

Some group-based cryptosystems

Enric Ventura

Departament de Matemàtica Aplicada III

Universitat Politècnica de Catalunya

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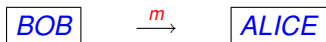
Outline

- 1 The origins of public key cryptography
- 2 A protocol based on the word problem
- 3 Protocols based on the conjugacy problem
- 4 Protocols based on the factorization problem
- 5 Anshel-Anshel-Goldfeld protocol
- 6 Some authentication protocols

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The goal

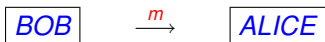


Bob wants to send a **secret** message, m , to **Alice** over an **open** channel (and **Eve** is trying to illegitimately discover m and break the system).

From Wikipedia: *"Diffie-Hellman key agreement was invented in 1976 ... and was the first practical method for establishing a shared secret over an unprotected communications channel"*.

A third author, **Merkle**, was also involved in the construction (U.S. Patent 4.200.770, now expired, describes the algorithms and credits **Diffie**, **Hellman** and **Merkle** as inventors).

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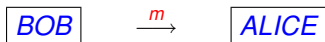


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Reduction to key establishment

- For simplicity, we assume that $m \in \{0, 1\}^n$.
- Let \mathcal{S} be a set and $H: \mathcal{S} \rightarrow \{0, 1\}^n$ a function (called the **key space** and a **Hash function**, respectively).
- Suppose Bob and Alice share a **secret key**, $K \in \mathcal{S}$.
- **Encryption:** Bob encrypts his message m as

$$E(m) = m + H(K).$$

- **Decryption:** Alice decrypts in the same way:

$$E(m) + H(K) = m + (H(K) + H(K)) = m.$$

- **Eavesdropper:** Eve needs to find $H(K)$, i.e. K .
- **Expansion factor** is 1.

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- **Public:** p (prime) and $g \notin p\mathbb{Z}$.
- **Alice:** picks a random $a \in \mathbb{N}$, and sends $g^a \bmod p$.
- **Bob:** picks a random $b \in \mathbb{N}$, and sends $g^b \bmod p$.
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Alice: $(g^b)^a = g^{ba} \bmod p$
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- **Eve:** knows p , g and g^a , $g^b \bmod p$, and needs $g^{ab} \bmod p$.
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In practical implementations, $|g|$ is typically about 10^{300} , so brute force attack is **computationally infeasible**.

This is **not** a problem for **Alice** and **Bob** because computing $g^a \bmod p$ for a particular a is much faster, $O(\log_2 a)$, by the **square-and-multiply** method:

$$g^{21} = g^{16} \cdot g^4 \cdot g = (((g^2)^2)^2)^2 \cdot (g^2)^2 \cdot g.$$

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The word problem in groups

Let $\langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a finite presentation of a group G .

- **Word Problem:** “given a word $w(x_1, \dots, x_n)$ decide whether $w =_G 1$ or not (i.e. whether $w \in \langle\langle R \rangle\rangle$)”.

There are finitely presented groups with **unsolvable Word Problem**.

A set of words Σ on X is said to have **no collision in G** if the natural map $\Sigma \rightarrow G$ is **injective**.

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Wagner-Magyarik protocol (1984)

- **Public:** A platform $G = \langle X \mid R \rangle$ and two words $\Sigma = \{w_0, w_1\}$.
- **Private:** A set of words S such that
 - the Word Problem is “difficult” in $G = \langle X \mid R \rangle$,
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 - Σ has no collision in G' (and so, in G).
- **Bob:** encodes each bit b in his message by an arbitrary (and changing) word w such that $w =_G w_b$.
- **Alice:** decodes w by solving the Word Problem in G' : decide whether $w =_{G'} w_0$ or $w =_{G'} w_1$.
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- **Or...:** find an alternative private key, T , with **easy** Word Problem in G/T , and **no collision** for Σ .

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Solvable Conjugacy Problem \implies solvable Word Problem.

Solvable Conjugacy Problem $\not\Leftarrow$ solvable Word Problem.

- **Conjugacy Search Problem:** “given $u, v \in G$ and the information that u and v are conjugate to each other in G , find an $x \in G$ such that $v =_G x^{-1}ux$ ”.

CSP is **always solvable** (brute force searching over all possible $x \in G$), but at **which complexity** this is a much more delicate question.

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Then, $a_1 w^b a_2 = a_1 (b^{-1} w b) a_2 = b^{-1} (a_1 w a_2) b = b^{-1} w^a b = w^{ab}$, and she finds the secret.

