# Commuting degree for infinite groups 

## Enric Ventura

Departament de Matemàtica Aplicada III Universitat Politècnica de Catalunya

## Dusseldorf, Algebra Seminar.

October 27th, 2014.

## Outline

(9) Motivation
(2) Main definition
(3) Finite index subgroups

4 Short exact sequences
(5) A Gromov-like theorem

6 Other related results

## Outline

Main definition3 Finite index subgroups

4 Short exact sequences

5 A Gromov-like theorem

6 Other related results

## Motivation

(Joint work with Y. Antolín and A. Martino.)
Theorem (Gustaison, 1973)
Let $G$ be a finite group. If the probability that two elements from $G$ commute is bigger than $5 / 8$, then $G$ is abelian.

Proof. Suppose G is not abelian. Then,

$$
\begin{aligned}
d c(G) & =\frac{\mid\{(u, v)|u v-v u|}{|G|^{2}}=\frac{1}{|G|^{2}} \sum_{u \in G}\left|C_{G}(u)\right|= \\
& =\frac{1}{|G|^{2}}\left(|Z(G)||G|+\sum_{u \in G \backslash Z(G)}\left|C_{G}(u)\right|\right) \leqslant \\
& \leqslant \frac{1}{|G|^{2}}\left(|Z(G)||G|+(|G|-|Z(G)|) \frac{|G|}{2}\right)=
\end{aligned}
$$

## Motivation

(Joint work with Y. Antolín and A. Martino.)

## Theorem (Gustafson, 1973)

Let $G$ be a finite group. If the probability that two elements from $G$ commute is bigger than $5 / 8$, then $G$ is abelian.

Proof. Suppose $G$ is not abelian. Then,


$$
\begin{aligned}
& =\frac{1}{|G|^{2}}\left(|Z(G)||G|+\sum_{u \in G \backslash Z(G)}\left|C_{G}(u)\right|\right) \leqslant \\
& \leqslant \frac{1}{|G|^{2}}\left(|Z(G)||G|+(|G|-|Z(G)|) \frac{|G|}{2}\right)=
\end{aligned}
$$

## Motivation

(Joint work with Y. Antolín and A. Martino.)

## Theorem (Gustafson, 1973)

Let $G$ be a finite group. If the probability that two elements from $G$ commute is bigger than $5 / 8$, then $G$ is abelian.

Proof. Suppose G is not abelian. Then,

$$
d c(G)=\frac{|\{(u, v) \mid u v=v u\}|}{|G|^{2}}=\frac{1}{|G|^{2}} \sum_{u \in G}\left|C_{G}(u)\right|=
$$

## Motivation

(Joint work with Y. Antolín and A. Martino.)

## Theorem (Gustafson, 1973)

Let $G$ be a finite group. If the probability that two elements from $G$ commute is bigger than $5 / 8$, then $G$ is abelian.

Proof. Suppose G is not abelian. Then,

$$
\begin{aligned}
d c(G) & =\frac{|\{(u, v) \mid u v=v u\}|}{|G|^{2}}=\frac{1}{|G|^{2}} \sum_{u \in G}\left|C_{G}(u)\right|= \\
& =\frac{1}{|G|^{2}}\left(|Z(G)||G|+\sum_{u \in G \backslash Z(G)}\left|C_{G}(u)\right|\right) \leqslant \\
& =\frac{1}{|G|^{2}}\left(|Z(G)| G \left\lvert\,+(|G|-|Z(G)|) \frac{|G|}{2}\right.\right)=
\end{aligned}
$$

## Motivation

(Joint work with Y. Antolín and A. Martino.)

## Theorem (Gustafson, 1973)

Let $G$ be a finite group. If the probability that two elements from $G$ commute is bigger than $5 / 8$, then $G$ is abelian.

Proof. Suppose G is not abelian. Then,

$$
\begin{aligned}
d c(G) & =\frac{|\{(u, v) \mid u v=v u\}|}{|G|^{2}}=\frac{1}{|G|^{2}} \sum_{u \in G}\left|C_{G}(u)\right|= \\
& =\frac{1}{|G|^{2}}\left(|Z(G)||G|+\sum_{u \in G \backslash Z(G)}\left|C_{G}(u)\right|\right) \leqslant \\
& \leqslant \frac{1}{|G|^{2}}\left(|Z(G)||G|+(|G|-|Z(G)|) \frac{|G|}{2}\right)=
\end{aligned}
$$

## Motivation

$$
\begin{aligned}
& =\frac{1}{|G|^{2}}\left(|Z(G)||G|+(|G|-|Z(G)|) \frac{|G|}{2}\right)= \\
& =\frac{|G|+|Z(G)|}{2|G|} \leqslant \frac{1}{2}+\frac{|G|}{4 \cdot 2|G|}=\frac{1}{2}+\frac{1}{8}=\frac{5}{8}
\end{aligned}
$$

## because $G / Z(G)$ cannot be cyclic and so, $|Z(G)| \leqslant|G| / 4$.

## Observation

The quaternion group has $d c(Q)=5 / 8$.

## "There is no live between 5/8 and 1"

## (Goal)

Is there a version of do for infinite groups?

## Motivation

$$
\begin{aligned}
& =\frac{1}{|G|^{2}}\left(|Z(G)||G|+(|G|-|Z(G)|) \frac{|G|}{2}\right)= \\
& =\frac{|G|+|Z(G)|}{2|G|} \leqslant \frac{1}{2}+\frac{|G|}{4 \cdot 2|G|}=\frac{1}{2}+\frac{1}{8}=\frac{5}{8}
\end{aligned}
$$

because $G / Z(G)$ cannot be cyclic and so, $|Z(G)| \leqslant|G| / 4$. $\square$

## Observation

The quaternion group has $d c(Q)=5 / 8$.
"There is no live between 5/8 and 1"

## (Goal)

## Motivation

$$
\begin{aligned}
& =\frac{1}{|G|^{2}}\left(|Z(G)||G|+(|G|-|Z(G)|) \frac{|G|}{2}\right)= \\
& =\frac{|G|+|Z(G)|}{2|G|} \leqslant \frac{1}{2}+\frac{|G|}{4 \cdot 2|G|}=\frac{1}{2}+\frac{1}{8}=\frac{5}{8}
\end{aligned}
$$

because $G / Z(G)$ cannot be cyclic and so, $|Z(G)| \leqslant|G| / 4$. $\square$

## Observation

The quaternion group has $d c(Q)=5 / 8$.

## (Goal)

## Motivation

$$
\begin{aligned}
& =\frac{1}{|G|^{2}}\left(|Z(G)||G|+(|G|-|Z(G)|) \frac{|G|}{2}\right)= \\
& =\frac{|G|+|Z(G)|}{2|G|} \leqslant \frac{1}{2}+\frac{|G|}{4 \cdot 2|G|}=\frac{1}{2}+\frac{1}{8}=\frac{5}{8}
\end{aligned}
$$

because $G / Z(G)$ cannot be cyclic and so, $|Z(G)| \leqslant|G| / 4$.

## Observation

The quaternion group has $d c(Q)=5 / 8$.
"There is no live between $5 / 8$ and 1 "

## (Goal)

Is there a version of do for infinite groups?

## Motivation

$$
\begin{aligned}
& =\frac{1}{|G|^{2}}\left(|Z(G)||G|+(|G|-|Z(G)|) \frac{|G|}{2}\right)= \\
& =\frac{|G|+|Z(G)|}{2|G|} \leqslant \frac{1}{2}+\frac{|G|}{4 \cdot 2|G|}=\frac{1}{2}+\frac{1}{8}=\frac{5}{8}
\end{aligned}
$$

because $G / Z(G)$ cannot be cyclic and so, $|Z(G)| \leqslant|G| / 4$.

## Observation

The quaternion group has $d c(Q)=5 / 8$.
"There is no live between 5/8 and 1"

## (Goal)

Is there a version of dc for infinite groups ?

