The lattice of subgroups of a free group

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Outline

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This is work done by different authors during several years, and in different contexts.

We'll mostly follow a version by Bartholdi-Silva.

Then, we'll see several applications.

Free monoid

Definition

- $A = \{a_1, \ldots, a_r\}$ *is a finite alphabet.*
- A^* *is the free monoid on* A.
- *A language is a subset* $L \subseteq A^*$.
- *An involutive alphabet* $\tilde{A} = \{a_1, \ldots, a_r, a_1^{-1}, \ldots, a_r^{-1}\}$.
- *Reduced words; reduction* $\sim$; $R(A)$.
- *Formal word definitions*

$$(a^{-1})^{-1} = a,$$
$$(a_{i_1}^{\epsilon_1} \cdots a_{i_k}^{\epsilon_k})^{-1} = a_{i_k}^{-\epsilon_k} \cdots a_{i_1}^{-\epsilon_1}.$$

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Free group

Definition

The free group on A, $F(A) = \tilde{A}^* / \sim$.

Lemma

For every $w \in \tilde{A}^*$, there is a unique $\overline{u} \in R(A)$, s.t. $u =_{F(A)} \overline{u}$.

Lemma

$F(A)$ is a quotient of $\tilde{A}^*$. The projection is denoted π:

$$\begin{array}{rcl} \pi : \tilde{A}^* & \to & F(A) \\ u & \mapsto & [u] = [\overline{u}]. \end{array}$$

Definition

For a subgroup $H \leqslant F(A)$, define $\overline{H} = \{\overline{u} \mid u \in H\} \subseteq \tilde{A}^*$.

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Let A be an alphabet. An A-automaton $\mathcal{A}$ is an oriented graph with labels from A at the edges, and with a basepoint, $\mathcal{A} = (V, E, q_0)$, where

- *V is a finite set (of vertices),*
- *$E \subseteq V \times A \times V$ is the set of edges,*
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such that the underlying undirected graph is connected.

Note that $\mathcal{A}$ admits loops, but no parallel edges with the same label.

Definition

An A-automata $\mathcal{A} = (V, E, q_0)$ is involutive if A is an involutive alphabet and $(p, a, q) \in E \Leftrightarrow (q, a^{-1}, p) \in E$.

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Paths

Form now on, all automata we consider will be involutive.

Definition

Let $\mathcal{A}$ be an A-automata.

- *A path γ in $\mathcal{A}$,*
- *the label of a path γ, label$(\gamma) \in \tilde{A}^*$,*
- *reduced path,*
- *notation: $p \xrightarrow{u} q$ means a path from p to q with label $u \in \tilde{A}^*$.*

Lemma

Let $p \xrightarrow{u} q$ be a path in $\mathcal{A}$. If u is reduced then $p \xrightarrow{u} q$ is reduced. The convers is not true.

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Trimness

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The language of an A-automata $\mathcal{A}$, is

$$L(\mathcal{A}) = \{u \in \tilde{A}^* \mid \exists\, q_0 \xrightarrow{u} q_0\} \subseteq \tilde{A}^*.$$

Definition

An A-automata $\mathcal{A}$ is trim if it has no vertices of degree 1 except maybe the basepoint.

Lemma

If $\mathcal{A}$ is trim then $\forall q \neq q_0$ there exists a reduced path $q_0 \to q \to q_0$.

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Determinism

Definition

An A-automata $\mathcal{A}$ is deterministic if $(p, a, q) \in E$ and $(p, a, q') \in E$ imply $q = q'$.

Lemma

Let $\mathcal{A}$ be a deterministic A-automaton. We have,

i) if $p \xrightarrow{u} q$ is reduced then u is reduced,

ii) if $\exists\ p \xrightarrow{u} q$, $\exists\ p \xrightarrow{u} q'$ then $q = q'$,

iii) if $\exists\ p \xrightarrow{u} q$, $\exists\ p' \xrightarrow{u} q$ then $p = p'$.

