1. Motivation	2. Main definition	3. Finite index subgroups	4. A Gromov-like theorem	5. The hyperbolic case	6. Generalizations	7. Degree of r-nilpotency

The degree of commutativity/nilpotency of an infinite group

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Bilbao

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- Finite index subgroups
- A Gromov-like theorem
- 5 The hyperbolic case
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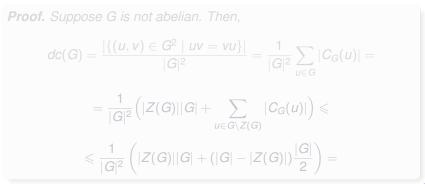
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Y. Antolín, A. Martino, E.V., "Degree of commutativity of infinite groups", Proc. Amer. Math. Soc. **145**(2) (2017), 479-485.

Theorem (Gustafson, 1973)

Let G be a finite group. If the probability that two elements from G commute is bigger than 5/8, then G is abelian.





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$$dc(G) = \frac{|\{(u, v) \in G^2 \mid uv = vu\}|}{|G|^2} = \frac{1}{|G|^2} \sum_{u \in G} |C_G(u)| = \frac{1}{|G|^2} \left(|Z(G)||G| + \sum_{u \in G \setminus Z(G)} |C_G(u)| \right) \leq \frac{1}{|G|^2} \left(|Z(G)||G| + (|G| - |Z(G)|)\frac{|G|}{2} \right) = \frac{1}{|G|^2} \left(|Z(G)||G| + (|G| - |Z(G)|)\frac{|G|}{2} \right)$$



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Observation

The quaternion group has dc(Q) = 5/8.

"There is no live between 5/8 and 1"

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Degree of commutativity

Definition

Let $G = \langle X \rangle$ be a f.g. group. The degree of commutativity of G w.r.t. X is

$$dc_X(G) = \limsup_{n \to \infty} \frac{|\{(u, v) \in \mathbb{B}_X(n) \times \mathbb{B}_X(n) \mid uv = vu\}|}{|\mathbb{B}_X(n)|^2} \in [0, 1],$$

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where $\mathbb{B}_X(n) = \{g \in G \mid |g|_X \leqslant n\}.$

- (i) Is this a real lim?
- (ii) Does it depend on X?
- (iii) What is the relation with the algebraic structure of G?

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Theorem (Antolín–Martino–V.)

Let $G = \langle X \rangle$ be of subexponential* growth and residually finite (this includes all groups of polynomial growth). Then, (i) $dc_X(G) > 5/8 \Leftrightarrow G$ is abelian; (ii) $dc_X(G) > 0 \Leftrightarrow G$ is virtually abelian; (iii) $dc_X(G) > 0 \Leftrightarrow G$ is virtually abelian;

Conjecture

The same is true for an arbitrary f.g. G.

Very recently

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Independence on X

Definition

A f.g. group $G = \langle X \rangle$ is of

- subexponential* growth if $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n+1)|}{|\mathbb{B}_X(n)|} = 1$;
- polynomial growth if $|\mathbb{B}_X(n)| \leq Dn^d$.

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Let $G = \langle X \rangle$. A map $f : G \to \mathbb{N}$ is an estimation of the X-metric if $\exists K > 0$ such that $\forall w \in G$

$$\frac{1}{K}f(w)\leqslant |w|_X\leqslant Kf(w).$$

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It is well known that, for $G = \langle X \rangle = \langle Y \rangle$, $| \cdot |_X$ is an estimation of the *Y*-metric, and $| \cdot |_Y$ is an estimation of the *X*-metric.

