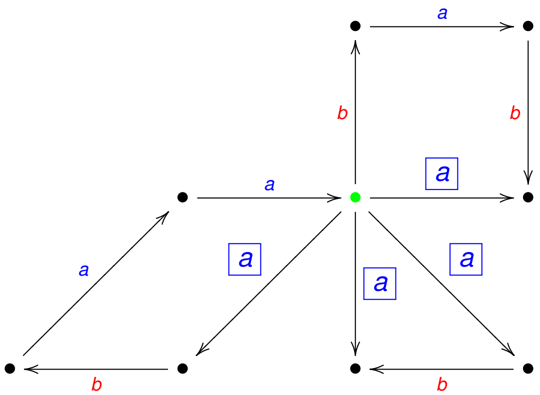

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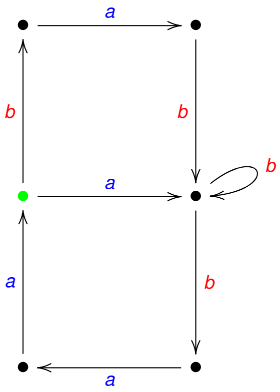

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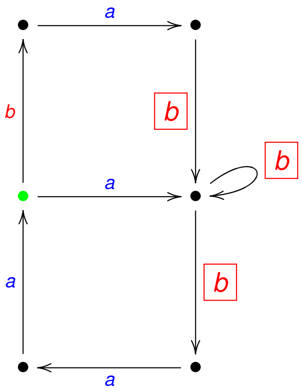

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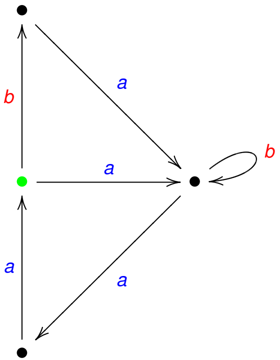

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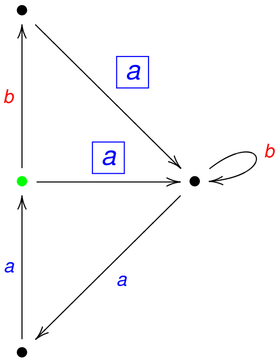

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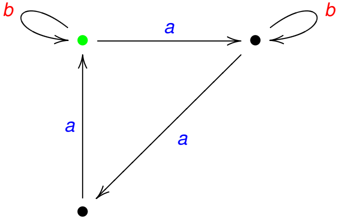

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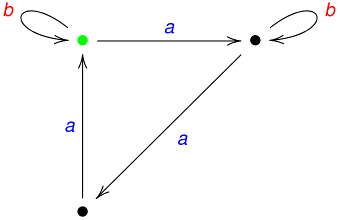

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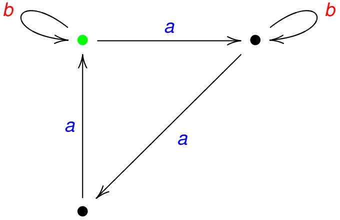

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By Stallings Lemma, $\pi(\Gamma(H), \bullet) = \langle baba^{-1}, aba^{-1}, aba^2 \rangle$
$= \langle b, aba^{-1}, a^3 \rangle$

Local confluence

It can be shown that

Proposition

The automaton $\Gamma(H)$ does not depend on the sequence of foldings.

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The automaton $\Gamma(H)$ does not depend on the generators of H.

Theorem

The following is a bijection:

$$\begin{array}{ccc} \{f.g.\ subgroups\ of\ F_A\} & \longleftrightarrow & \{Stallings\ automata\} \\ H & \to & \Gamma(H) \\ \pi(X,v) & \leftarrow & (X,v) \end{array}$$

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Every subgroup of F_A is free.

- Finite automata work for the finitely generated case, but everything extends easily to the general case (using infinite graphs).
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Outline

Stallings' graphs as partial injections

Definition

Let Γ be a Stallings graph. Every letter in A determines a partial injection of the set of vertices $V\Gamma$: $a(i) = j$ *iff* $i \xrightarrow{a} j$.

$\Gamma=$

$a: V$	$\to$	V	$b: V$	$\to$	V	$c: V$	$\to$	V
1	$\mapsto$	1	1	$\mapsto$	2	1		
2	$\mapsto$	3	2			2		
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And the r partial injections $a_1, \ldots, a_r$ determine back the graph Γ.

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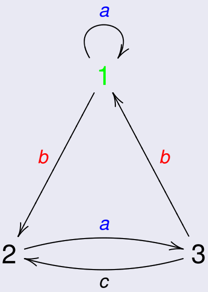

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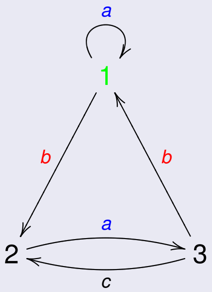

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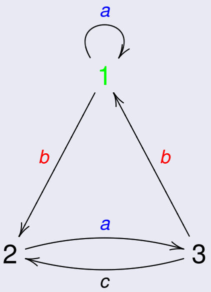
$\Gamma =$ 

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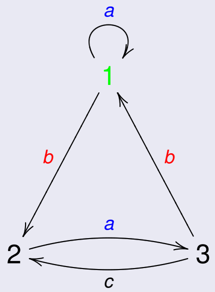

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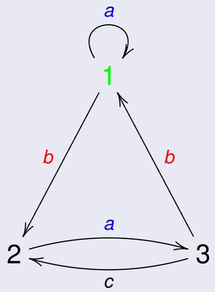
$\Gamma=$ 

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A Stallings graph (over A) with n vertices can be thought as a r-tuple of partial injections, plus a base-point, $\sigma \in I_n^r \times [n]$, such that

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Counting partial injections

Observation

Any partial injection $\sigma \in I_n$ decomposes in orbits of two types: closed and open (i.e. cycles and segments).