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Morphisms

Definition

Let $\mathcal{A} = (V, E, q_0)$ and $\mathcal{A}' = (V', E', q_0')$ be two A-automata. A morphism $\mathcal{A} \to \mathcal{A}'$ is a map $\varphi\colon V \to V'$ such that $q_0\varphi = q_0'$ and

$$(p, a, q) \in E \;\Rightarrow\; (p\varphi, a, q\varphi) \in E'.$$

Proposition

Let $\mathcal{A} = (V, E, q_0)$ and $\mathcal{A}' = (V', E', q_0')$ be two A-automata, $\mathcal{A}'$ deterministic. Then,

$$L(\mathcal{A}) \subseteq L(\mathcal{A}') \;\Leftrightarrow\; \exists \text{ morphism } \varphi\colon \mathcal{A} \to \mathcal{A}'.$$

In this case, φ is unique.

This proof will be repeated with variations later.

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Corollary

If $\mathcal{A}$ is deterministic then the only morphism $\mathcal{A} \to \mathcal{A}$ is the identity.

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If $\mathcal{A}$ and $\mathcal{A}'$ are deterministic and $L(\mathcal{A}) = L(\mathcal{A}')$ then $\mathcal{A} \simeq \mathcal{A}'$.

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Flower automaton

GOAL: *to construct an algorithmic bijection between the lattice of finitely generated subgroups of $F(A)$, and the set of A-automata deterministic and trim.*

Definition

Given a finite set of reduced words $W \subseteq R(A) \subseteq F(A)$, we define the flower automaton $\mathcal{F}(W)$ in the natural way.

Observation

The flower automaton $\mathcal{F}(W)$ is

i) *involutive (by construction),*
ii) *trim,*
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Stallings folding

We want to make $\mathcal{F}(W)$ deterministic.

Definition

Let $\mathcal{A}$ be an A-automaton, and suppose $e = (p, a, q)$ and $e' = (p, a, q')$ are two different edges (so, $q \neq q'$).

Consider $\mathcal{L} = \{\{f, f'\} \neq \{e, e'\} \mid f = (p', b, q),\ f' = (p', b, q')$ for some $p' \in V, b \in \tilde{A}\}$.

Consider the automata $\mathcal{A}'$ to be $\mathcal{A}$ identifying $q = q'$ (and so, $e = e'$, and $f = f'$ for every $\{f, f'\} \in \mathcal{L}$, if any). We define $\mathcal{A} \rightsquigarrow \mathcal{A}'$ to be a Stallings folding.

The number $\ell = |\mathcal{L}| \geqslant 0$ is called the lost of the folding. A folding is called critical when it has positive lost.

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Consider the automata $\mathcal{A}'$ to be $\mathcal{A}$ identifying $q = q'$ (and so, $e = e'$, and $f = f'$ for every $\{f, f'\} \in \mathcal{L}$, if any). We define $\mathcal{A} \rightsquigarrow \mathcal{A}'$ to be a Stallings folding.

The number $\ell = |\mathcal{L}| \geqslant 0$ is called the lost of the folding. A folding is called critical when it has positive lost.

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We want to make $\mathcal{F}(W)$ deterministic.

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Observation

Let $\mathcal{A} = (V, E, q_0) \rightsquigarrow \mathcal{A}' = (V', E', q_0)$ be a folding with lost $\ell \geqslant 0$. Then, $|V'| = |V| - 1$ and $|E'| = |E| - 1 - \ell$.

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Applying enough foldings to any given A-automata $\mathcal{A}$,

$$\mathcal{A} \rightsquigarrow \mathcal{A}' \rightsquigarrow \cdots \rightsquigarrow \mathcal{A}^k,$$

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Lemma

Let $\mathcal{A} \rightsquigarrow \mathcal{A}'$ to be a Stallings folding. Then

i) *If, in $\mathcal{A}$, $|\{a \in \tilde{A} | q \xrightarrow{a} \}| \geqslant 2\ \forall q \neq q_0$, then the same is true in $\mathcal{A}'$.*

ii) *$\exists$ a morphism $\varphi \colon \mathcal{A} \to \mathcal{A}'$ (and so, $L(\mathcal{A}) \subseteq L(\mathcal{A}')$).*

iii) *$L(\mathcal{A})\pi = L(\mathcal{A}')\pi$.*

Corollary

Let $W \subset R(A)$, $|W| < \infty$, and consider a sequence of foldings $\mathcal{F}(W) \rightsquigarrow \cdots \rightsquigarrow \mathcal{A}$ to a deterministic $\mathcal{A}$. Then,

i) *$\mathcal{A}$ is deterministic and trim,*

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satisfies

i) $label(\gamma)\pi = label(\gamma')\pi$,

ii) *in the non-critical case, it is injective.*

Corollary

If all foldings in $\mathcal{F}(W) \rightsquigarrow \cdots \rightsquigarrow \mathcal{A}$ are non-critical, then the above map, $\pi\mathcal{F}(W)(q_0, q_0) \to \pi\mathcal{A}(q_0, q_0)$, $\gamma \mapsto \gamma'$, is injective.

