Commuting degree for infinite groups

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Outline

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Motivation

(Joint work with Y. Antolín and A. Martino.)

Theorem (Gustafson, 1973)

Let G be a finite group. If the probability that two elements from G commute is bigger than 5/8, then G is abelian.

Proof. *Suppose G is not abelian. Then,*

$$dc(G) = \frac{|\{(u,v) \mid uv = vu\}|}{|G|^2} = \frac{1}{|G|^2} \sum_{u \in G} |C_G(u)| =$$

$$= \frac{1}{|G|^2} \Big(|Z(G)||G| + \sum_{u \in G \setminus Z(G)} |C_G(u)| \Big) \leqslant$$

$$\leqslant \frac{1}{|G|^2} \Big(|Z(G)||G| + (|G| - |Z(G)|) \frac{|G|}{2} \Big) =$$

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because $G/Z(G)$ cannot be cyclic and so, $|Z(G)| \leqslant |G|/4$. □

Observation

The quaternion group has $dc(Q) = 5/8$.

"There is no live between $5/8$ and 1"

(Goal)

Is there a version of dc for infinite groups ?

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Degree of commutativity

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Let $G = \langle X \rangle$ be a f.g. group. The degree of commutativity of G w.r.t. X is

$$dc_X(G) = \limsup_{n \to \infty} \frac{|\{(u,v) \in \mathbb{B}_X(n) \times \mathbb{B}_X(n) \mid uv = vu\}|}{|\mathbb{B}_X(n)|^2} \in [0,1],$$

where $\mathbb{B}_X(n) = \{g \in G \mid |g|_X \leqslant n\}$.

Question

Is this a real lim ? Does it depend on X ?

About limsup we have no idea:

- *No example where lim doesn't exist;*
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A f.g. group $G = \langle X \rangle$ *is of*

- *subexponential growth* if $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n+1)|}{|\mathbb{B}_X(n)|} = 1$;
- *polynomial growth (of degree d)* if $0 < Cn^d \leqslant |\mathbb{B}_X(n)| \leqslant Dn^d$.

Definition

Let $G = \langle X \rangle$. *A map* $f\colon G \to \mathbb{N}$ *is an estimation of the* X*-metric if* $\exists K > 0$ *such that* $\forall w \in G$

$$\frac{1}{K} f(w) \leqslant |w|_X \leqslant K\, f(w).$$

Example

It is well known that, for $G = \langle X \rangle = \langle Y \rangle$, $|\cdot|_X$ *is an estimation of the* Y*-metric, and* $|\cdot|_Y$ *is an estimation of the* X*-metric.*

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Define the f-ball and the f-dc:

$$\mathbb{B}_f(n) = \{w \in G \mid f(w) \leqslant n\},$$

$$dc_f(G) = \limsup_{n\to\infty} \frac{|\{(u,v) \in \mathbb{B}_f(n) \times \mathbb{B}_f(n) \mid uv = vu\}|}{|\mathbb{B}_f(n)|^2}.$$

Proposition

Let $G = \langle X \rangle$ be of polynomial growth, and $f\colon G \to \mathbb{N}$ be an estimation of the X-metric. Then,

$$dc_X(G) > 0 \iff dc_f(G) > 0.$$

Proof. *Clearly, $\mathbb{B}_f(n) \subseteq \mathbb{B}_X(Kn) \subseteq \mathbb{B}_f(K^2 n)$ so,*

$$|\{(u,v) \in (\mathbb{B}_f(n))^2 \mid uv = vu\}| \leqslant |\{(u,v) \in (\mathbb{B}_X(Kn))^2 \mid uv = vu\}|.$$

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So, $dc_X(G)=0 \quad \Rightarrow \quad dc_f(G)=0$, because

$$\frac{|\mathbb{B}_f(n)|}{|\mathbb{B}_X(Kn)|} \geqslant \frac{|\mathbb{B}_X(n/K)|}{|\mathbb{B}_X(Kn)|} \geqslant \frac{C(n/K)^d}{D(Kn)^d} = \frac{C}{DK^{2d}} > 0. \quad \square$$

Corollary

If $G=\langle X\rangle=\langle Y\rangle$ is of polynomial growth, then

$$dc_X(G)=0 \iff dc_Y(G)=0.$$

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Finite index subgroups

Lemma (Burillo–Ventura, 2002)

If $H \leqslant_{f.i.} G = \langle X \rangle$ and G has subexponential growth then there exists $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n) \cap H|}{|\mathbb{B}_X(n)|} = \frac{1}{[G:H]}$.

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Let $\langle Y \rangle = H \leqslant_{f.i.} G = \langle X \rangle$ be of polynomial growth. Then, $dc_X(G) \geqslant \frac{1}{[G:H]^2} dc_Y(H)$.

Proposition (Gallagher, 1970)

Let G be a finite group and $H \trianglelefteq G$. Then, $dc(G) \leqslant dc(H) \cdot dc(G/H)$.

Corollary

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Short exact sequences

Proposition

Let $G = \langle X \rangle$, $H \trianglelefteq G$, and let $\pi \colon G \twoheadrightarrow Q = G/H = \langle \overline{X} \rangle$. Put

$$0 \leqslant \lambda = \left(\liminf \frac{|\mathbb{B}_X(n)|}{|\mathbb{B}_{\overline{X}}(n)| \cdot |\mathbb{B}_X(2n) \cap H|} \right)^2 \leqslant 1.$$

Then, $\lambda \cdot dc_X(G) \leqslant dc_{\overline{X}}(Q) \cdot dc_X(H)$.

