The degree of commutativity/nilpotency of an infinite group

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Outline

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Motivation

Y. Antolín, A. Martino, E.V., "Degree of commutativity of infinite groups", Proc. Amer. Math. Soc. **145**(2) (2017), 479-485.

Theorem (Gustafson, 1973)

Let G be a finite group. If the probability that two elements from G commute is bigger than $5/8$, then G is abelian.

Proof. *Suppose G is not abelian. Then,*

$$dc(G) = \frac{|\{(u,v) \in G^2 \mid uv = vu\}|}{|G|^2} = \frac{1}{|G|^2} \sum_{u \in G} |C_G(u)| =$$

$$= \frac{1}{|G|^2} \Big(|Z(G)||G| + \sum_{u \in G \setminus Z(G)} |C_G(u)| \Big) \leqslant$$

$$\leqslant \frac{1}{|G|^2} \Big(|Z(G)||G| + (|G| - |Z(G)|) \frac{|G|}{2} \Big) =$$

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because $G/Z(G)$ cannot be cyclic and so, $|Z(G)| \leqslant |G|/4$. □

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The quaternion group has $dc(Q) = 5/8$.

"There is no live between $5/8$ and 1"

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Is there a version of dc for infinite groups ?

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Degree of commutativity

Definition

Let $G = \langle X \rangle$ be a f.g. group. The degree of commutativity of G w.r.t. X is

$$dc_X(G) = \limsup_{n\to\infty} \frac{|\{(u,v) \in \mathbb{B}_X(n) \times \mathbb{B}_X(n) \mid uv = vu\}|}{|\mathbb{B}_X(n)|^2} \in [0,1],$$

where $\mathbb{B}_X(n) = \{g \in G \mid |g|_X \leqslant n\}$.

Question

(i) *Is this a real lim ?*

(ii) *Does it depend on X ?*

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Main result

Theorem (Antolín–Martino–V.)

Let $G = \langle X \rangle$ be of subexponential growth and residually finite (this includes all groups of polynomial growth). Then,*

(i) $dc_X(G) > 5/8 \Leftrightarrow G$ *is abelian;*

(ii) $dc_X(G) > 0 \Leftrightarrow G$ *is virtually abelian;*

(iii) $dc_X(G)$ *is a real limit and does not depend on* X.

Conjecture

The same is true for an arbitrary f.g. G.

Very recently

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Independence on X

Definition

A f.g. group $G = \langle X \rangle$ is of

- *subexponential* growth if* $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n+1)|}{|\mathbb{B}_X(n)|} = 1$;
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Definition

Let $G = \langle X \rangle$. A map $f \colon G \to \mathbb{N}$ is an estimation of the X-metric if $\exists$ $K > 0$ such that $\forall w \in G$

$$\frac{1}{K} f(w) \leqslant |w|_X \leqslant K\, f(w).$$

Example

It is well known that, for $G = \langle X \rangle = \langle Y \rangle$, $|\cdot|_X$ is an estimation of the Y-metric, and $|\cdot|_Y$ is an estimation of the X-metric.

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Define the f-ball and the f-dc:

$$\mathbb{B}_f(n) = \{w \in G \mid f(w) \leqslant n\},$$

$$dc_f(G) = \limsup_{n\to\infty} \frac{|\{(u,v) \in \mathbb{B}_f(n) \times \mathbb{B}_f(n) \mid uv = vu\}|}{|\mathbb{B}_f(n)|^2}.$$

Proposition

Let $G = \langle X \rangle$ be of polynomial growth, and $f\colon G \to \mathbb{N}$ be an estimation of the X-metric. Then,

$$dc_X(G) > 0 \iff dc_f(G) > 0.$$

Proof. *Clearly, $\mathbb{B}_f(n) \subseteq \mathbb{B}_X(Kn) \subseteq \mathbb{B}_f(K^2 n)$ so,*

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So, $dc_X(G)=0 \;\Rightarrow\; dc_f(G)=0$ since ...

$$\frac{|\mathbb{B}_f(n)|}{|\mathbb{B}_X(Kn)|} \geqslant \frac{|\mathbb{B}_X(n/K)|}{|\mathbb{B}_X(Kn)|} \geqslant \frac{C(n/K)^d}{D(Kn)^d} = \frac{C}{DK^{2d}} > 0. \quad \square$$

Corollary

If $G=\langle X\rangle=\langle Y\rangle$ is of polynomial growth, then

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Corollary

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Definition

Let $\langle Y\rangle = H \leqslant G = \langle X\rangle$. The subgroup H is undistorted if $\exists\ K > 0$ s.t. $\forall\ h \in H$, $|h|_Y/K \leqslant |h|_X \leqslant K|h|_Y$.
In this case, $|\cdot|_X$ restricted to H is an estimation of the Y-metric for H.