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Membership problem

Theorem

The membership problem is solvable in $F(A)$: given $g, h_1, \ldots, h_n \in F(A)$, one can decide whether $g \in H = \langle h_1, \ldots, h_n \rangle$.

(1) *can assume $W = \{h_1, \ldots, h_n\} \subseteq R(A)$;*

(2) *draw the flower automaton $\mathcal{F}(W)$;*

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Independence of the process

In a sequence of Stallings foldings, $\mathcal{F}(W) \rightsquigarrow \cdots \rightsquigarrow \mathcal{A}$, the result will not depend on the process, and even on W, but only on the subgroup $H = \langle W \rangle \leqslant F(A)$.

Theorem

$\mathcal{A}$ depends only on $H = \langle W \rangle$, and is called the Schreier graph, $\Gamma(H)$.

Proposition

Let $H \leqslant_{f.g} F(A)$, choose a finite set of generators W, $\langle W \rangle = H$, and let $\mathcal{F}(W) \rightsquigarrow \cdots \rightsquigarrow \mathcal{A}$ be an arbitrary sequence of Stallings foldings, with $\mathcal{A}$ deterministic. Then,

$$L(\mathcal{A}) = \bigcap_{\mathcal{B}} L(\mathcal{B}),$$

where $\mathcal{B}$ runs over all possible automata deterministic, trim, and such that $\overline{H} \subseteq L(\mathcal{B}) \subseteq \tilde{A}^*$.

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Outline

Nielsen-Schreier Theorem

Let $\mathcal{A}$ be an A-automata deterministic and trim, and let $H = L(A)\pi \leqslant_{f.g.} F(A)$.

Take a maximal tree T in $\mathcal{A}$ and for every $e \in E\mathcal{A} \setminus ET$ take

$$h_e = label(T[q_0, \iota e]eT[\tau e, q_0])\pi \in H.$$

Proposition

$\{h_e \mid e \in E\mathcal{A} \setminus ET\}$ is a free basis for H.

Theorem (Nielsen)

Every finitely generated subgroup of a free group is free.

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Computability of rank and basis

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There is an algorithm which, given $h_1 \ldots, h_n \in F(A)$, computes the rank and a basis of $H = \langle h_1 \ldots, h_n \rangle \leqslant F(A)$. More specifically,

$$rg(H) = 1 - |V\Gamma(H)| + |E\Gamma(H)| = |W| - \ell,$$

where ℓ is the total lost in the chain $\mathcal{F}(W) \rightsquigarrow \cdots \rightsquigarrow \Gamma(H)$.

Theorem

Free groups are hopfian.

Corollary

$F(A) \simeq F(B)$ if and only if $|A| = |B|$.

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$$rg(H) = 1 - |V\Gamma(H)| + |E\Gamma(H)| = |W| - \ell,$$

where ℓ is the total lost in the chain $\mathcal{F}(W) \rightsquigarrow \cdots \rightsquigarrow \Gamma(H)$.

Theorem

Free groups are hopfian.

Corollary

$F(A) \simeq F(B)$ if and only if $|A| = |B|$.

Computability of rank and basis

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Outline

The full Schreier graph

Definition

An A-automata $\mathcal{A}$ is complete if every vertex has an edge going in and an edge going out with each label.

Definition

For $H \leqslant_{f.g.} F(A)$, define $\tilde{\Gamma}(H)$ to be $\Gamma(H)$ with infinite trees attached in order to make it complete.

Observation

i) $\Gamma(H)$ *is complete* $\Leftrightarrow \tilde{\Gamma}(H) = \Gamma(H)$.

ii) $\forall\, u \in \tilde{A}^*,\ \forall\, q \in V,\ \exists\, q \xrightarrow{u}$ in $\tilde{\Gamma}(H)$.

iii) $L(\tilde{\Gamma}(H)) = H\pi^{-1}$.

