Orbit decidability, applications and variations

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Outline

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Orbit decidability

Definition

Let X be a set. A collection of maps $A \subseteq \mathrm{Map}(X, X)$ is said to be orbit decidable (O.D.) if there is an algorithm s.t., given $x, y \in X$, it decides whether $x\alpha = y$ for some $\alpha \in A$ (and, if so, finds such an α).

Definition

For X and $A \subseteq \mathrm{Map}(X, X)$, the orbit of $x \in X$ is $\mathcal{O}(x) = \{x\alpha \mid \alpha \in A\}$.

Observation

O.D. is membership in a given orbit of A (in X).

(Zoom into the problem)

- *Geometry: take X = space, A = action;*
- *Algebra: take X = algebraic structure, $A \subseteq \mathrm{End}(X)$;*
    - *Our case: $X = G$ group, $A \subseteq \mathrm{End}(G)$.*

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First examples: $G = \mathbb{Z}^d$

Observation (folklore)

The full group $\mathrm{Aut}(\mathbb{Z}^d) = \mathrm{GL}_d(\mathbb{Z})$ *is orbit decidable.*

Proof. *For* $u, v \in \mathbb{Z}^d$, *there exists* $A \in \mathrm{GL}_d(\mathbb{Z})$ *such that* $v = Au$ *if and only if* $gcd(u_1, \ldots, u_d) = gcd(v_1, \ldots, v_d)$.

Proposition (Bogopolski-Martino-V., 2008)

Finite index subgroups of $GL_d(\mathbb{Z})$ *are O.D.*

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- *Denote λ the eigenvalue of A with maximum modulus. The projection of uA^n to E_λ grows faster than all other projections.*
- *So we can compute n_0 such that either u, uA, $uA^2, uA^3, \ldots, uA^{n_0}$ hits v, or either $uA^n \neq v$ for all n.*

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Brinkmann's result

Theorem (Brinkmann, 2006)

Cyclic groups of $\mathrm{Aut}(F_r)$ *are orbit decidable. That is, given* $\varphi \in \mathrm{Aut}(F_r)$ *and* $u, v \in F_r$*, one can decide whether* $v = u\varphi^n$ *for some* $n \in \mathbb{Z}$*.*

Proof.

- *Same idea as before: there is a computable* n_0 *such that either* $u, u\varphi, u\varphi^2, u\varphi^3, \ldots, u\varphi^{n_0}$ *hits* v*, or either* $u\varphi^n \neq v$ *for all* n*.*
- *The computation of* n_0 *is quite complicated, making strong use of train-tracks.*

Theorem (Brinkmann, 2006)

Cyclic groups of $\mathrm{Aut}(F_r)$ *are orbit decidable up to conjugacy. That is, given* $\varphi \in \mathrm{Aut}(F_r)$ *and* $u, v \in F_r$*, one can decide whether* $v \sim u\varphi^n$ *for some* $n \in \mathbb{Z}$ *(i.e.,* $\langle \varphi \rangle \cdot \mathrm{Inn}(F_r)$ *is O.D.).*

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Whitehead problem and variations

Theorem (Whitehead'30)

The full group $\mathrm{Aut}(F_r)$ is orbit decidable. That is, given $u, v \in F_r$ one can decide whether $v = u\alpha$ for some $\alpha \in \mathrm{Aut}(F_r)$.

This is a classical and very influential result.

Proposition (Bogopolski-Martino-V., 2008)

Finite index subgroups of $\mathrm{Aut}(F_r)$ are O.D.

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Every finitely generated subgroup of $\mathrm{Aut}(F_2)$ is O.D.

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Theorem (Makanin, 1982)

The full $\operatorname{End}(F_r)$ *is orbit decidable. That is, given* $u, v \in F_r$ *one can decide whether* $v = u\alpha$ *for some* $\alpha \in \operatorname{End}(F_r)$ *(also for tuples).*

Proof. *It reduces to solving (a system of) equation over* F_r.

Theorem (Ciobanu-Houcine, 2010)

$\operatorname{Mon}(F_r)$ *is orbit decidable. That is, given* $u, v \in F_r$ *one can decide whether* $v = u\alpha$ *for some injective endomorphism* $\alpha \in \operatorname{Mon}(F_r)$ *(also for tuples).*

Theorem

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A virtual endomorphism of G is a homomorphism $\varphi\colon H \to K$ between finite index subgroups $H, K \leqslant_{\text{fi}} G$.

Theorem (Rubió-V., w.p.)

The collection of virtual endos (resp. virtual monos, virtual autos) of F_r is O.D. (also for tuples).