Corollary

Let $G = \langle X\rangle$ be of polynomial growth, and $\langle Y\rangle = H \leqslant G$ be a non-distorted subgroup. Then,

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Outline

Finite index subgroups

Lemma (Burillo–V., 2002)

If $H \leqslant_{f.i.} G = \langle X \rangle$ and G has subexponential growth then, for every $g \in G$, there exists $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n) \cap gH|}{|\mathbb{B}_X(n)|} = \lim_{n\to\infty} \frac{|\mathbb{B}_X(n) \cap Hg|}{|\mathbb{B}_X(n)|} = \frac{1}{[G:H]}$.*

Remark

This is false in the free group: $H = \{\text{even words}\} \leqslant_2 F_r$.

Proposition*

Let $\langle Y \rangle = H \leqslant_{f.i.} G = \langle X \rangle$ be of polynomial growth. Then,

$$dc_X(G) \geqslant \frac{1}{[G:H]^2} dc_X(H).$$

In particular, $dc_Y(H) > 0 \Rightarrow dc_X(H) > 0 \Rightarrow dc_X(G) > 0$.

Finite index subgroups

Lemma (Burillo–V., 2002)

If $H \leqslant_{f.i.} G = \langle X \rangle$ and G has subexponential growth then, for every $g \in G$, there exists $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n) \cap gH|}{|\mathbb{B}_X(n)|} = \lim_{n\to\infty} \frac{|\mathbb{B}_X(n) \cap Hg|}{|\mathbb{B}_X(n)|} = \frac{1}{[G:H]}$.*

Remark

This is false in the free group: $H = \{\text{even words}\} \leqslant_2 F_r$.

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Let $\langle Y \rangle = H \leqslant_{f.i.} G = \langle X \rangle$ be of polynomial growth. Then,

$$dc_X(G) \geqslant \frac{1}{[G:H]^2} dc_X(H).$$

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Proof. *Clearly,*

$|\{(u,v) \in (\mathbb{B}_X(n))^2 \mid uv = vu\}| \geqslant |\{(u,v) \in (H \cap \mathbb{B}_X(n))^2 \mid uv = vu\}|$.

Therefore, given $\varepsilon > 0$*, we have for* $n \gg 0$

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Taking limsups, $dc_X(G) \geqslant dc_X(H)\left(\frac{1}{[G:H]} - \varepsilon\right)^2$*. And this is true* $\forall \varepsilon > 0$ *so,* $dc_X(G) \geqslant \frac{1}{[G:H]^2} dc_X(H)$. □

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Proposition* (Gallagher, 1970)

Let G be a finite group and $H \trianglelefteq G$. Then, $dc(G) \leqslant dc(H) \cdot dc(G/H)$.

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Proof. *Let $N \trianglelefteq G$ with $[G : N] = d$.*
By B–V, $\forall g \in G$ $\lim_{n\to\infty} |gN \cap \mathbb{B}_X(n)|/|\mathbb{B}_X(n)| = 1/d$, indep. X and g.
But $|G/N| < \infty$, so this lim is uniform on g, i.e.,
$\forall \varepsilon > 0,\ \exists n_0,\ \forall n \geqslant n_0$ *and $\forall g \in G$,*

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$\exists \delta > 0$ *s.t.* $|\{(u,v) \in (\mathbb{B}_X(n))^2 \mid uv = vu\}| / |\mathbb{B}_X(n)|^2 > dc(G/N) + \delta$ *for infinitely many n's.*

In the above inequality, take $\varepsilon > 0$ *small enough so that* $2\varepsilon d + \varepsilon^2 d^2 < \delta$, *and* $\exists n \gg 0$ *such that*

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Outline

Proof of the main result

Theorem (Antolín–Martino–V.)