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Finite index subgroups

Proposition

Let $H \leqslant_{f.g.} F(A)$ and let T be a maximal tree in $\tilde{\Gamma}(H)$. Then,

$$\begin{array}{rccc} \varphi\colon & V\tilde{\Gamma}(H) & \to & H\backslash F(A) \\ & v & \mapsto & H \cdot l_v \end{array}$$

is bijective, where $l_v = label(T[q_0, v])\pi$.

Corollary

Let $H \leqslant_{f.g.} F(A)$. Then, $H \leqslant_{f.i.} F(A) \Leftrightarrow \Gamma(H)$ is complete. In this case, $[F(A) : H] = |V\Gamma(H)|$.

Corollary (Schreier index formula)

Every $H \leqslant_{f.i.} F(A)$ is finitely generated and $r(H) - 1 = [F(A) : H](|A| - 1)$.

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Corollary

There is an algorithm which, given $h_1, \ldots, h_n, k_1, \ldots, k_m \in F(A)$, decides whether $\langle h_1, \ldots, h_n \rangle = H \leqslant_{f.i.} K = \langle k_1, \ldots, k_m \rangle$ and, in this case, computes the index and a set of coset representatives.

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Normality

Corollary

If $1 \neq H \leqslant_{f.g.} F(A)$ is normal then $H \leqslant_{f.i.} F(A)$.

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M. Hall's theorem

Definition

Let $H \leqslant K \leqslant F(A)$. We say that H is a free factor of K, denoted $H \leqslant_{f.f.} K$, if it is possible to extend a basis of H to a basis of K.

Lemma

For $H \leqslant_{f.g.} K \leqslant_{f.g.} F(A)$, if $\Gamma(H)$ is a subautomaton of $\Gamma(K)$ then $H \leqslant_{f.f.} K$. The convers is not true.

Theorem (M. Hall)

For every $H \leqslant_{f.g.} F(A)$ there exists $K \leqslant_{f.i.} F(A)$ such that $H \leqslant_{f.f.} K \leqslant_{f.i.} F(A)$.

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Residual finiteness

Definition

A group G is said to be residually finite if $\forall 1 \neq g \in G$ there exists a finite quotient G/H where $1 \neq \overline{g} \in G/H$.

Theorem

Free groups are residually finite.

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Free groups are residually p, for every prime p.

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Outline

Howson property

Definition

A group G satisfies the **Howson property** *if the intersection of two finitely generated subgroups is again finitely generated.*

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There is an algorithm which, given $h_1, \ldots, h_n, k_1, \ldots, k_m \in F(A)$, computes a basis of $H \cap K$, where $H = \langle h_1, \ldots, h_n \rangle$ and $K = \langle k_1, \ldots, k_m \rangle$.

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Hanna Neumann "Conjecture"

Definition

Define the reduced rank of $H \leqslant F(A)$ *as* $\tilde{r}(H) = \max\{0, r(H) - 1\}$.

Theorem

For $H, K \leqslant_{f.g.} F(A)$, $\tilde{r}(H \cap K) \leqslant 2\tilde{r}(H)\tilde{r}(K)$.

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Malnormality

Definition

A subgroup of a group $H \leqslant G$ is malnormal *if $H^g \cap H = 1$ for all $g \notin H$.*

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There is an algorithm which, given $h_1, \ldots, h_n \in F(A)$, decides whether $H = \langle h_1, \ldots, h_n \rangle$ is malnormal in $F(A)$.

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Outline

Motivation

- In basic linear algebra:

$$U \leqslant V \leqslant K^n \quad \Rightarrow \quad V = U \oplus L.$$

- In $\mathbb{Z}^n$, the analog is almost true:

$$U \leqslant V \leqslant \mathbb{Z}^n \quad \Rightarrow \quad \exists\ U \leqslant_{fi} U' \leqslant V \text{ s.t. } V = U' \oplus L.$$

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almost true again, ... in the sense of Takahasi.

Algebraic and transcendental elements

Mimicking field theory...