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Define the *f*-ball and the *f*-dc:

 $\mathbb{B}_f(n) = \{ w \in G \mid f(w) \leqslant n \},$

 $dc_f(G) = \limsup_{n \to \infty} \frac{|\{(u, v) \in \mathbb{B}_f(n) \times \mathbb{B}_f(n) \mid uv = vu\}|}{|\mathbb{B}_f(n)|^2}$

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Let $G = \langle X \rangle$ be of polynomial growth, and $f : G \to \mathbb{N}$ be an estimation of the X-metric. Then,

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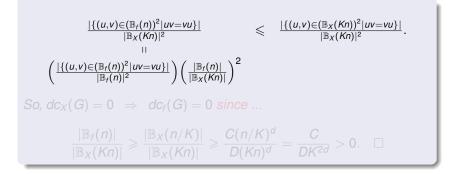
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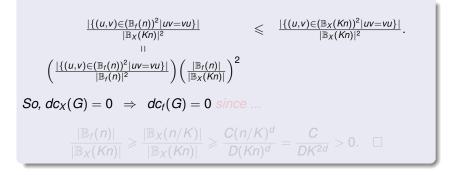


Corollary

If $G = \langle X \rangle = \langle Y \rangle$ is of polynomial growth, then

 $dc_X(G) = 0 \iff dc_Y(G) = 0.$



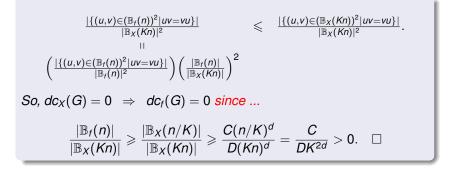


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$$\frac{\frac{|\{(u,v)\in (\mathbb{B}_{f}(n))^{2}|uv=vu\}|}{|\mathbb{B}_{X}(Kn)|^{2}}}{(1-||\mathbb{B}_{f}(n)||^{2}} \leqslant \frac{\frac{|\{(u,v)\in (\mathbb{B}_{X}(Kn))^{2}|uv=vu\}|}{|\mathbb{B}_{X}(Kn)|^{2}}}{(1-||\mathbb{B}_{X}(Kn)||^{2})} \cdot \frac{(1-||\mathbb{B}_{f}(n)||^{2})}{(1-||\mathbb{B}_{f}(n)||^{2})} \circ \frac{dc_{f}(G) = 0 \text{ since } \dots}{dc_{f}(G) = 0 \text{ since } \dots}}$$

$$\frac{|\mathbb{B}_{f}(n)|}{|\mathbb{B}_{X}(Kn)|} \geqslant \frac{|\mathbb{B}_{X}(n/K)|}{|\mathbb{B}_{X}(Kn)|} \geqslant \frac{C(n/K)^{d}}{D(Kn)^{d}} = \frac{C}{DK^{2d}} > 0. \quad \Box$$

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If $G = \langle X \rangle = \langle Y \rangle$ is of polynomial growth, then

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Independence on X

Definition

Let $\langle Y \rangle = H \leq G = \langle X \rangle$. The subgroup H is undistorted if $\exists K > 0$ s.t. $\forall h \in H$, $|h|_Y/K \leq |h|_X \leq K|h|_Y$. In this case, $|\cdot|_X$ restricted to H is an estimation of the Y-metric for H.

Corollary

Let $G = \langle X \rangle$ be of polynomial growth, and $\langle Y \rangle = H \leqslant G$ be a non-distorted subgroup. Then,

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Outline



- Main definition and results
- Finite index subgroups
- A Gromov-like theorem
- 5 The hyperbolic case
- 6 Generalizations
- Degree of r-nilpotency



Lemma (Burillo–V., 2002)

If $H \leq_{f.i.} G = \langle X \rangle$ and G has subexponential* growth then, for every $g \in G$, there exists $\lim_{n \to \infty} \frac{|\mathbb{B}_X(n) \cap \mathcal{G}H|}{|\mathbb{B}_X(n)|} = \lim_{n \to \infty} \frac{|\mathbb{B}_X(n) \cap \mathcal{H}g|}{|\mathbb{B}_X(n)|} = \frac{1}{[G:H]}$.

Remark

This is false in the free group: $H = \{even words\} \leq_2 F_r$.

Proposition³

Let $\langle Y \rangle = H \leq_{f.i.} G = \langle X \rangle$ be of polynomial growth. Then,

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In particular, $dc_Y(H) > 0 \Rightarrow dc_X(H) > 0 \Rightarrow dc_X(G) > 0$.

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Proof. Clearly, $|\{(u, v) \in (\mathbb{B}_{X}(n))^{2} \mid uv = vu\}| \ge |\{(u, v) \in (H \cap \mathbb{B}_{X}(n))^{2} \mid uv = vu\}|.$



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Let G be a finite group and $H \leq G$. Then, $dc(G) \leq dc(H) \cdot dc(G/H)$.