## Outline

## (9) Motivation

3 Finite index subgroups

4 Short exact sequences
(5) A Gromov-like theorem

6 Other related results

## Degree of commutativity

## Definition

Let $G=\langle X\rangle$ be a f.g. group. The degree of commutativity of $G$ w.r.t. $X$ is

$$
d c_{X}(G)=\limsup _{n \rightarrow \infty} \frac{\left|\left\{(u, v) \in \mathbb{B}_{X}(n) \times \mathbb{B}_{X}(n) \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(n)\right|^{2}} \in[0,1],
$$

where $\mathbb{B}_{X}(n)=\left\{\left.g \in G| | g\right|_{x} \leqslant n\right\}$.

## Question

Is this a real lim ? Does it depend on $X$ ?

About limsup we have no idea:

- No example where lim doesn t exist;
- No proof it is always a real limit.


## Degree of commutativity

## Definition

Let $G=\langle X\rangle$ be a f.g. group. The degree of commutativity of $G$ w.r.t. $X$ is

$$
d c_{X}(G)=\limsup _{n \rightarrow \infty} \frac{\left|\left\{(u, v) \in \mathbb{B}_{X}(n) \times \mathbb{B}_{X}(n) \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(n)\right|^{2}} \in[0,1],
$$

where $\mathbb{B}_{X}(n)=\left\{\left.g \in G| | g\right|_{X} \leqslant n\right\}$.

## Question

Is this a real lim ?

About limsup we have no idea:

- No example where lim doesn't exist;
- No proof it is always a real limit.


## Degree of commutativity

## Definition

Let $G=\langle X\rangle$ be a f.g. group. The degree of commutativity of $G$ w.r.t. $X$ is

$$
d c_{X}(G)=\limsup _{n \rightarrow \infty} \frac{\left|\left\{(u, v) \in \mathbb{B}_{X}(n) \times \mathbb{B}_{X}(n) \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(n)\right|^{2}} \in[0,1],
$$

where $\mathbb{B}_{X}(n)=\left\{\left.g \in G| | g\right|_{X} \leqslant n\right\}$.

## Question

Is this a real lim ? Does it depend on $X$ ?

About limsup we have no idea:

- No example where lim doesn't exist;
- No proof it is always a real limit.


## Degree of commutativity

## Definition

Let $G=\langle X\rangle$ be a f.g. group. The degree of commutativity of $G$ w.r.t. $X$ is

$$
d c_{X}(G)=\limsup _{n \rightarrow \infty} \frac{\left|\left\{(u, v) \in \mathbb{B}_{X}(n) \times \mathbb{B}_{X}(n) \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(n)\right|^{2}} \in[0,1],
$$

where $\mathbb{B}_{X}(n)=\left\{\left.g \in G| | g\right|_{X} \leqslant n\right\}$.

## Question

Is this a real lim ? Does it depend on $X$ ?

About limsup we have no idea:

- No example where lim doesn't exist;
- No proof it is always a real limit.


## Degree of commutativity

## Definition

Let $G=\langle X\rangle$ be a f.g. group. The degree of commutativity of $G$ w.r.t. $X$ is

$$
d c_{X}(G)=\limsup _{n \rightarrow \infty} \frac{\left|\left\{(u, v) \in \mathbb{B}_{X}(n) \times \mathbb{B}_{X}(n) \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(n)\right|^{2}} \in[0,1],
$$

where $\mathbb{B}_{X}(n)=\left\{\left.g \in G| | g\right|_{X} \leqslant n\right\}$.

## Question

Is this a real lim ? Does it depend on $X$ ?

About limsup we have no idea:

- No example where lim doesn't exist;
- No proof it is always a real limit.


## Independence on $X$

## Definition

A f.g. group $G=\langle X\rangle$ is of

- subexponential growth if $\lim _{n \rightarrow \infty} \frac{|\mathbb{B} x(n+1)|}{\left|\mathbb{B}_{x}(n)\right|}=1$;



## Definition

Let $G=\langle X\rangle$. A map $f: G \rightarrow \mathbb{N}$ is an estimation of the $X$-metric if $\exists$
$K>0$ such that $\forall w \in G$


## Example

$\square$ $Y$-metric, and $|\cdot|_{\gamma}$ is an estimation of the $X$-metric.

## Independence on $X$

## Definition

A f.g. group $G=\langle X\rangle$ is of

- subexponential growth if $\lim _{n \rightarrow \infty} \frac{|\mathbb{B} x(n+1)|}{\left|\mathbb{B}_{x}(n)\right|}=1$;
- polynomial growth (of degree d) if $0<C n^{d} \leqslant\left|\mathbb{B}_{X}(n)\right| \leqslant D n^{d}$.


## Definition

Let $G=\langle X\rangle$. A map $f: G \rightarrow \mathbb{N}$ is an estimation of the $X$-metric if $\exists$
$K>0$ such that $\forall w \in G$


## Example

It is well known that, for $G=\langle X\rangle=\langle Y\rangle,|\cdot| x$ is an estimation of the
$Y$-metric, and $|\cdot|_{\gamma}$ is an estimation of the $X$-metric.

## Independence on $X$

## Definition

A f.g. group $G=\langle X\rangle$ is of

- subexponential growth if $\lim _{n \rightarrow \infty} \frac{\left|\mathbb{B}_{x}(n+1)\right|}{\left|\mathbb{B}_{x}(n)\right|}=1$;
- polynomial growth (of degree d) if $0<C n^{d} \leqslant\left|\mathbb{B}_{X}(n)\right| \leqslant D n^{d}$.


## Definition

Let $G=\langle X\rangle$. A map $f: G \rightarrow \mathbb{N}$ is an estimation of the $X$-metric if $\exists$ $K>0$ such that $\forall w \in G$

$$
\frac{1}{K} f(w) \leqslant|w|_{X} \leqslant K f(w)
$$

Example
It is well known that, for $G=\langle X\rangle=\langle Y\rangle,|\cdot| x$ is an estimation of the
$Y$-metric, and $|\cdot|_{Y}$ is an estimation of the $X$-metric.

## Independence on $X$

## Definition

A f.g. group $G=\langle X\rangle$ is of

- subexponential growth if $\lim _{n \rightarrow \infty} \frac{\left|\mathbb{B}_{x}(n+1)\right|}{\left|\mathbb{B}_{x}(n)\right|}=1$;
- polynomial growth (of degree d) if $0<C n^{d} \leqslant\left|\mathbb{B}_{X}(n)\right| \leqslant D n^{d}$.


## Definition

Let $G=\langle X\rangle$. A map $f: G \rightarrow \mathbb{N}$ is an estimation of the $X$-metric if $\exists$ $K>0$ such that $\forall w \in G$

$$
\frac{1}{K} f(w) \leqslant|w|_{X} \leqslant K f(w)
$$

## Example

It is well known that, for $G=\langle X\rangle=\langle Y\rangle,|\cdot| X$ is an estimation of the $Y$-metric, and $|\cdot|_{Y}$ is an estimation of the $X$-metric.

## Independence on $X$

## Definition

Define the $f$-ball and the $f$-dc:

$$
\begin{gathered}
\mathbb{B}_{f}(n)=\{w \in G \mid f(w) \leqslant n\} \\
d c_{f}(G)=\limsup _{n \rightarrow \infty} \frac{\left|\left\{(u, v) \in \mathbb{B}_{f}(n) \times \mathbb{B}_{f}(n) \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{f}(n)\right|^{2}} .
\end{gathered}
$$

## Proposition

Let $G=\langle X\rangle$ be of polynomial growth, and $f: G \rightarrow \mathbb{N}$ be an estimation of the $X$-metric. Then,
$d c_{X}(G)>0 \quad \Longleftrightarrow \quad d c_{f}(G)>0$.