Definition

Let $H \leqslant F(A)$ and $w \in F(A)$. We say that w is

- *algebraic over H if $\exists\, 1 \neq e_H(x) \in H * \langle x \rangle$ such that $e_H(w) = 1$;*
- *transcendental over H otherwise.*

Observation

w is transcendental over H $\iff \langle H, w \rangle \simeq H * \langle w \rangle$
$\iff$ *H is contained in a proper f.f. of $\langle H, w \rangle$.*

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w_1, w_2 algebraic over H $\not\Rightarrow$ $w_1 w_2$ algebraic over H.

$H = \langle a, \overline{b}ab, \overline{c}ac \rangle \leqslant \langle a, b, c \rangle$, and $w_1 = b$, $w_2 = \overline{c}$.

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A relative notion works better...

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- *K-algebraic over H if $\forall$ free factorization $K = K_1 * K_2$ with $H \leqslant K_1$, we have $w \in K_1$;*
- *K-transcendental over H otherwise.*

Observation

w is algebraic over H if and only if it is $\langle H, w \rangle$-algebraic over H.

Observation

If w_1 and w_2 are K-algebraic over H, then so is $w_1 w_2$.

Algebraic and free extensions

Definition

Let $H \leqslant K \leqslant F(A)$.

We say that $H \leqslant K$ is an algebraic extension, denoted $H \leqslant_{alg} K$,

$\iff$ *every $w \in K$ is K-algebraic over H,*

$\iff$ *H is not contained in any proper free factor of K,*

$\iff$ *$H \leqslant K_1 \leqslant K_1 * K_2 = K$ implies $K_2 = 1$.*

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- if $r(H) \geqslant 2$ and $r(K) \leqslant 2$ then $H \leqslant_{alg} K$.
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Can we compute them all ?

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Takahasi's Theorem

Theorem (Takahasi, 1951)

For every $H \leqslant_{fg} F(A)$, the set of algebraic extensions, denoted $\mathcal{AE}(H)$, is finite.

- Original proof by Takahasi was combinatorial and technical,
- Modern proof, using Schreier automata, is much simpler, and due independently to Ventura (1997), Margolis-Sapir-Weil (2001) and Kapovich-Miasnikov (2002).
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Fringe of a subgroup

Definition

Let $\mathcal{A}$ be a deterministic and trim A-automata, and let $\sim$ an eq. rel. on $V\mathcal{A}$. We denote by $\mathcal{A}/\sim$ the new (deterministic and trim) A-automata resulting from identifying the vertices according to $\sim$, plus foldings.

Definition

The fringe of $\mathcal{A}$ is the (finite) collection of A-automata of the form $\mathcal{A}/\sim$.

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Let $H \leqslant_{fg} F(A)$. The fringe of H is $\mathcal{O}(H) = \{L(\Gamma(H)/\sim)\pi \mid \sim \text{ eq. rel. on } V\mathcal{A}\}$.

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Observation

For $H \leqslant_{fg} F(A)$, we have $\mathcal{O}(H) = \{H_0, H_1, \ldots, H_k\}$, all of them f.g., and with $H_0 = H$ and $H_k = \langle A' \rangle$ ($A' \subseteq A$ the set of used letters).

Observation

For $H \leqslant_{fg} F(A)$, $\mathcal{O}(H)$ is finite and computable.

Proposition

For $H \leqslant_{fg} F(A)$, $\mathcal{AE}(H) \subseteq \mathcal{O}(H)$.

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For $H \leqslant_{fg} F(A)$, $\mathcal{AE}(H)$ is computable.

1) Compute $\Gamma(H)$,
2) Compute $\Gamma(H)/\sim$ for all eq. rel. $\sim$ of $V\Gamma(H)$,
3) Compute $\mathcal{O}(H)$,
4) Clean $\mathcal{O}(H)$ by detecting all pairs $K_1, K_2 \in \mathcal{O}(H)$ such that $K_1 \leqslant_{ff} K_2$ and deleting K_2.
5) The resulting set is $\mathcal{AE}(H)$. □

For the cleaning step we need:

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Proposition

Given $H, K \leqslant F(A)$, it is algorithmically decidable whether $H \leqslant_{ff} K$ or not.