Definition

A partial injection $\sigma \in I_n$ is called a

- *permutation if all its orbits are closed,*
- *fragmented permutation if all its orbits are open.*

Let S_n and J_n, resp., be the sets of permutations and fragmented permutations in I_n.

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Every partial injection is the disjoint union of a permutation and a fragmented permutation. In particular, $|I_n| = \sum_{k=0}^{n} \binom{n}{k} |S_k||J_{n-k}| = \sum_{k=0}^{n} \frac{n!}{(n-k)!} |J_{n-k}|$.

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Counting partial injections

Definition

a) The EGS for partial injections: $I(z) = \sum_{n=0}^{\infty} \frac{|I_n|}{n!} z^n$.

b) The EGS for permutations: $S(z) = \sum_{n=0}^{\infty} \frac{|S_n|}{n!} z^n = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

c) The EGS for fragmented permutations: $J(z) = \sum_{n=0}^{\infty} \frac{|J_n|}{n!} z^n$.

Theorem

a) $I(z) = \frac{1}{1-z} e^{\frac{z}{1-z}} = 1 + 2z + \frac{7}{2} z^2 + \frac{17}{3} z^3 + \cdots$.

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Outline

1. A claim due to Gromov
2. Arzhantseva-Ol'shanskii's proof
3. A new point of view
4. Stallings' graphs
5. Counting Stallings' graphs: partial injections
6. **Most groups are trivial**
7. Proof of the combinatorial theorem

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Definition

Let $\sigma \in I_n$. Define $\gcd(\sigma)$ as the gcd *of the lengths of the closed orbits of σ (if $\sigma \in J_n$, put $\gcd(\sigma) = \infty$).*

Key observation

Let $\sigma = (\sigma_1, \ldots, \sigma_r, j) \in {I_n}^r \times [n]$, let $\Gamma(\sigma)$ be the corresponding (Stallings) graph, and let $G = \langle a_1, \ldots, a_r \mid \pi(\Gamma(\sigma)) \rangle$. We have,

- *if $\gcd(\sigma_i) = 1$ then $a_i = 1$ in G,*
- *if $\gcd(\sigma_1) = \cdots = \gcd(\sigma_r) = 1$ then $G = 1$.*

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Definition

For a prime p, let $S_n^{(p)}$ be the set of permutations $\sigma \in S_n$ with all its cycles having length multiple of p. Clearly, $S_n^{(p)} \neq \emptyset \quad \Rightarrow p|n$.

Lemma

Let $n \geqslant 2$, and p be a prime divisor of n. Then,

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$$\leqslant \frac{1}{|I_n|} \sum_{k=0}^{\lfloor n^{1/3} \rfloor} \frac{n!}{(n-k)!} |J_{n-k}| + \frac{1}{|I_n|} \sum_{k=\lceil n^{1/3} \rceil}^{n} \frac{n!}{(n-k)!} \frac{M}{\sqrt{k}} |J_{n-k}|$$

Proof of the combinatorial theorem

Theorem (Bassino, Martino, Nicaud, V., Weil, 2010)

$$\frac{|\{\sigma \in I_n \mid \gcd(\sigma) > 1\}|}{|I_n|} = \mathcal{O}(\frac{1}{n^{1/6}})$$

Proof.

- *Every such $\sigma \in I_n$ is the disjoint union of a permutation in S_k and a fragmented permutation in J_{n-k}, for some $k = 0, \ldots, n$.*
- *Let's distinguish between k "short" and k "long".*

$$\frac{|\{\sigma \in I_n \mid \gcd(\sigma) > 1\}|}{|I_n|} = \frac{1}{|I_n|} \sum_{k=0}^{n} \binom{n}{k} |Q_k||J_{n-k}|$$

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Proof of the combinatorial theorem

$$\leqslant \frac{1}{|I_n|} n!(1+\lfloor n^{1/3}\rfloor)\frac{|J_n|}{n!} + \frac{M}{|I_n|\cdot n^{1/6}} \sum_{k=\lceil n^{1/3}\rceil}^{n} \frac{n!}{(n-k)!}|J_{n-k}|$$

$$\leqslant (1+\lfloor n^{1/3}\rfloor)\frac{|J_n|}{|I_n|} + \frac{M}{|I_n|\cdot n^{1/6}} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} k!|J_{n-k}|$$

$$\leqslant \mathcal{O}(\frac{n^{1/3}}{n^{1/2}}) + \mathcal{O}(\frac{1}{n^{1/6}})$$

$$= \mathcal{O}(\frac{1}{n^{1/6}}). \quad \square$$

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Thanks