Proof. Clearly, $\mathbb{B}_{f}(n) \subseteq \mathbb{B}_{x}(K n) \subseteq \mathbb{B}_{f}\left(K^{2} n\right)$ so,

$$
\left|\left\{(u, v) \in\left(\mathbb{B}_{f}(n)\right)^{2} \mid u v=v u\right\}\right| \leqslant\left|\left\{(u, v) \in\left(\mathbb{B}_{x}(K n)\right)^{2} \mid u v=v u\right\}\right| .
$$

## Independence on $X$

## Definition

Define the $f$-ball and the $f$-dc:

$$
\begin{gathered}
\mathbb{B}_{f}(n)=\{w \in G \mid f(w) \leqslant n\} \\
d c_{f}(G)=\limsup _{n \rightarrow \infty} \frac{\left|\left\{(u, v) \in \mathbb{B}_{f}(n) \times \mathbb{B}_{f}(n) \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{f}(n)\right|^{2}} .
\end{gathered}
$$

## Proposition

Let $G=\langle X\rangle$ be of polynomial growth, and $f: G \rightarrow \mathbb{N}$ be an estimation of the $X$-metric. Then,

$$
d c_{X}(G)>0 \Longleftrightarrow d c_{f}(G)>0
$$

Proof. Clearly, $\mathbb{B}_{f}(n) \subseteq \mathbb{B}_{X}(K n) \subseteq \mathbb{B}_{f}\left(K^{2} n\right)$ so,

$$
\left|\left\{(u, v) \in\left(\mathbb{B}_{f}(n)\right)^{2} \mid u v=v u\right\}\right| \leqslant\left|\left\{(u, v) \in\left(\mathbb{B}_{x}(K n)\right)^{2} \mid u v=v u\right\}\right| .
$$

## Independence on $X$

## Definition

Define the $f$-ball and the $f$-dc:

$$
\begin{gathered}
\mathbb{B}_{f}(n)=\{w \in G \mid f(w) \leqslant n\} \\
d c_{f}(G)=\limsup _{n \rightarrow \infty} \frac{\left|\left\{(u, v) \in \mathbb{B}_{f}(n) \times \mathbb{B}_{f}(n) \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{f}(n)\right|^{2}} .
\end{gathered}
$$

## Proposition

Let $G=\langle X\rangle$ be of polynomial growth, and $f: G \rightarrow \mathbb{N}$ be an estimation of the $X$-metric. Then,

$$
d c_{X}(G)>0 \Longleftrightarrow d c_{f}(G)>0
$$

Proof. Clearly, $\mathbb{B}_{f}(n) \subseteq \mathbb{B}_{X}(K n) \subseteq \mathbb{B}_{f}\left(K^{2} n\right)$ so,

## Independence on $X$

## Definition

Define the $f$-ball and the $f$-dc:

$$
\begin{gathered}
\mathbb{B}_{f}(n)=\{w \in G \mid f(w) \leqslant n\} \\
d c_{f}(G)=\limsup _{n \rightarrow \infty} \frac{\left|\left\{(u, v) \in \mathbb{B}_{f}(n) \times \mathbb{B}_{f}(n) \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{f}(n)\right|^{2}} .
\end{gathered}
$$

## Proposition

Let $G=\langle X\rangle$ be of polynomial growth, and $f: G \rightarrow \mathbb{N}$ be an estimation of the $X$-metric. Then,

$$
d c_{X}(G)>0 \Longleftrightarrow d c_{f}(G)>0
$$

Proof. Clearly, $\mathbb{B}_{f}(n) \subseteq \mathbb{B}_{X}(K n) \subseteq \mathbb{B}_{f}\left(K^{2} n\right)$ so,

$$
\left|\left\{(u, v) \in\left(\mathbb{B}_{f}(n)\right)^{2} \mid u v=v u\right\}\right| \leqslant\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(K n)\right)^{2} \mid u v=v u\right\}\right| .
$$

## Independence on $X$

$$
\begin{array}{cc}
\frac{\left|\left\{(u, v) \in\left(\mathbb{B}_{f}(n)\right)^{2} \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(K n)\right|^{2}} & \leqslant \frac{\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(K n)\right)^{2} \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(K n)\right|^{2}} . \\
1 \mid & \left.\frac{\left|\left\{(u, v) \in\left(\mathbb{B}_{f}(n)\right)^{2} \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{f}(n)\right|^{2}}\right)\left(\frac{\left|\mathbb{B}_{f}(n)\right|}{\left|\mathbb{B}_{X}(K n)\right|}\right)^{2}
\end{array}
$$

$$
\text { So, } d c_{X}(G)=0 \Rightarrow d c_{f}(G)=0 \text {, because }
$$

$\frac{\left|\mathbb{B}_{f}(n)\right|}{\left|\mathbb{B}_{X}(K n)\right|}$


## Corollary

If $G=\langle X\rangle=\langle Y\rangle$ is of polynomial growth, then

## Independence on $X$

$$
\begin{gathered}
\frac{\left|\left\{(u, v) \in\left(\mathbb{B}_{f}(n)\right)^{2} \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(K n)\right|^{2}} \leqslant \frac{\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(K n)\right)^{2} \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(K n)\right|^{2}} . \\
\left.\| \frac{\left|\left\{(u, v) \in\left(\mathbb{B}_{f}(n)\right)^{2} \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{f}(n)\right|^{2}}\right)\left(\frac{\left|\mathbb{B}_{f}(n)\right|}{\left|\mathbb{B}_{X}(K n)\right|}\right)^{2}
\end{gathered}
$$

So, $d c_{x}(G)=0 \quad \Rightarrow \quad d c_{f}(G)=0$, because

$$
\frac{\left|\mathbb{B}_{f}(n)\right|}{\left|\mathbb{B}_{X}(K n)\right|} \geqslant \frac{\left|\mathbb{B}_{X}(n / K)\right|}{\left|\mathbb{B}_{X}(K n)\right|} \geqslant \frac{C(n / K)^{d}}{D(K n)^{d}}=\frac{C}{D K^{2 d}}>0 .
$$

## Corollary

If $G=\langle X\rangle=\langle Y\rangle$ is of polynomial growth, then


## Independence on $X$

$$
\begin{gathered}
\frac{\left|\left\{(u, v) \in\left(\mathbb{B}_{f}(n)\right)^{2} \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(K n)\right|^{2}} \leqslant \frac{\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(K n)\right)^{2} \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(K n)\right|^{2}} . \\
\left.\| \frac{\left|\left\{(u, v) \in\left(\mathbb{B}_{f}(n)\right)^{2} \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{f}(n)\right|^{2}}\right)\left(\frac{\left|\mathbb{B}_{f}(n)\right|}{\left|\mathbb{B}_{X}(K n)\right|}\right)^{2}
\end{gathered}
$$

So, $d c_{x}(G)=0 \quad \Rightarrow \quad d c_{f}(G)=0$, because

$$
\frac{\left|\mathbb{B}_{f}(n)\right|}{\left|\mathbb{B}_{X}(K n)\right|} \geqslant \frac{\left|\mathbb{B}_{X}(n / K)\right|}{\left|\mathbb{B}_{X}(K n)\right|} \geqslant \frac{C(n / K)^{d}}{D(K n)^{d}}=\frac{C}{D K^{2 d}}>0 .
$$

Corollary
If $G=\langle X\rangle=\langle Y\rangle$ is of polynomial growth, then

$$
d c_{X}(G)=0 \quad \Longleftrightarrow \quad d c_{Y}(G)=0
$$

## Outline

## (9) Motivation

(2) Main definition
(3) Finite index subgroups

4 Short exact sequences
(5) A Gromov-like theorem

6 Other related results

## Finite index subgroups

Lemma (Burillo-Ventura, 2002)
If $H \leqslant_{\text {f.i. }} G=\langle X\rangle$ and $G$ has subexponential growth then there exists $\lim _{n \rightarrow \infty} \frac{\left|\mathbb{B}_{X}(n) \cap H\right|}{\left|\mathbb{B}_{X}(n)\right|}=\frac{1}{[G: H]}$.