Proved by:

- Whitehead 1930's (classical and exponential),
- Silva-Weil 2006 (faster but still exponential),
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The algebraic closure

Observation

If $H \leqslant_{alg} K_1$ and $H \leqslant_{alg} K_2$ then $H \leqslant_{alg} \langle K_1 \cup K_2 \rangle$.

Corollary

For every $H \leqslant_{fg} K \leqslant_{fg} F(A)$, $\mathcal{AE}_K(H)$ has a unique maximal element, called the K-algebraic closure of H, and denoted $Cl_K(H)$.

Theorem

Every extension $H \leqslant K$ of f.g. subgroups of $F(A)$ splits, in a unique way, in an algebraic part and a free part, $H \leqslant_{alg} Cl_K(H) \leqslant_{ff} K$.

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Outline

Pseudo-varieties

Definition

A pseudo-variety of groups $\mathcal{V}$ is a class of finite groups closed under taking subgroups, quotients and finite direct products.

i) $\mathcal{G}$ = all finite groups,
ii) $\mathcal{G}_p$ = all finite p-groups,
iii) $\mathcal{G}_{nil}$ = all finite nilpotent groups,
iv) $\mathcal{G}_{sol}$ = all finite soluble groups,
v) $\mathcal{G}_{ab}$ = all finite abelian groups,
vi) for a finite group V, $[V]$ = all quotients of subgroups of V^k, $k \geqslant 1$.
vii) $\cdots$

Definition

$\mathcal{V}$ is extension-closed if $V \lhd W$ with $V, W/V \in \mathcal{V}$ imply $W \in \mathcal{V}$.

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A pseudo-variety of groups $\mathcal{V}$ is a class of finite groups closed under taking subgroups, quotients and finite direct products.

i) $\mathcal{G}$ = all finite groups,
ii) $\mathcal{G}_p$ = all finite p-groups,
iii) $\mathcal{G}_{nil}$ = all finite nilpotent groups,
iv) $\mathcal{G}_{sol}$ = all finite soluble groups,
v) $\mathcal{G}_{ab}$ = all finite abelian groups,
vi) for a finite group V, $[V]$ = all quotients of subgroups of V^k, $k \geqslant 1$.
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$\mathcal{V}$ is extension-closed if $V \lhd W$ with $V, W/V \in \mathcal{V}$ imply $W \in \mathcal{V}$.

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The pro-$\mathcal{V}$ topology

Definition

Let G be a group, and $\mathcal{V}$ be a pseudo-variety of finite groups. The pro-$\mathcal{V}$ topology on G can be defined in several equivalent ways:

i) *it is the smallest topology making all the morphisms from G into all $V \in \mathcal{V}$ (with the discrete topology) continuous,*

ii) *a basis of open sets is given by $\varphi^{-1}(x)$, for all morphism $\varphi \colon G \to V \in \mathcal{V}$,*

iii) *the normal (finite index) subgroups $K \trianglelefteq G$ such that $G/K \in \mathcal{V}$ form a basis of neighborhoods of 1,*

iv) *it is the topology given by the pseudo-ultra-metric $d(x,y) = 2^{-r(x,y)}$, where $r(x,y) = \min\{|V| \mid V \in \mathcal{V}$ and separates x and y $\}$.*

Observation

This topology is Hausdorf $\Longleftrightarrow$ d is an ultra-metric $\Longleftrightarrow$ G is residually-$\mathcal{V}$.

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The $\mathcal{V}$-closure

Proposition

Let G be a group equipped with the pro-$\mathcal{V}$ topology, and let $H \leq G$. Then, TFAE:

(a) *H is open*

(b) *H is clopen (i.e. open and closed)*

(c) *$H \leqslant_{fi} G$ and $G/H_G \in \mathcal{V}$.*

Furthermore,

$$cl_{\mathcal{V}}(H) = \bigcap_{H \leqslant K,\ open} K = \bigcap_{\varphi:\ G \to V \in \mathcal{V}} \varphi^{-1}(\varphi(H)).$$

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The extension-closed case

Proposition (Ribes, Zaleskiĭ)

Let $\mathcal{V}$ be an extension-closed pseudo-variety, and consider $F(A)$ the free group on A with the pro-$\mathcal{V}$ topology. For a given $H \leqslant_{fg} F(A)$,

$$H \text{ is closed} \iff H \text{ is a free factor of a clopen subgroup.}$$

Corollary

For an extension-closed $\mathcal{V}$ and a $H \leqslant_{fg} F(A)$, we have $H \leqslant_{alg} cl_{\mathcal{V}}(H)$.