Let $G = \langle X \rangle$ be of subexponential growth and residually finite. Then,*

(i) $dc_X(G) > 5/8 \Leftrightarrow G$ *is abelian;*

(ii) $dc_X(G) > 0 \Leftrightarrow G$ *is virtually abelian;*

Proof: *(i). Suppose $dc_X(G) > 5/8$. Then, $dc(G/N) > 5/8$ for every $N \trianglelefteq_{f.i.} G$. Hence, by Gustafson's thm, every finite quotient of G is abelian. Residual finiteness implies G abelian.*

(ii, $\Leftarrow$). Suppose $G = \langle X \rangle$ is virtually abelian, $\langle Y \rangle = H \leqslant_{f.i.} G$ with H abelian. Then G is polynomially growing and $dc_Y(H) = 1 > 0$ so, $dc_X(G) > 0$.

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Proof: *(i). Suppose* $dc_X(G) > 5/8$. *Then,* $dc(G/N) > 5/8$ *for every* $N \trianglelefteq_{f.i.} G$. *Hence, by Gustafson's thm, every finite quotient of G is abelian.* *Residual finiteness implies G abelian.*

(ii, $\Leftarrow$*). Suppose* $G = \langle X \rangle$ *is virtually abelian,* $\langle Y \rangle = H \leqslant_{f.i.} G$ *with H abelian. Then G is polynomially growing and* $dc_Y(H) = 1 > 0$ *so,* $dc_X(G) > 0$.

(ii, $\Rightarrow$*). Suppose G is not virtually abelian and let us prove that* $dc_X(G) = 0$.

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Then $\forall r, \quad K_r \trianglelefteq G, \quad (G/K_r)/(K_{r-1}/K_r) = G/K_{r-1}$ and, by Gallagher,

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By induction, $dc(G/K_r) \leqslant (5/8)^r$ and so,

$$dc_X(G) \leqslant dc(G/K_r) \leqslant (5/8)^r,$$

for every r. Therefore, $dc_X(G) = 0$. $\square$

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Let $G = \langle X \rangle$ be of subexponential growth and residually finite. Then,*

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Let $G = \langle X \rangle$ be of subexponential growth and residually finite. Then, $dc_X(G)$ is a real limit and does not depend on X.*

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Proposition (Antolín–Martino–V.)

Let G be f.g., and $A \trianglelefteq_{f.i.} G$, with $\mathbb{Z}^k \simeq A$ (via $\mathbf{u} \mapsto t^{\mathbf{u}}$). Then,

$$\left\{(g,h) \in G^2 \mid gh = hg\right\} = \bigsqcup_{i=1}^{r} (g_i A, h_i A) \sqcup \bigsqcup_{i=1}^{s} P_i,$$

where $P_i = \{(g_i t^{\mathbf{u}}, h_i t^{\mathbf{v}}) \mid (\mathbf{u},\mathbf{v}) \in L_i\}$, and L_i is a proper direct summand of $A^2 \simeq \mathbb{Z}^{2k}$.

Proof: *Consider the action of G by right conjugation on A,*

$$\begin{array}{rcl} \gamma\colon G & \to & \mathrm{Aut}(A) = GL_k(\mathbb{Z}) \\ g & \mapsto & \gamma_g \colon A \to A \\ & & \quad t^{\mathbf{u}} \mapsto g^{-1} t^{\mathbf{u}} g = t^{\mathbf{u} M_g}. \end{array}$$

Now split into a finite union

$$\left\{(g,h) \in G^2 \mid gh = hg\right\} = \bigsqcup_{C, C' \in G/A} \left\{(g,h) \in (C, C') \mid gh = hg\right\}.$$

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So, $gt^{\mathbf{u}}$ and $ht^{\mathbf{v}}$ commute $\Leftrightarrow (\mathbf{u},\mathbf{v}) \cdot (M_h - Id, Id - M_g)^t = 0$.

- *If $M_g = M_h = Id$ then we get pairs of full cosets modulo A.*
- *Otherwise, $(M_h - Id, Id - M_g) \neq (0)$ and we get a block of the form $P = \{(gt^{\mathbf{u}}, ht^{\mathbf{v}}) \mid (\mathbf{u},\mathbf{v}) \in L\}$, where $L \leqslant_{\oplus} A^2 \simeq \mathbb{Z}^{2k}$ is proper,*

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Outline

The hyperbolic case

Theorem (Antolín–Martino–V.)