## Proposition



## Proposition (Gallagher, 1970)

## Let $G$ be a finite group and $H \triangleleft G$. Then, $d c(G) \leqslant d c(H) \cdot d c(G / H)$

## Corollary

Let $\langle Y\rangle=H \quad G=\langle X\rangle$ be of polynomial growth. Then, $d c x(G)>0$
if and only if $\mathrm{dC}_{Y}(H)>0$

## Finite index subgroups

Lemma (Burillo-Ventura, 2002)
If $H \leqslant_{\text {f.i. }} G=\langle X\rangle$ and $G$ has subexponential growth then there exists $\lim _{n \rightarrow \infty} \frac{\left|\mathbb{B}_{X}(n) \cap H\right|}{\left|\mathbb{B}_{X}(n)\right|}=\frac{1}{[G: H]}$.

## Proposition

Let $\langle Y\rangle=H \leqslant$ f.i. $G=\langle X\rangle$ be of polynomial growth. Then, $d c_{X}(G) \geqslant \frac{1}{[G: H]^{2}} d c_{Y}(H)$.

## Proposition (Gallagher, 1970)

Let $G$ be a finite group and $H \unlhd G$. Then, $d c(G) \leqslant d c(H) \cdot d c(G / H)$

## Corollary

Let $\langle Y\rangle=H \leqslant f i . G=\langle X\rangle$ be of polynomial growth. Then, $d c_{x}(G)>0$
if and only if $d_{Y}(H)>0$

## Finite index subgroups

Lemma (Burillo-Ventura, 2002)
If $H \leqslant_{\text {f.i. }} G=\langle X\rangle$ and $G$ has subexponential growth then there exists $\lim _{n \rightarrow \infty} \frac{\left|\mathbb{B}_{X}(n) \cap H\right|}{\left|\mathbb{B}_{X}(n)\right|}=\frac{1}{[G: H]}$.

## Proposition

Let $\langle Y\rangle=H \leqslant$ f.i. $G=\langle X\rangle$ be of polynomial growth. Then, $d c_{X}(G) \geqslant \frac{1}{[G: H]^{2}} d c_{Y}(H)$.

## Proposition (Gallagher, 1970)

Let $G$ be a finite group and $H \unlhd G$. Then, $d c(G) \leqslant d c(H) \cdot d c(G / H)$.


## Finite index subgroups

Lemma (Burillo-Ventura, 2002)
If $H \leqslant$ f.i. $G=\langle X\rangle$ and $G$ has subexponential growth then there exists $\lim _{n \rightarrow \infty} \frac{\left|\mathbb{B}_{X}(n) \cap H\right|}{\left|\mathbb{B}_{X}(n)\right|}=\frac{1}{[G: H]}$.

## Proposition

Let $\langle Y\rangle=H \leqslant$ f.i. $G=\langle X\rangle$ be of polynomial growth. Then,
$d c_{X}(G) \geqslant \frac{1}{[G: H]^{2}} d c_{Y}(H)$.

## Proposition (Gallagher, 1970)

Let $G$ be a finite group and $H \unlhd G$. Then, $d c(G) \leqslant d c(H) \cdot d c(G / H)$.

## Corollary

Let $\langle Y\rangle=H \leqslant$ f.i. $G=\langle X\rangle$ be of polynomial growth. Then, $d c_{X}(G)>0$
if and only if $d c_{Y}(H)>0$.

## Outline

(2) Main definition

3 Finite index subgroups

4 Short exact sequences

5 A Gromov-like theorem

6 Other related results

## Short exact sequences

## Proposition

Let $G=\langle X\rangle, H \unlhd G$, and let $\pi: G \rightarrow Q=G / H=\langle\bar{X}\rangle$. Put

$$
0 \leqslant \lambda=\left(\liminf \frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{x}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}\right)^{2} \leqslant 1 .
$$

Then, $\lambda \cdot d c_{x}(G) \leqslant d c_{\bar{x}}(Q) \cdot d c_{x}(H)$.

## Proof. Write $d c_{X}(G)=\lim \sup d c_{X}(G, n)$, where



We have,

## Short exact sequences

## Proposition

Let $G=\langle X\rangle, H \unlhd G$, and let $\pi: G \rightarrow Q=G / H=\langle\bar{X}\rangle$. Put

$$
0 \leqslant \lambda=\left(\liminf \frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{x}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}\right)^{2} \leqslant 1 .
$$

Then, $\lambda \cdot d c_{X}(G) \leqslant d c_{\bar{x}}(Q) \cdot d c_{\chi}(H)$.

Proof. Write $d c_{X}(G)=\lim \sup d c_{X}(G, n)$, where

$$
d c_{X}(G, n)=\frac{\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(n)\right)^{2} \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(n)\right|^{2}} .
$$

We have,

## Short exact sequences



## Short exact sequences

$$
\begin{gathered}
\left|\mathbb{B}_{X}(n)\right|^{2} d c_{X}(G, n)=\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(n)\right)^{2} \mid u v=v u\right\}\right|= \\
=\sum_{u \in \mathbb{B}_{X}(n)}\left|C_{G}(u) \cap \mathbb{B}_{X}(n)\right|=\sum_{q \in \mathbb{B}_{\bar{X}}(n)} \sum_{\substack{u \in \mathbb{B}_{X}(n) \\
\pi(u)=q}}\left|C_{G}(u) \cap \mathbb{B}_{X}(n)\right| \leqslant
\end{gathered}
$$



## Short exact sequences

$$
\begin{aligned}
& \quad\left|\mathbb{B}_{X}(n)\right|^{2} d c_{X}(G, n)=\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(n)\right)^{2} \mid u v=v u\right\}\right|= \\
& =\sum_{u \in \mathbb{B}_{X}(n)}\left|C_{G}(u) \cap \mathbb{B}_{X}(n)\right|=\sum_{q \in \mathbb{B}_{\bar{X}}(n)} \sum_{\substack{u \in \mathbb{B}_{X}(n) \\
\pi(u)=q}}\left|C_{G}(u) \cap \mathbb{B}_{X}(n)\right| \leqslant \\
& \leqslant \sum_{q \in \mathbb{B}_{\bar{X}}(n)} \sum_{\substack{u \in \mathbb{B}_{X}(n) \\
\pi(u)=q}}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \cdot\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|= \\
& \left.=\sum_{q \in \mathbb{B}_{X}(n)}\left|C_{Q}(q) \cap \mathbb{B}_{X}(n)\right| \sum_{\substack{u \in \mathbb{B}_{X}(n) \\
\pi(u)=q}}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|=(1)\right)
\end{aligned}
$$

## Short exact sequences

$$
\begin{aligned}
& \left|\mathbb{B}_{X}(n)\right|^{2} d c_{X}(G, n)=\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(n)\right)^{2} \mid u v=v u\right\}\right|= \\
= & \sum_{u \in \mathbb{B}_{X}(n)}\left|C_{G}(u) \cap \mathbb{B}_{X}(n)\right|=\sum_{q \in \mathbb{B}_{\bar{X}}(n)} \sum_{\substack{u \in \mathbb{B}^{\prime}(n) \\
\pi(u)=q}}\left|C_{G}(u) \cap \mathbb{B}_{X}(n)\right| \leqslant \\
\leqslant & \sum_{q \in \mathbb{B}_{\bar{X}}(n)} \sum_{\substack{u \in \mathbb{B}_{X}(n) \\
\pi(u)=q}}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \cdot\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|= \\
= & \sum_{q \in \mathbb{B}_{X}(n)}\left(\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \sum_{\substack{u \in \mathbb{B}_{X}(n) \\
\pi(u)=q}}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|\right)=(1)
\end{aligned}
$$

## Short exact sequences

But, fixing $u_{0} \in \mathbb{B}_{X}(n)$ with $\pi\left(u_{0}\right)=q$,

$$
\sum_{\mathbb{X}_{x}(n), \pi(u)=q}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|=
$$



## Short exact sequences

But, fixing $u_{0} \in \mathbb{B}_{X}(n)$ with $\pi\left(u_{0}\right)=q$,

$$
\begin{gathered}
\sum_{u \in \mathbb{B}_{X}(n), \pi(u)=q}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|= \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right)\left|v \in C_{H}\left(u_{0} h\right),\left|u_{0} h\right|_{X} \leq n\right\} \mid=\right.\right.
\end{gathered}
$$


$C_{H}(v) \cap \mathbb{B}_{X}(2 n) \mid$.