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Finite index subgroups

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Outline



- 2 Main definition and results
- Finite index subgroups
- 4 Gromov-like theorem
- 5 The hyperbolic case
- 6 Generalizations
- Degree of r-nilpotency

1. Motivation	2. Main definition	3. Finite index subgroups	4. A Gromov-like theorem	5. The hyperbolic case	6. Generalizations	7. Degree of r-nilpotency
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Theorem (Antolín–Martino–V.)

Let $G = \langle X \rangle$ be of subexponential* growth and residually finite. Then, (i) $dc_X(G) > 5/8 \Leftrightarrow G$ is abelian; (ii) $dc_X(G) > 0 \Leftrightarrow G$ is virtually abelian;

Proof: (i). Suppose $dc_X(G) > 5/8$. Then, dc(G/N) > 5/8 for every $N \leq_{f.i.} G$. Hence, by Gustafson's thm, every finite quotient of G is abelian. Residual finiteness implies G abelian.

(ii, \Leftarrow). Suppose $G = \langle X \rangle$ is virtually abelian, $\langle Y \rangle = H \leq_{f.i.} G$ with H abelian. Then G is polynomially growing and $dc_Y(H) = 1 > 0$ so, $dc_X(G) > 0$.

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Let $G = \langle X \rangle$ be of subexponential^{*} growth and residually finite. Then,

(i) $dc_X(G) > 5/8 \Leftrightarrow G$ is abelian;

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Proof: (i). Suppose $dc_X(G) > 5/8$. Then, dc(G/N) > 5/8 for every $N \leq_{f.i.} G$. Hence, by Gustafson's thm, every finite quotient of G is abelian. Residual finiteness implies G abelian.

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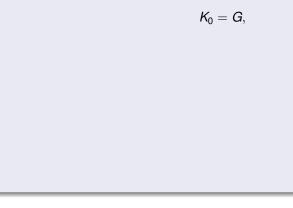
Claim: If H is f.g., r.f., not virtually abelian then $\exists K \leq_{ch.} H$ such that H/K is (finite) not abelian.

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Proof of the main result

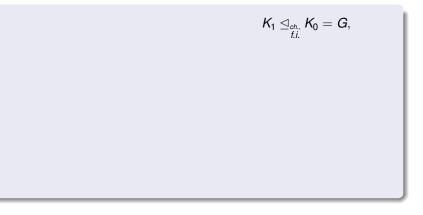
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$$K_2 \leq_{ch., f.i.} K_1 \leq_{ch., f.i.} K_0 = G,$$

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such that K_{r-1}/K_r is not abelian so, $dc(K_{r-1}/K_r) \leq 5/8 \quad \forall r$.

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such that K_{r-1}/K_r is not abelian so, $dc(K_{r-1}/K_r) \leq 5/8 \quad \forall r$. Then $\forall r$, $K_r \leq G$, $(G/K_r)/(K_{r-1}/K_r) = G/K_{r-1}$ and, by Gallagher,

 $dc(G/K_r) \leqslant dc(K_{r-1}/K_r) \cdot dc(G/K_{r-1}) \leqslant 5/8 \cdot dc(G/K_{r-1}).$

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 $dc(G/K_r) \leqslant dc(K_{r-1}/K_r) \cdot dc(G/K_{r-1}) \leqslant 5/8 \cdot dc(G/K_{r-1}).$

By induction, $dc(G/K_r) \leq (5/8)^r$ and so,

$$dc_X(G) \leqslant dc(G/K_r) \leqslant (5/8)^r$$
,

for every r. Therefore, $dc_X(G) = 0$. \Box

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Independence from X

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Corollary

Let $G = \langle X \rangle$ be of subexponential^{*} growth and residually finite. Then, $dc_X(G)$ is a real limit and does not depend on X.

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Independence from X

Proposition (Antolín-Martino-V.)