For every non-elementary hyperbolic group G and every X, $dc_X(G) = 0$.

Outline

Generalizations

- *We can replace $xy = yx$ by any system of equations $\mathcal{E}$.*
- *We can replace the uniform measures on balls to any sequence of measures μ_n with increasing compact support (coming from random walks, amenability, etc).*

Definition

Let G, $\mathcal{E}$ and μ_n be as above. We define the degree of satisfiability of $\mathcal{E}$ in G w.r.t. μ_n as

$$ds(G, \mathcal{E}, \{\mu_n\}_n) =$$

$$\limsup_{n\to\infty} \mu_n^{\times k}(\{(g_1, \ldots, g_k) \in G^k \mid (g_1, \ldots, g_k) \text{ sol. } \mathcal{E}\}) \in [0, 1].$$

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Generalizations

Meta-conjecture

Let G, $\mathcal{E}$, and $\{\mu_n\}_n$ be as above, with $\mathcal{E}$ having a gap for finite groups, and μ_n being "reasonable". Then,

$$ds(G, \mathcal{E}, \{\mu_n\}_n) > 0 \iff \mathcal{E} \text{ is a virtual law in } G.$$

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$\mathcal{E}$ is a *law* in G if every $(g_1, \ldots, g_k) \in G^k$ is a solution of $\mathcal{E}$ in G.

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Outline

Degree of r-nilpotency

Let us consider the r-equation: $[[\ [[x_0, x_1], x_2] \cdots], x_r]$.

Notation: $\mathbf{u} = (u_0, \ldots, u_r)$, $[\mathbf{u}] = [u_0, \ldots, u_r] = [[\ [[u_0, u_1], u_2] \cdots], u_r]$.

Definition

For a finite group G, the degree of r-nilpotency is

$$dn_r(G) = \frac{|\{\mathbf{u} \in G^{r+1} \mid [[\ [[u_0, u_1], u_2] \cdots], u_r] = 1\}|}{|G|^{r+1}}.$$

Proposition (indep. by R. Rezaei–Russo for compact groups)

For $r \geqslant 1$, any finite group G, if $dn_r(G) > 1 - \frac{3}{2^{r+2}}$ then $dn_r(G) = 1$.

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For $r \geqslant 1$, any finite group G, if $dn_r(G) > 1 - \frac{3}{2^{r+2}}$ then $dn_r(G) = 1$.

Degree of r-nilpotency

Let us consider the r-equation: $[[\ [[x_0, x_1], x_2] \cdots], x_r]$.

Notation: $\mathbf{u} = (u_0, \ldots, u_r)$, $[\mathbf{u}] = [u_0, \ldots, u_r] = [[\ [[u_0, u_1], u_2] \cdots], u_r]$.

Definition

For a finite group G, the degree of r-nilpotency is

$$dn_r(G) = \frac{|\{\mathbf{u} \in G^{r+1} \mid [[\ [[u_0, u_1], u_2] \cdots], u_r] = 1\}|}{|G|^{r+1}}.$$

Proposition (indep. by R. Rezaei–Russo for compact groups)

For $r \geqslant 1$, any finite group G, if $dn_r(G) > 1 - \frac{3}{2^{r+2}}$ then $dn_r(G) = 1$.

The meta-conjecture for r-nilpotence

Definition

Let $G = \langle X \rangle$ be f.g. The degree of r-nilpotency of G w.r.t. X is

$$dn_{r,X}(G) = \limsup_{n \to \infty} \frac{|\{\mathbf{u} \in \mathbb{B}_X(n)^{r+1} \mid [[\,[[u_0, u_1], u_2] \cdots], u_r] = 1\}|}{|\mathbb{B}_X(n)|^{r+1}},$$

where $\mathbb{B}_X(n) = \{g \in G \mid |g|_X \leqslant n\}$.

Theorem (Antolín–Martino–V.)

Let $G = \langle X \rangle$ be of subexponential growth and residually-p for infinitely many primes. Then,*

(i) $dn_{r,X}(G) > 1 - \frac{3}{2^{r+2}} \Leftrightarrow$ *G is r-nilpotent;*

(ii) $dn_{r,X}(G) > 0 \Leftrightarrow$ *G is virtually r-nilpotent.*

The problem here is that we still don't know whether
$dn_r(G) \leqslant dn_r(H) \cdot dn_r(G/H)$?

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THANKS

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