## Short exact sequences

But, fixing $u_{0} \in \mathbb{B}_{X}(n)$ with $\pi\left(u_{0}\right)=q$,

$$
\begin{gathered}
\sum_{u \in \mathbb{B}_{X}(n), \pi(u)=q}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|= \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right)\left|v \in C_{H}\left(u_{0} h\right),\left|u_{0} h\right|_{X} \leq n\right\} \mid=\right.\right. \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right) \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right|=
\end{gathered}
$$

## Short exact sequences

But, fixing $u_{0} \in \mathbb{B}_{X}(n)$ with $\pi\left(u_{0}\right)=q$,

$$
\begin{gathered}
\sum_{u \in \mathbb{B}_{X}(n), \pi(u)=q}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|= \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right)\left|v \in C_{H}\left(u_{0} h\right),\left|u_{0} h\right| x \leq n\right\} \mid=\right.\right. \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right) \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right|= \\
=\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|\left\{h \in H \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right| \leqslant
\end{gathered}
$$

## Short exact sequences

But, fixing $u_{0} \in \mathbb{B}_{X}(n)$ with $\pi\left(u_{0}\right)=q$,

$$
\begin{gathered}
\sum_{u \in \mathbb{B}_{X}(n), \pi(u)=q}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|= \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right)\left|v \in C_{H}\left(u_{0} h\right),\left|u_{0} h\right| x \leq n\right\} \mid=\right.\right. \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right) \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right|= \\
=\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|\left\{h \in H \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right| \leqslant \\
\leqslant \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right| .
\end{gathered}
$$

## Short exact sequences

But, fixing $u_{0} \in \mathbb{B}_{X}(n)$ with $\pi\left(u_{0}\right)=q$,

$$
\begin{gathered}
\sum_{u \in \mathbb{B}_{X}(n), \pi(u)=q}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|= \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right)\left|v \in C_{H}\left(u_{0} h\right),\left|u_{0} h\right| x \leq n\right\} \mid=\right.\right. \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right) \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right|= \\
=\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|\left\{h \in H \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right| \leqslant \\
\leqslant \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right| .
\end{gathered}
$$

Hence,

## Short exact sequences

$$
(1) \leqslant \sum_{q \in \mathbb{B}_{X}(n)}\left(\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)=
$$



$$
\left|\mathbb{B}_{\bar{X}}(n)\right|^{2} \cdot d c_{X}(Q, n) \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|^{2} \cdot d c_{X}(H, 2 n) .
$$

## It follows that



Finally, taking limits, we get

## Short exact sequences

$$
\begin{aligned}
& (1) \leqslant \sum_{q \in \mathbb{B}_{X}(n)}\left(\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
& =\left(\sum_{q \in \mathbb{B}_{\bar{X}}(n)}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right|\right)\left(\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)=
\end{aligned}
$$

$$
\left|\mathbb{B}_{X}(n)\right|^{2} \cdot d c_{X}(Q, n) \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|^{2} \cdot d c_{X}(H, 2 n) .
$$

## It follows that



Finally, taking limits, we get

## Short exact sequences

$$
\begin{gathered}
(1) \leqslant \sum_{q \in \mathbb{B}_{X}(n)}\left(\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
=\left(\sum_{q \in \mathbb{B}_{X}(n)}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right|\right)\left(\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
\left|\mathbb{B}_{\bar{X}}(n)\right|^{2} \cdot d c_{X}(Q, n) \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|^{2} \cdot d c_{X}(H, 2 n) .
\end{gathered}
$$

## It follows that



## Short exact sequences

$$
\begin{gathered}
(1) \leqslant \sum_{q \in \mathbb{B}_{\bar{X}}(n)}\left(\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
=\left(\sum_{q \in \mathbb{B}_{X}(n)}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right|\right)\left(\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
\left|\mathbb{B}_{\bar{X}}(n)\right|^{2} \cdot d c_{\bar{X}}(Q, n) \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|^{2} \cdot d c_{X}(H, 2 n) .
\end{gathered}
$$

It follows that

$$
\left(\frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}\right)^{2} \cdot d c_{X}(G, n) \leqslant d c_{X}(Q, n) \cdot d c_{X}(H, 2 n) .
$$

## Short exact sequences

$$
\begin{gathered}
(1) \leqslant \sum_{q \in \mathbb{B}_{\bar{X}}(n)}\left(\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
=\left(\sum_{q \in \mathbb{B}_{\bar{X}}(n)}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right|\right)\left(\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
\left|\mathbb{B}_{\bar{X}}(n)\right|^{2} \cdot d c_{\bar{X}}(Q, n) \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|^{2} \cdot d c_{X}(H, 2 n) .
\end{gathered}
$$

It follows that

$$
\left(\frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}\right)^{2} \cdot d c_{X}(G, n) \leqslant d c_{\bar{X}}(Q, n) \cdot d c_{X}(H, 2 n) .
$$

Finally, taking limits, we get

## Short exact sequences

$$
\begin{gathered}
(1) \leqslant \sum_{q \in \mathbb{B}_{\bar{X}}(n)}\left(\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
=\left(\sum_{q \in \mathbb{B}_{\bar{X}}(n)}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right|\right)\left(\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
\left|\mathbb{B}_{\bar{X}}(n)\right|^{2} \cdot d c_{\bar{X}}(Q, n) \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|^{2} \cdot d c_{X}(H, 2 n) .
\end{gathered}
$$

It follows that

$$
\left(\frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}\right)^{2} \cdot d c_{X}(G, n) \leqslant d c_{\bar{X}}(Q, n) \cdot d c_{X}(H, 2 n)
$$

Finally, taking limits, we get

$$
\lambda \cdot d c_{X}(G) \leqslant d c_{\bar{X}}(Q) \cdot d c_{X}(H)
$$

## Short exact sequences

## Proposition

Let $G=\langle X\rangle$ be polynomially growing with degree d. Then, $\forall H \unlhd_{\text {fi. }} G$, we have

$$
\left(\frac{C}{D \cdot 4^{d}}\right)^{2} \cdot d c_{X}(G) \leq d c_{\bar{X}}(G / H) \cdot d c_{X}(H)
$$

Proof. Clearly, $\left|\mathbb{B}_{X}(2 n)\right| \geqslant\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(n) \cap H\right|$. Now fix $H$ and, $\forall \epsilon>0, \exists n_{0}$ s.t. $\forall n \geqslant n_{0}$,


## Short exact sequences

## Proposition

Let $G=\langle X\rangle$ be polynomially growing with degree d. Then, $\forall H \unlhd_{\text {f.i. }} G$, we have

$$
\left(\frac{C}{D \cdot 4^{d}}\right)^{2} \cdot d c_{X}(G) \leq d c_{\bar{X}}(G / H) \cdot d c_{X}(H)
$$

Proof. Clearly, $\left|\mathbb{B}_{X}(2 n)\right| \geqslant\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(n) \cap H\right|$.
Now fix $H$ and, $\forall \epsilon>0, \exists n_{0}$ s.t. $\forall n \geqslant n_{0}$,


## Short exact sequences

## Proposition

Let $G=\langle X\rangle$ be polynomially growing with degree d. Then, $\forall H \unlhd_{\text {f.i. }} G$, we have

$$
\left(\frac{C}{D \cdot 4^{d}}\right)^{2} \cdot d c_{X}(G) \leq d c_{\bar{X}}(G / H) \cdot d c_{X}(H)
$$

Proof. Clearly, $\left|\mathbb{B}_{X}(2 n)\right| \geqslant\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(n) \cap H\right|$. Now fix $H$ and, $\forall \epsilon>0, \exists n_{0}$ s.t. $\forall n \geqslant n_{0}$,

$$
\frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|} \geqslant \frac{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right| \cdot\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}=
$$