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Other groups

Theorem (Collins, Zieschang, 1984)

Let $G_1, \ldots, G_n$ be freely indecomposable groups with $\mathrm{Aut}(G_i)$ being O.D. Then, its free product $G = G_1 * G_2 * \cdots * G_n$ has $\mathrm{Aut}(G)$ O.D.

Theorem (Levitt-Vogtman, 2000)

For a surface group G, $\mathrm{Aut}(G)$ is O.D. (also for tuples).

Theorem (Dahmani, Girardel, 2010)

For a hyperbolic group G, $\mathrm{Aut}(G)$ is O.D. (also for tuples).

Theorem (Kharlampovich-V., 2012)

For G torsion-free relatively hyperbolic with abelian parabolic subgroups, $\mathrm{Aut}(G)$ is O.D. (also for tuples).

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Theorem (Day, 2013)

For G a PC group $\mathrm{Aut}(G)$ is O.D. (also for tuples modulo conjugation).

Theorem (Delgado-V., 2013)

For $G = \mathbb{Z}^m \times F_n$, $\mathrm{Aut}(G)$, $\mathrm{Mon}(G)$ and $\mathrm{End}(G)$ are all O.D.

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Outline

Finding orbit undecidable subgroups

Proposition (Bogopolski-Martino-V., 2008)

Let F be a group, and let $A \leqslant B \leqslant \mathrm{Aut}(F)$ and $v \in F$ be such that $B \cap Stab^*(v) = 1$. Then,

$$OD(A) \text{ solvable} \quad \Rightarrow \quad MP(A, B) \text{ solvable.}$$

Proof. *Given $\varphi \in B \leq \mathrm{Aut}(F)$, let $w = v\varphi$ and*

$$\{\phi \in B \mid v\phi = w\} = B \cap (Stab(v) \cdot \varphi) = (B \cap Stab(v)) \cdot \varphi = \{\varphi\}.$$

$$\{\phi \in B \mid v\phi \sim w\} = B \cap (Stab^*(v) \cdot \varphi) = (B \cap Stab^*(v)) \cdot \varphi = \{\varphi\}.$$

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Proposition (Bogopolski-Martino-V., 2008)

Let F be a group, and let $A \leqslant B \leqslant \mathrm{Aut}(F)$ and $v \in F$ be such that $B \cap Stab^*(v) = 1$. Then,

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Corollary (Bogopolski-Martino-V., 2008)

$\mathrm{Aut}(F_r)$ contains f.g. orbit undecidable subgroups, for $r \geqslant 3$.

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$\mathrm{GL}_d(\mathbb{Z})$ contains f.g. orbit undecidable subgroups, for $d \geqslant 4$.

Question

Does there exist an orbit undecidable subgroup of $GL_3(\mathbb{Z})$?

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Connection to semidirect products

Observation (B-M-V)

Let F be f.g., and $\Gamma \leqslant_{\mathrm{fg}} \mathrm{Aut}(F)$. If $F \rtimes \Gamma$ has solvable CP, then $\Gamma \cdot \mathrm{Inn}(F) \leqslant \mathrm{Aut}(F)$ is orbit decidable.

Proof. $G = F \rtimes \Gamma$ *contains elements* $(x, \gamma) \in F \times \Gamma$ *operated like*

$$(x_1, \gamma_1) \cdot (x_2, \gamma_2) = (x_1 \gamma_1(x_2), \gamma_1 \gamma_2)$$

$$(x, \gamma)^{-1} = (\gamma^{-1}(x^{-1}), \gamma^{-1}).$$

For $x_1, x_2 \in F \leqslant G$, *we have* $x_1 \sim_G x_2 \;\Leftrightarrow\; \exists (x, \gamma) \in F \rtimes \Gamma$ *s.t.*

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Hence, $x_1 \sim_G x_2 \;\Leftrightarrow\; \exists \gamma \in \Gamma$ *and* $x \in F$ *s.t.* $x_1 = x \gamma(x_2) x^{-1}$. $\square$

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In fact, for the free and free abelian cases (among others), the converse is also true after "erasing the relations from Γ":

Let F be a group, $\alpha_1, \ldots, \alpha_m \in \mathrm{Aut}(F)$, and consider $\langle \alpha_1, \ldots, \alpha_m \rangle \leqslant$ $\leqslant \mathrm{Aut}(F)$ and the semidirect product $G = F \rtimes_{\alpha_1, \ldots, \alpha_m} F_m$.

Theorem (B-M-V, 2008)

Let F be $\mathbb{Z}^d$ or F_r. Then $G = F \rtimes_{\alpha_1, \ldots, \alpha_m} F_m$ has solvable CP if and only if $\Gamma = \langle \alpha_1, \ldots, \alpha_m \rangle \cdot \mathrm{Inn}(F) \leqslant \mathrm{Aut}(F)$ is orbit decidable.