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Outline

- 1 The origins of public key cryptography
- 2 A protocol based on the word problem
- 3 Protocols based on the conjugacy problem
- 4 Protocols based on the factorization problem**
- 5 Anshel-Anshel-Goldfeld protocol
- 6 Some authentication protocols

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- **Factorization Problem:** “given $u \in G$ and $A, B \leq G$, decide whether $u =_G ab$ for some $a \in A$ and $b \in B$ ”.
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- **Public:** $G = \langle X \mid R \rangle$ and $A, B \leq G$ such that $[a, b] = 1 \forall a \in A, \forall b \in B$.
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- **Eve:** knows $a_1 a_2$ and $b_1 b_2$, and needs $a_2 a_1 b_1 b_2$.
This can be done by solving the **Factorization Search Problem in A (or B)**.

Note that **Eve** can compute

$$(a_1 b_1)(a_2 b_2) = a_1 a_2 b_1 b_2 \quad \text{and} \quad (a_2 b_2)(a_1 b_1) = a_2 a_1 b_2 b_1,$$

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Kurt's protocol (2006)

- Public:** $G = \langle X \mid R \rangle$, 10 subgroups $A_1, A_2, A_3, X_1, X_2, B_1, B_2, B_3, Y_1, Y_2 \leq G$ such that $[A_2, Y_1] = [A_3, Y_2] = [B_1, X_1] = [B_2, X_2] = 1$.
- Alice:** picks a random $a_1 \in A_1, a_2 \in A_2, a_3 \in A_3, x_1 \in X_1, x_2 \in X_2$, and sends $a_1 x_1, x_1^{-1} a_2 x_2$ and $x_2^{-1} a_3$.
- Bob:** picks a random $b_1 \in B_1, b_2 \in B_2, b_3 \in B_3, y_1 \in Y_1, y_2 \in Y_2$, and sends $b_1 y_1, y_1^{-1} b_2 y_2$ and $y_2^{-1} b_3$.
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Stickel's protocol (2005)

- **Public:** A finite group G , $w \in G$, and $a, b \in G$ with $ab \neq ba$ (of order N and M , respectively).

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- **Bob:** picks a random $0 < n' < N$ and $0 < m' < M$, and sends $a^{n'} w b^{m'}$.

- **Common secret:** Alice: $a^n (a^{n'} w b^{m'}) b^m = a^{n+n'} w b^{m+m'}$
Bob: $a^{n'} (a^n w b^m) b^{m'} = a^{n+n'} w b^{m+m'}$.

- **Eve:** knows $a, b, a^n w b^m$ and $a^{n'} w b^{m'}$, and needs $a^{n+n'} w b^{m+m'}$.

This can be done by solving a variation of the **Discrete Logarithm Problem** (in G).

Or... finding alternative $x, y \in G$ such that $xa = ax, yb = by$ and $xwy = a^n w b^m$. Then,

$$x(a^{n'} w b^{m'})y = a^{n'} xwyb^{m'} = a^{n'} (a^n w b^m) b^{m'} = a^{n+n'} w b^{m+m'}.$$

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- These are protocols to ensure that **somebody is really who is claiming to be**.
- **General setting**: Every player has a public **name**, and a secret **key**. When I call somebody by his name, he must provide me a proof that he knows the corresponding secret **key** (so, he is who is supposed to be), but without revealing any information about the **key** itself.
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- **Repeat** these last three steps, k times.

- **Eve:** has to send the commitment before knowing the future values of α ; so, acting like before, she only has probability $\frac{1}{2^k}$ to succeed.
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The Twisted Conjugacy Problem

One can use the same idea, but replacing the **Conjugacy Search Problem** to the harder **Twisted Conjugacy Search Problem**.

- **Twisted Conjugacy Problem:** “given $u, v \in G$ and $\varphi: G \rightarrow G$, decide whether $v =_G (x\varphi)^{-1}ux$ for some $x \in G$ ”.

Solv. Twisted Conjugacy Problem \implies solv. Conjugacy Problem.

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TCSP is **always solvable** (brute force searching over all possible $x \in G$), but at **which complexity** this is a much more delicate question.

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- **Twisted Conjugacy Search Problem:** “given $u, v \in G$, $\varphi: G \rightarrow G$, and the information that u and v are φ -twisted conjugated to each other in G , find an $x \in G$ such that $v =_G (x\varphi)^{-1}ux$ ”.

TCSP is **always solvable** (brute force searching over all possible $x \in G$), but at **which complexity** this is a much more delicate question.

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- **Repeat** these last three steps, k times.
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1. The origins
○○○○○

2. Word problem
○○○

3. Conjugacy problem
○○○○○

4. Factorization problem
○○○○

5. Anshel-Anshel-Goldfeld protocol
○○

6. Authentication protocols
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THANKS