## Short exact sequences

## Proposition

Let $G=\langle X\rangle$ be polynomially growing with degree d. Then, $\forall H \unlhd_{\text {f.i. }} G$, we have

$$
\left(\frac{C}{D \cdot 4^{d}}\right)^{2} \cdot d c_{X}(G) \leq d c_{\bar{X}}(G / H) \cdot d c_{X}(H)
$$

Proof. Clearly, $\left|\mathbb{B}_{X}(2 n)\right| \geqslant\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(n) \cap H\right|$. Now fix $H$ and, $\forall \epsilon>0, \exists n_{0}$ s.t. $\forall n \geqslant n_{0}$,

$$
\begin{gathered}
\frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{x}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|} \geqslant \frac{\left|\mathbb{B}_{\bar{x}}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right| \cdot\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{\bar{x}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}= \\
=\frac{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{X}(2 n) \cap H\right|}=\frac{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right|} \cdot \frac{\left|\mathbb{B}_{x}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right|}{\left|\mathbb{B}_{X}(2 n)\right|} \cdot \frac{\left|\mathbb{B}_{x}(2 n)\right|}{\left|\mathbb{B}_{x}(2 n) \cap H\right|}
\end{gathered}
$$

## Short exact sequences

## Proposition

Let $G=\langle X\rangle$ be polynomially growing with degree d. Then, $\forall H \unlhd_{\text {f.i. }} G$, we have

$$
\left(\frac{C}{D \cdot 4^{d}}\right)^{2} \cdot d c_{X}(G) \leq d c_{\bar{X}}(G / H) \cdot d c_{X}(H)
$$

Proof. Clearly, $\left|\mathbb{B}_{X}(2 n)\right| \geqslant\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(n) \cap H\right|$. Now fix $H$ and, $\forall \epsilon>0, \exists n_{0}$ s.t. $\forall n \geqslant n_{0}$,

$$
\begin{gathered}
\frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|} \geqslant \frac{\left|\mathbb{B}_{\bar{X}}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right| \cdot\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}= \\
=\frac{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{X}(2 n) \cap H\right|}=\frac{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right|} \cdot \frac{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right|}{\left|\mathbb{B}_{X}(2 n)\right|} \cdot \frac{\left|\mathbb{B}_{X}(2 n)\right|}{\left|\mathbb{B}_{X}(2 n) \cap H\right|} \geqslant
\end{gathered}
$$

## Short exact sequences

$$
\geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot \frac{C \cdot\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^{d}}{D \cdot(2 n)^{d}} \cdot([G: H]-\epsilon) \geqslant
$$



And this is true for every $\epsilon>0$ hence, $\lambda \geqslant\left(\frac{C}{D \cdot 4^{d}}\right)^{2}$, Which happens to be independent from $\mathrm{H} . \square$

## Short exact sequences

$$
\begin{aligned}
& \geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot \frac{C \cdot\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^{d}}{D \cdot(2 n)^{d}} \cdot([G: H]-\epsilon) \geqslant \\
& \geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot\left(\frac{C}{D \cdot 4^{d}}-\epsilon\right) \cdot([G: H]-\epsilon)
\end{aligned}
$$

So, $\lambda=(\lim \inf \cdots)^{2} \geqslant\left(\frac{1}{[G: H]}-\epsilon\right)^{2} \cdot\left(\frac{C}{D \cdot 4^{s}}-\epsilon\right)^{2} \cdot([G: H]-\epsilon)^{2}$.
And this is true for every $\epsilon>0$ hence, $\lambda \geqslant\left(\frac{C}{D \cdot 4^{d}}\right)^{2}$, Which happens to be independent from $H$. $\square$

## Short exact sequences

$$
\begin{aligned}
& \geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot \frac{C \cdot\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^{d}}{D \cdot(2 n)^{d}} \cdot([G: H]-\epsilon) \geqslant \\
& \geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot\left(\frac{C}{D \cdot 4^{d}}-\epsilon\right) \cdot([G: H]-\epsilon)
\end{aligned}
$$

So, $\lambda=(\liminf \cdots)^{2} \geqslant\left(\frac{1}{[G: H]}-\epsilon\right)^{2} \cdot\left(\frac{C}{D \cdot 4^{s}}-\epsilon\right)^{2} \cdot([G: H]-\epsilon)^{2}$.

## Short exact sequences

$$
\begin{aligned}
& \geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot \frac{C \cdot\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^{d}}{D \cdot(2 n)^{d}} \cdot([G: H]-\epsilon) \geqslant \\
& \geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot\left(\frac{C}{D \cdot 4^{d}}-\epsilon\right) \cdot([G: H]-\epsilon)
\end{aligned}
$$

So, $\lambda=(\liminf \cdots)^{2} \geqslant\left(\frac{1}{[G: H]}-\epsilon\right)^{2} \cdot\left(\frac{C}{D \cdot 4^{s}}-\epsilon\right)^{2} \cdot([G: H]-\epsilon)^{2}$.
And this is true for every $\epsilon>0$ hence, $\lambda \geqslant\left(\frac{C}{D \cdot 4^{d}}\right)^{2}$, Which happens to be independent from H. $\square$

## Short exact sequences

$$
\begin{aligned}
& \geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot \frac{C \cdot\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^{d}}{D \cdot(2 n)^{d}} \cdot([G: H]-\epsilon) \geqslant \\
& \geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot\left(\frac{C}{D \cdot 4^{d}}-\epsilon\right) \cdot([G: H]-\epsilon)
\end{aligned}
$$

So, $\lambda=(\liminf \cdots)^{2} \geqslant\left(\frac{1}{[G: H]}-\epsilon\right)^{2} \cdot\left(\frac{C}{D \cdot 4^{s}}-\epsilon\right)^{2} \cdot([G: H]-\epsilon)^{2}$.
And this is true for every $\epsilon>0$ hence, $\lambda \geqslant\left(\frac{C}{D \cdot 4^{d}}\right)^{2}$,
Which happens to be independent from $\mathrm{H} . \square$

## Outline

## (9) Motivation

(2) Main definition
(3) Finite index subgroups

4 Short exact sequences
(5) A Gromov-like theorem

6 Other related results

## t.f. nilpotent groups

## Proposition

Let $G=\langle X\rangle$ be t.f. nilpotent. Then, either $G$ is abelian, or $d c_{X}(G)=0$.

## Proof. Assume $G$ is not abelian and $d_{X}(G)>0$ and let us find a

 contradiction.- We have a uniform $\lambda>0$ s.t., for every $H \unlhd_{\text {f.i. }} G$,

$$
\lambda \cdot d c_{X}(G) \leqslant d c_{\bar{X}}(G / H) \cdot d c_{X}(H)
$$

- Choose $n$ s.t. $\lambda \cdot d c_{X}(G) \cdot\left(\frac{8}{5}\right)^{n}>1$.
- Take $\left\{p_{1}, \ldots, p_{n}\right\}$ be n pairwise different primes.
- By Grumbergs' classical result, $G$ is residually- $p_{i}$.
- Hence, $G$ has a non-abelian, finite $p_{i}$-quotient $\pi_{i}: G \rightarrow Q_{i}$; in particular, $d c\left(Q_{i}\right) \leqslant \frac{5}{8}$.


## t.f. nilpotent groups

## Proposition

Let $G=\langle X\rangle$ be t.f. nilpotent. Then, either $G$ is abelian, or $d c_{X}(G)=0$.