This comes from a more general result:

- replace F to any group with solvable TCP,
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Theorem (Bogopolski-Martino-V., 2008)

Let

$$1 \longrightarrow F \stackrel{\alpha}{\longrightarrow} G \stackrel{\beta}{\longrightarrow} H \longrightarrow 1$$

be an algorithmic short exact sequence of groups such that

(i) *$TCP(F)$ is solvable,*

(ii) *$CP(H)$ is solvable,*

(iii) *there is an algorithm which, given an input $1 \neq h \in H$, computes a finite set of elements $z_{h,1}, \ldots, z_{h,t_h} \in H$ such that*

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

Then,

$$CP(G) \text{ is solvable} \iff A_G = \left\{ \begin{array}{rccc} \gamma_g : & F & \to & F \\ & x & \mapsto & g^{-1}xg \end{array} \,\middle|\, g \in G \right\}$$

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Twisted conjugacy

Definition

For $\varphi \in \mathrm{End}(G)$, two elements $u, v \in G$ are said to be φ-twisted conjugated, denoted $u \sim_\varphi v$, if $v = (g\varphi)^{-1}ug$ for some $g \in G$.

Definition

The twisted conjugacy problem for G, denoted $TCP(G)$: "Given $\varphi \in \mathrm{Aut}(G)$ and $u, v \in G$ decide whether $u \sim_\varphi v$".

Observation

$TCP(\mathbb{Z}^d)$ is solvable.

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For free abelian-by-free groups:

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$\mathbb{Z}^d$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

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If $\Gamma = \langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_d(\mathbb{Z})$ then $\mathbb{Z}^d \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

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If $\Gamma = \langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_d(\mathbb{Z})$ then $\mathbb{Z}^d \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Corollary

Every $\mathbb{Z}^2$-by-free group has solvable conjugacy problem.

Positive applications

For free-by-free groups:

Corollary (Bogopolski-Martino-Maslakova-V., 2006)

Free-by-cyclic groups have solvable conjugacy problem.

Corollary

If $\Gamma = \langle \varphi_1, \ldots, \varphi_m \rangle$ has finite index in $\mathrm{Aut}(F_r)$ then $F_r \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

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Taking the copy B of $F_2 \times F_2$ in $\mathrm{Aut}(F_3)$ *via the embedding*

$$\begin{array}{rcl} F_2 \times F_2 & \hookrightarrow & Aut(F_3), \\ (u,v) & \mapsto & {}_u\theta_v \colon \begin{array}{rcl} F_3 & \to & F_3 \\ q & \mapsto & u^{-1}qv \\ a & \mapsto & a \\ b & \mapsto & b \end{array} \end{array}$$

and a Mihailova subgroup in there $A \leqslant B \leqslant$ $\mathrm{Aut}(F_3)$ *(taking $v = qaqbq$) one obtains precisely the orbit undecidable subgroups corresponding to Miller's examples.*

Theorem (Miller, 70's)

There exist free-by-free groups ($F_3 \rtimes F_{14}$) with unsolvable conjugacy problem.

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Question

Does there exist a $\mathbb{Z}^3$-by-free group with unsolvable conjugacy problem ?

Theorem (Burillo-Matucci-V., 2012)

There exists a Thompson-by-free group with unsolvable conjugacy problem.

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Playing with 2 extra dimensions...

These orbit undecidable examples $\Gamma \leqslant GL_4(\mathbb{Z})$ come from Mihailova's construction, so they are not finitely presented...

Proposition (Sunic-V.)

For $d \geqslant 6$, $GL_d(\mathbb{Z})$ contains f.g., orbit undecidable, free, subgroups.

Proof. *Let $d \geqslant 6$.*

- *Since $d-2 \geqslant 4$, there exists $\langle g_1, \ldots, g_m \rangle = \Gamma \leqslant GL_{d-2}(\mathbb{Z})$ being orbit undecidable.*
- *Let $F_m = \langle f_1, \ldots, f_m \rangle$, and choose matrices $s_1, \ldots, s_m \in GL_2(\mathbb{Z})$ such that $\langle s_1, \ldots, s_m \rangle \simeq F_m$.*
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In summary,

For $d \geqslant 6$*, there exists a* ***free*** $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$ *such that* $\mathbb{Z}^d \rtimes \Gamma$ *has unsolvable CP.*

Theorem (Sunic-V., 2012)

There exist automaton groups (i.e. self-similar groups generated by finite self-similar sets) with unsolvable conjugacy problem.

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