Proof. *Write $dc_X(G) = \limsup dc_X(G, n)$, where*

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$$|\mathbb{B}_X(n)|^2 dc_X(G,n) = |\{(u,v) \in (\mathbb{B}_X(n))^2 \mid uv = vu\}| =$$

$$= \sum_{u \in \mathbb{B}_X(n)} |C_G(u) \cap \mathbb{B}_X(n)| = \sum_{q \in \mathbb{B}_{\overline{X}}(n)} \sum_{\substack{u \in \mathbb{B}_X(n) \\ \pi(u)=q}} |C_G(u) \cap \mathbb{B}_X(n)| \leqslant$$

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Short exact sequences

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$$\left(\frac{|\mathbb{B}_X(n)|}{|\mathbb{B}_{\overline{X}}(n)| \cdot |\mathbb{B}_X(2n) \cap H|} \right)^2 \cdot dc_X(G, n) \leqslant dc_{\overline{X}}(Q, n) \cdot dc_X(H, 2n).$$

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Short exact sequences

Proposition

Let $G = \langle X \rangle$ be polynomially growing with degree d. Then, $\forall\ H \trianglelefteq_{f.i.} G$, we have

$$\left(\frac{C}{D \cdot 4^d}\right)^2 \cdot dc_X(G) \leq dc_{\overline{X}}(G/H) \cdot dc_X(H).$$

Proof. Clearly, $|\mathbb{B}_X(2n)| \geqslant |\mathbb{B}_{\overline{X}}(n)| \cdot |\mathbb{B}_X(n) \cap H|$.
Now fix H and, $\forall\ \epsilon > 0$, $\exists\ n_0$ s.t. $\forall\ n \geqslant n_0$,

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Short exact sequences

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So, $\lambda = (\liminf \cdots)^2 \geqslant \left(\frac{1}{[G:H]} - \epsilon\right)^2 \cdot \left(\frac{C}{D \cdot 4^s} - \epsilon\right)^2 \cdot ([G:H] - \epsilon)^2$.

And this is true for every $\epsilon > 0$ *hence,* $\lambda \geqslant \left(\frac{C}{D \cdot 4^d}\right)^2$,
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So, $\lambda = (\liminf \cdots)^2 \geqslant \left(\frac{1}{[G:H]} - \epsilon\right)^2 \cdot \left(\frac{C}{D \cdot 4^s} - \epsilon\right)^2 \cdot ([G:H] - \epsilon)^2$.

And this is true for every $\epsilon > 0$ *hence,* $\lambda \geqslant \left(\frac{C}{D \cdot 4^d}\right)^2$,
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Outline

t.f. nilpotent groups

Proposition

Let $G = \langle X \rangle$ be t.f. nilpotent. Then, either G is abelian, or $dc_X(G) = 0$.

Proof. *Assume G is not abelian and $dc_X(G) > 0$ and let us find a contradiction.*

- *We have a uniform $\lambda > 0$ s.t., for every $H \trianglelefteq_{f.i.} G$,*

$$\lambda \cdot dc_X(G) \leqslant dc_{\overline{X}}(G/H) \cdot dc_X(H).$$

- *Choose n s.t. $\lambda \cdot dc_X(G) \cdot (\frac{8}{5})^n > 1$.*
- *Take $\{p_1, \ldots, p_n\}$ be n pairwise different primes.*
- *By Grumbergs' classical result, G is residually-p_i.*
- *Hence, G has a non-abelian, finite p_i-quotient $\pi_i \colon G \twoheadrightarrow Q_i$; in particular, $dc(Q_i) \leqslant \frac{5}{8}$.*

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$$\lambda \cdot dc_X(G) \leqslant dc_X(H) \cdot dc_{\overline{X}}(Q_1 \times \cdots \times Q_n) \leqslant dc_X(H) \cdot (\frac{5}{8})^n.$$

- *Hence,*

$$1 < \lambda \cdot dc_X(G) \cdot (\frac{8}{5})^n \leqslant dc_X(H) \leqslant 1,$$

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A Gromov-like theorem

Theorem

Let G be a polynomially growing group. Then,

$$G \text{ is virtually abelian } \iff dc_X(G) > 0 \text{ for some (and hence all) } X.$$

Proof. *($\Rightarrow$) Ok.*
($\Leftarrow$)

- *By Gromov result, $\exists$ a nilpotent $H \leqslant_{f.i.} G$.*
- *So, $\exists$ a t.f. nilpotent $K \leqslant_{f.i.} H \leqslant_{f.i.} G$.*
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Other related results

Theorem

Let G be non-elementary hyperbolic. Then $dc_X(G) = 0$ for every X.

Theorem

Let $G = G(X)$ be a pc group. Then,

$$dc_X(G(X)) = \begin{cases} 0 & \text{if } X \text{ is not complete} \\ 1 & \text{if } X \text{ is complete} \end{cases}$$

Theorem

Let $G = \langle X \rangle$ be a f.g. residually finite group with sub-exponential growth. If $dc_X(G) > 5/8$ for some X the G is abelian.

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THANKS