Proof. Assume $G$ is not abelian and $d c_{X}(G)>0$ and let us find a contradiction.
> - We have a uniform $\lambda>0$ s.t., for every $H \unlhd_{\text {f.i. }} G$, $d c_{X}(G) \leqslant d c_{\bar{X}}(G / H) \cdot d c_{X}(H)$
> - Choose $n$ s.t. $\lambda \cdot d c_{X}(G) \cdot\left(\frac{8}{5}\right)^{n}>1$.
> - Take $\left\{p_{1}, \ldots, p_{n}\right\}$ be n pairwise different primes.
> - By Grumbergs' classical result, $G$ is residually-pi
> - Hence, $G$ has a non-abelian, finite $p_{i}$-quotient $\pi_{i}: G \rightarrow Q_{i}$; in particular, $d c\left(Q_{i}\right)$

## t.f. nilpotent groups

## Proposition

Let $G=\langle X\rangle$ be t.f. nilpotent. Then, either $G$ is abelian, or $d c_{X}(G)=0$.

Proof. Assume $G$ is not abelian and $d c_{X}(G)>0$ and let us find a contradiction.

- We have a uniform $\lambda>0$ s.t., for every $H \unlhd_{\text {f.i. }} G$,

$$
\lambda \cdot d c_{X}(G) \leqslant d c_{\bar{X}}(G / H) \cdot d c_{X}(H) .
$$

- Choose $n$ s.t. $\lambda \cdot d c_{X}(G) \cdot\left(\frac{8}{5}\right)^{n}>1$.
- Take $\left\{p_{1}, \ldots, p_{n}\right\}$ be $n$ pairwise different primes.
- By Grumbergs' classical result, $G$ is residually-pi
- Hence, $G$ has a non-abelian, finite $p_{i}$-quotient $\pi_{i}: G \rightarrow Q_{i}$; in particular, $d c\left(Q_{i}\right)$


## t.f. nilpotent groups

## Proposition

Let $G=\langle X\rangle$ be t.f. nilpotent. Then, either $G$ is abelian, or $d c_{X}(G)=0$.

Proof. Assume $G$ is not abelian and $d c_{X}(G)>0$ and let us find a contradiction.

- We have a uniform $\lambda>0$ s.t., for every $H \unlhd_{\text {f.i. }} G$,

$$
\lambda \cdot d c_{X}(G) \leqslant d c_{\bar{X}}(G / H) \cdot d c_{X}(H) .
$$

- Choose $n$ s.t. $\lambda \cdot d c_{X}(G) \cdot\left(\frac{8}{5}\right)^{n}>1$.
- Take $\left\{p_{1}, \ldots, p_{n}\right\}$ be n pairwise different primes.
- By Grumbergs' classical result, $G$ is residually- $p_{i}$
- Hence, $G$ has a non-abelian, finite $p_{i}$-auotient $\pi_{i}: G \rightarrow Q_{i} ;$ in particular, dc $\left(Q_{i}\right)$


## t.f. nilpotent groups

## Proposition

Let $G=\langle X\rangle$ be t.f. nilpotent. Then, either $G$ is abelian, or $d c_{X}(G)=0$.

Proof. Assume $G$ is not abelian and $d c_{X}(G)>0$ and let us find a contradiction.

- We have a uniform $\lambda>0$ s.t., for every $H \unlhd_{\text {f.i. }} G$,

$$
\lambda \cdot d c_{X}(G) \leqslant d c_{\bar{X}}(G / H) \cdot d c_{X}(H)
$$

- Choose $n$ s.t. $\lambda \cdot d c_{X}(G) \cdot\left(\frac{8}{5}\right)^{n}>1$.
- Take $\left\{p_{1}, \ldots, p_{n}\right\}$ be $n$ pairwise different primes.
- By Grumbergs' classical result, $G$ is residually-p
- Hence, $G$ has a non-abelian, finite $p_{i}$-quotient $\pi_{i}: G \rightarrow Q_{i}$; in particular, dc $\left(Q_{i}\right)$


## t.f. nilpotent groups

## Proposition

Let $G=\langle X\rangle$ be t.f. nilpotent. Then, either $G$ is abelian, or $d c_{X}(G)=0$.

Proof. Assume $G$ is not abelian and $d c_{X}(G)>0$ and let us find a contradiction.

- We have a uniform $\lambda>0$ s.t., for every $H \unlhd_{\text {f.i. }} G$,

$$
\lambda \cdot d c_{X}(G) \leqslant d c_{\bar{X}}(G / H) \cdot d c_{X}(H) .
$$

- Choose $n$ s.t. $\lambda \cdot d c_{X}(G) \cdot\left(\frac{8}{5}\right)^{n}>1$.
- Take $\left\{p_{1}, \ldots, p_{n}\right\}$ be $n$ pairwise different primes.
- By Grumbergs' classical result, $G$ is residually- $p_{i}$.
- Hence, G has a non-abelian, finite $p_{i}$-quotient $\tau$


## t.f. nilpotent groups

## Proposition

Let $G=\langle X\rangle$ be t.f. nilpotent. Then, either $G$ is abelian, or $d c_{X}(G)=0$.

Proof. Assume $G$ is not abelian and $d c_{X}(G)>0$ and let us find a contradiction.

- We have a uniform $\lambda>0$ s.t., for every $H \unlhd_{\text {f.i. }} G$,

$$
\lambda \cdot d c_{X}(G) \leqslant d c_{\bar{X}}(G / H) \cdot d c_{X}(H)
$$

- Choose $n$ s.t. $\lambda \cdot d c_{X}(G) \cdot\left(\frac{8}{5}\right)^{n}>1$.
- Take $\left\{p_{1}, \ldots, p_{n}\right\}$ be $n$ pairwise different primes.
- By Grumbergs' classical result, $G$ is residually- $p_{i}$.
- Hence, $G$ has a non-abelian, finite $p_{i}$-quotient $\pi_{i}: G \rightarrow Q_{i}$; in particular, $d c\left(Q_{i}\right) \leqslant \frac{5}{8}$.


## t.f. nilpotent groups

- Now, the morphism $\times{ }_{i=1}^{n} \pi_{i}: G \rightarrow Q_{1} \times \cdots \times Q_{n}$ is onto (because $\left.\operatorname{gcd}\left(p_{j}, p_{1} \cdots p_{j-1} p_{j+1} \cdots p_{n}\right)=1\right)$.
- Take $H=\operatorname{ker} x_{i=1}^{n} \pi_{i} \unlhd_{\text {fi. }}$ G; we have,
- Hence,



## t.f. nilpotent groups

- Now, the morphism $\times{ }_{i=1}^{n} \pi_{i}: G \rightarrow Q_{1} \times \cdots \times Q_{n}$ is onto (because $\left.\operatorname{gcd}\left(p_{j}, p_{1} \cdots p_{j-1} p_{j+1} \cdots p_{n}\right)=1\right)$.
- Take $H=\operatorname{ker} \times_{i=1}^{n} \pi_{i} \unlhd_{\text {fi. }}$ G; we have,

$$
\lambda \cdot d c_{X}(G) \leqslant d c_{x}(H) \cdot d c_{\bar{x}}\left(Q_{1} \times \cdots \times Q_{n}\right) \leqslant d c_{X}(H) \cdot\left(\frac{5}{8}\right)^{n} .
$$

- Hence,


## t.f. nilpotent groups

- Now, the morphism $\times{ }_{i=1}^{n} \pi_{i}: G \rightarrow Q_{1} \times \cdots \times Q_{n}$ is onto (because $\left.\operatorname{gcd}\left(p_{j}, p_{1} \cdots p_{j-1} p_{j+1} \cdots p_{n}\right)=1\right)$.
- Take $H=\operatorname{ker} \times_{i=1}^{n} \pi_{i} \unlhd_{\text {fi. }}$ G; we have,

$$
\lambda \cdot d c_{X}(G) \leqslant d c_{x}(H) \cdot d c_{\bar{X}}\left(Q_{1} \times \cdots \times Q_{n}\right) \leqslant d c_{x}(H) \cdot\left(\frac{5}{8}\right)^{n}
$$

- Hence,

$$
1<\lambda \cdot d c_{X}(G) \cdot\left(\frac{8}{5}\right)^{n} \leqslant d c_{X}(H) \leqslant 1,
$$

a contradiction.