Furthermore, it can also be proven that

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In this situation, $r(cl_{\mathcal{V}}(H)) \leqslant r(H)$.

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p-closure, *nil*-closure

Theorem (Margolis-Sapir-Weil)

The p-closure of $H \leqslant_{fg} F(A)$, $cl_p(H)$, is effectively computable, for every prime p.

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For $H \leqslant_{fg} F(A)$, $cl_{nil}(H) = \cap_p cl_p(H)$. Thus, $cl_{nil}(H)$ is effectively computable.

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Find an algorithm to compute the solvable closure of a given $H \leqslant_{fg} F(A)$.

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Outline

Fixed subgroups are complicated

$$\begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & ba \\ c & \mapsto & ca^2 \end{array}$$

$$\mathrm{Fix}\,\phi = \langle a, bab^{-1}, cac^{-1} \rangle$$

$$\begin{array}{rcl} \varphi\colon F_4 & \to & F_4 \\ a & \mapsto & dac \\ b & \mapsto & c^{-1}a^{-1}d^{-1}ac \\ c & \mapsto & c^{-1}a^{-1}b^{-1}ac \\ d & \mapsto & c^{-1}a^{-1}bc \end{array}$$

$\mathrm{Fix}\,\varphi = \langle w \rangle$, where...

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What is known about fixed subgroups ?

Theorem (Dyer-Scott, 75)

Let $\phi \in Aut(F(A))$ be a finite order automorphism of $F(A)$. Then, $Fix(\phi) \leqslant_{\rm ff} F_n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) < \infty$.

Theorem (Bestvina-Handel, 88 (published 92))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

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Inertia

Definition

A subgroup $H \leqslant F_n$ is called inert if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F_n$.

Theorem (Dicks-V, 96)

Let $G \subseteq Mon(F_n)$ be an arbitrary set of monomorphisms of F_n. Then, $Fix(G)$ is inert; in particular, $r(Fix(G)) \leqslant n$.

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A subgroup $H \leqslant F_n$ is said to be

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- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
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Easy to see that 1-mono-fixed = 1-auto-fixed.

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Relations between them

$$\begin{array}{ccc} \boxed{1-auto-fixed} & \subseteq & \boxed{1-endo-fixed} \\ \cap\!| & & \cap\!| \\ \boxed{auto-fixed} & \subseteq & \boxed{endo-fixed} \end{array}$$

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Example (Martino-V., 03; Ciobanu-Dicks, 06)

Let $F_3 = \langle a, b, c \rangle$ and $H = \langle b, cacbab^{-1}c^{-1} \rangle \leqslant F_3$. Then, $H = Fix(a \mapsto 1,\ b \mapsto b,\ c \mapsto cacbab^{-1}c^{-1})$, but H is NOT the fixed subgroup of any set of automorphism of F_3.

Relations between them

$$\boxed{1-auto-fixed} \quad \subsetneq \quad \boxed{1-endo-fixed}$$

$$\cap| \quad \| ? \qquad\qquad \cap| \quad \| ?$$

$$\boxed{auto-fixed} \quad \subsetneq \quad \boxed{endo-fixed}$$

Conjecture (V.)

1-auto-fixed = auto-fixed, and 1-endo-fixed = endo-fixed.
That is, the families of 1-auto-fixed and 1-endo-fixed subgroups are closed under intersections.

It is true up to free factors

Theorem (Martino-V., 00)

Let $S \subseteq End(F_n)$. Then, $\exists \phi \in \langle S \rangle$ such that $Fix(S) \leqslant_{\rm ff} Fix(\phi)$.

However... free factors of 1-endo-fixed (1-auto-fixed) subgroups need not be even endo-fixed (auto-fixed).

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Definition

A subgroup $H \leqslant F(A)$ is compressed when $r(H) \leqslant r(K)$ for every $H \leqslant K \leqslant F(A)$.

Observation

H inert $\Rightarrow$ H compressed.

Is every compressed subgroup, inert?

Proposition

There is an algorithm which, given $H \leqslant_{fg} F(A)$, decides whether H is compressed.

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