Let G be f.g., and $A \leq_{f.i.} G$, with $\mathbb{Z}^k \simeq A$ (via $\boldsymbol{u} \mapsto t^{\boldsymbol{u}}$). Then,

$$\left\{(g,h)\in G^2\mid gh=hg
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where $P_i = \{(g_i t^{u}, h_i t^{v}) \mid (u, v) \in L_i\}$, and L_i is a proper direct summand of $A^2 \simeq \mathbb{Z}^{2k}$.

Proof: Consider the action of G by right conjugation on A,

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Now split into a finite union

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So, $gt^{\mathbf{u}}$ and $ht^{\mathbf{v}}$ commute $\Leftrightarrow (\mathbf{u}, \mathbf{v}) \cdot (M_h - Id, Id - M_g)^t = 0$.

• If $M_g = M_h = Id$ then we get pairs of full cosets modulo A.

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Hence, $dc_X(G) = \frac{r}{[G:A]^2} + s \cdot 0$ as a <u>real limit</u>, and independent from *X*.

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Hence, $dc_X(G) = \frac{r}{[G:A]^2} + s \cdot 0$ as a <u>real limit</u>, and <u>independent</u> from *X*.

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The hyperbolic case

Theorem (Antolín–Martino–V.)

For every non-elementary hyperbolic group G and every X, $dc_X(G) = 0$.



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• We can replace xy = yx by any system of equations \mathcal{E} .

• We can replace the uniform measures on balls to any sequence of measures μ_n with increasing compact support (coming from random walks, amenability, etc).

Definition

Let G, \mathcal{E} and μ_n be as above. We define the degree of satisfiability of \mathcal{E} in G w.r.t. μ_n as

$ds(G, \mathcal{E}, \{\mu_n\}_n) =$

 $\limsup_{n\to\infty}\mu_n^{\times k}\big(\{(g_1,\ldots,g_k)\in G^k\mid (g_1,\ldots,g_k) \text{ sol. } \mathcal{E}\}\big)\in[0,1].$

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Meta-conjecture

Let G, \mathcal{E} , and $\{\mu_n\}_n$ be as above, with \mathcal{E} having a gap for finite groups, and μ_n being "reasonable". Then,

 $ds(G, \mathcal{E}, {\mu_n}_n) > 0 \iff \mathcal{E}$ is a virtual law in G.

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Definition

 ${\mathcal E}$ is a law in G if every $(g_1,\ldots,g_k)\in G^k$ is a solution of ${\mathcal E}$ in G.

 \mathcal{E} is a virtual law in G if $\exists H \leq_{f.i.} G$ such that \mathcal{E} is a law in H.

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Let us consider the r-equation: $[[x_0, x_1], x_2] \cdots], x_r]$.

Notation: $\mathbf{u} = (u_0, \dots, u_r), [\mathbf{u}] = [u_0, \dots, u_r] = [[[u_0, u_1], u_2] \cdots], u_r].$

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For a finite group G, the degree of r-nilpotency is

$$dn_r(G) = \frac{|\{\mathbf{u} \in G^{r+1} \mid [[[[u_0, u_1], u_2] \cdots], u_r] = 1\}|}{|G|^{r+1}}$$

Proposition (indep. by R. Rezaei–Russo for compact groups) For $r \ge 1$, any finite group G, if $dn_r(G) > 1 - \frac{3}{2(r+2)}$ then $dn_r(G) = 1$

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Definition

Let $G = \langle X \rangle$ be f.g. The degree of *r*-nilpotency of *G* w.r.t. X is

$$dn_{r,X}(G) = \limsup_{n \to \infty} \frac{|\{\mathbf{u} \in \mathbb{B}_X(n)^{r+1} \mid [[[[u_0, u_1], u_2] \cdots], u_r] = 1\}|}{|\mathbb{B}_X(n)|^{r+1}}$$

where
$$\mathbb{B}_X(n) = \{g \in G \mid |g|_X \leqslant n\}$$
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Theorem (Antolín–Martino–V.)

Let $G = \langle X \rangle$ be of subexponential* growth and residually-p for infinitely many primes. Then,

- (i) $dn_{r,X}(G) > 1 \frac{3}{2^{r+2}} \Leftrightarrow G \text{ is } r\text{-nilpotent};$
- (ii) $dn_{r,X}(G) > 0 \Leftrightarrow G$ is virtually r-nilpotent.



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