## A Gromov-like theorem

## Theorem

Let $G$ be a polynomially growing group. Then,
$G$ is virtually abelian $\Longleftrightarrow d c_{X}(G)>0$ for some (and hence all) $X$.

## Proof. ( $\Rightarrow$ ) Ok.

$(\Leftarrow)$

- By Gromov result, $\exists$ a nilpotent $H \leqslant f . i$. G.
- So, $\exists$ a t.f. nilpotent $K \leqslant_{\text {f.i. }} H \leqslant_{\text {fi.i }} G$.
- By hypotesis, $d c_{X}(G)>0$.
- Hence, $d c_{Y}(K)>0$ for every $\langle Y\rangle=K$.
- Then, $K$ is abelian.
- So, G is virtually abelian.


## A Gromov-like theorem

## Theorem

Let $G$ be a polynomially growing group. Then,
$G$ is virtually abelian $\Longleftrightarrow d c_{X}(G)>0$ for some (and hence all) $X$.

## Proof. ( $\Rightarrow$ ) Ok.

- By Gromov result, $\exists$ a nilpotent $H \leqslant_{\text {fi. }}$ G.
- So, $\exists$ a t.f. nilpotent $K \leqslant_{f i} H \leqslant_{f i} G$.
- By hypotesis, dcx $(G)>0$.
- Hence, $d c_{Y}(K)>0$ for every $\langle Y\rangle=K$.
- Then, $K$ is abelian.
- So, G is virtually abelian.


## A Gromov-like theorem

## Theorem

Let $G$ be a polynomially growing group. Then,
$G$ is virtually abelian $\Longleftrightarrow d c_{X}(G)>0$ for some (and hence all) $X$.

Proof. ( $\Rightarrow$ ) Ok.
$(\Leftarrow)$

- By Gromov result, $\exists$ a nilpotent $H \leqslant$ f.i. $G$.
- So, $\exists$ a t.f. nilpotent $K \leqslant$ fi. $H \leqslant$ fi. G.
- By hypotesis, $d_{X}(G)>0$
- Hence, $d c_{Y}(K)>0$ for every $\langle Y\rangle=K$.
- Then, K is abelian.
- So, G is virtually abelian.


## A Gromov-like theorem

## Theorem

Let $G$ be a polynomially growing group. Then,
$G$ is virtually abelian $\Longleftrightarrow d c_{X}(G)>0$ for some (and hence all) $X$.

Proof. ( $\Rightarrow$ ) Ok.
$(\Leftarrow)$

- By Gromov result, $\exists$ a nilpotent $H \leqslant$ f.i. $G$.
- So, $\exists$ a t.f. nilpotent $K \leqslant_{\text {f.i. }} H \leqslant_{\text {f.i. }} G$.
- By hypotesis, $d_{X}(G)>0$
- Hence, $d c_{Y}(K)>0$ for every $\langle Y\rangle=K$.
- Then, $K$ is abelian.
- So, G is virtually abelian.


## A Gromov-like theorem

## Theorem

Let $G$ be a polynomially growing group. Then,
$G$ is virtually abelian $\Longleftrightarrow d c_{X}(G)>0$ for some (and hence all) $X$.

Proof. ( $\Rightarrow$ ) Ok.
$(\Leftarrow)$

- By Gromov result, $\exists$ a nilpotent $H \leqslant$ f.i. $G$.
- So, $\exists$ a t.f. nilpotent $K \leqslant_{\text {f.i. }} H \leqslant_{\text {f.i. }} G$.
- By hypotesis, $d c_{X}(G)>0$.
- Hence, $d_{Y}(K)>0$ for every $\langle Y\rangle=K$.
- Then, $K$ is abelian.
- So, G is virtually abelian.


## A Gromov-like theorem

## Theorem

Let $G$ be a polynomially growing group. Then,
$G$ is virtually abelian $\Longleftrightarrow d c_{x}(G)>0$ for some (and hence all) $X$.

Proof. ( $\Rightarrow$ ) Ok.
$(\Leftarrow)$

- By Gromov result, $\exists$ a nilpotent $H \leqslant$ f.i. $G$.
- So, $\exists$ a t.f. nilpotent $K \leqslant_{\text {f.i. }} H \leqslant_{\text {f.i. }} G$.
- By hypotesis, $d c_{X}(G)>0$.
- Hence, $d c_{Y}(K)>0$ for every $\langle Y\rangle=K$.
- Then, $K$ is abelian.
- So, G is virtually abelian.


## A Gromov-like theorem

## Theorem

Let $G$ be a polynomially growing group. Then,
$G$ is virtually abelian $\Longleftrightarrow d c_{x}(G)>0$ for some (and hence all) $X$.

Proof. ( $\Rightarrow$ ) Ok.
$(\Leftarrow)$

- By Gromov result, $\exists$ a nilpotent $H \leqslant$ f.i. $G$.
- So, $\exists$ a t.f. nilpotent $K \leqslant_{\text {f.i. }} H \leqslant_{\text {f.i. }} G$.
- By hypotesis, $d c_{X}(G)>0$.
- Hence, $d c_{Y}(K)>0$ for every $\langle Y\rangle=K$.
- Then, $K$ is abelian.
- So, G is virtually abelian.


## A Gromov-like theorem

## Theorem

Let $G$ be a polynomially growing group. Then,
$G$ is virtually abelian $\Longleftrightarrow d c_{x}(G)>0$ for some (and hence all) $X$.

Proof. ( $\Rightarrow$ ) Ok.
$(\Leftarrow)$

- By Gromov result, $\exists$ a nilpotent $H \leqslant$ f.i. $G$.
- So, $\exists$ a t.f. nilpotent $K \leqslant_{\text {f.i. }} H \leqslant_{\text {f.i. }} G$.
- By hypotesis, $d c_{X}(G)>0$.
- Hence, $d c_{Y}(K)>0$ for every $\langle Y\rangle=K$.
- Then, $K$ is abelian.
- So, G is virtually abelian.


## A conjecture

## Conjecture

Every f.g. group $G$ with super-polinomial growth has $d c_{X}(G)=0$ for every $X$.

## Conjecture



## A conjecture

## Conjecture

Every f.g. group $G$ with super-polinomial growth has $d c_{X}(G)=0$ for every $X$.

## Conjecture

For any f.g. group $G=\langle X\rangle$,

$$
d c_{X}(G)>0 \Longleftrightarrow G \text { is virtually abelian. }
$$

## Outline

Main definition3 Finite index subgroups

4 Short exact sequences

5 A Gromov-like theorem

6 Other related results

## Other related results

## Theorem

Let $G$ be non-elementary hyperbolic. Then $d c_{X}(G)=0$ for every $X$.

## Theorem

Let $G=G(X)$ be a pc group. Then,


Theorem
let $G=|\boldsymbol{X}\rangle$ be a f.g. residually finite group with sub-exponential growth. If $d c_{X}(G)>5 / 8$ for some $X$ the $G$ is abelian.

## Other related results

## Theorem

Let $G$ be non-elementary hyperbolic. Then $d c_{X}(G)=0$ for every $X$.

## Theorem

Let $G=G(X)$ be a pc group. Then,

$$
d c_{X}(G(X))= \begin{cases}0 & \text { if } X \text { is not complete } \\ 1 & \text { if } X \text { is complete }\end{cases}
$$

## Theorem

Let $G=\langle X\rangle$ be a f.g. residually finite group with sub-exponential growth. If $d c_{X}(G)>5 / 8$ for some $X$ the $G$ is abelian.

## Other related results

## Theorem

Let $G$ be non-elementary hyperbolic. Then $d c_{X}(G)=0$ for every $X$.

## Theorem

Let $G=G(X)$ be a pc group. Then,

$$
d c_{X}(G(X))= \begin{cases}0 & \text { if } X \text { is not complete } \\ 1 & \text { if } X \text { is complete }\end{cases}
$$

## Theorem

Let $G=\langle X\rangle$ be a f.g. residually finite group with sub-exponential growth. If $d c_{X}(G)>5 / 8$ for some $X$ the $G$ is abelian.

THANKS

