Existence of finitely presented intersection-saturated groups

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XVI Barcelona Weekend in Group Theory

(joint work with J. Delgado and M. Roy)

May 5th, 2023.

Outline

1. Our main results
2. Free-times-free-abelian groups
3. Realizable / unrealizable k-configurations
4. The free case
5. Open questions

Outline

Free groups

It is well known that subgroups of free groups are free ...

$$H \leqslant \mathbb{F}_n \quad \Rightarrow \quad H \text{ is free}$$

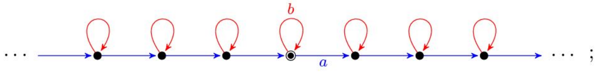
but not necessarily of rank $\leq n$.

Example

Consider $\mathbb{F}_2 = \langle x, y \mid \ \rangle$ and the normal closure of x,

$$\ll x \gg = \langle \ldots, y^2xy^{-2}, yxy^{-1}, x, y^{-1}xy, y^{-2}xy^2, \ldots \rangle.$$

Looking at its Stallings graph



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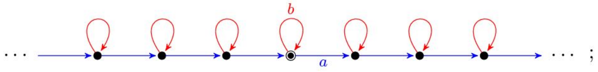

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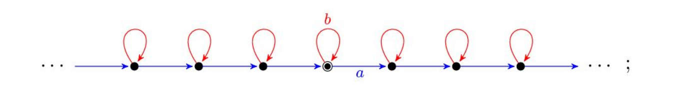

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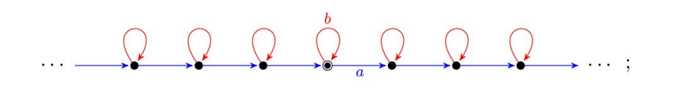

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The Howson property

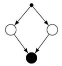
Definition

A group G is Howson if, for any finitely generated $H, K \leqslant_{fg} G$, the intersection $H \cap K$ is, again, finitely generated.

Theorem (Howson, 1954)

Free groups are Howson.

In other words... the configuration



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Observation

Out of $2^3 = 8$ possible such configurations this is the only one forbidden in free groups.

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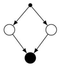

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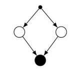

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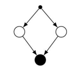

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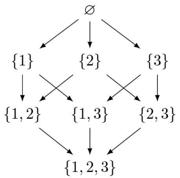

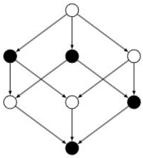

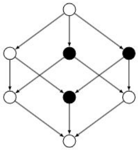

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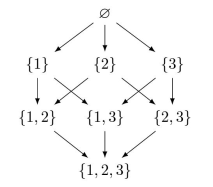

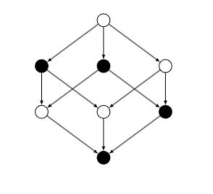

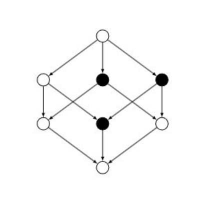

Our main results

Theorem (Delgado–Roy–V., '22)

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Formal definitions

Definition

A (intersection) k-configuration is a map $\chi : \mathcal{P}([k]) \setminus \{\emptyset\} \to \{0,1\}$. If $\mathcal{I} = (1)\chi^{-1}$ is the support of χ, we write $\chi = \chi_{\mathcal{I}}$. Notation:

- $\mathbf{0} = \chi_\emptyset$ is the *zero-configuration*;
- $\mathbf{1} = \chi_{\mathcal{P}([k])\setminus\{\emptyset\}}$ is the *one-configuration*;
- $\chi_{\mathcal{I}}$ is an *almost-zero* k-configuration if $\mathcal{I} = \{I\}$.

Definition

A k-configuration χ is realizable in a group G if there exists subgroups $H_1, \ldots, H_k \leq G$ such that, for every $\emptyset \neq I \subseteq [k]$, $H_I = \cap_{i \in I} H_i$ if f.g. $\Leftrightarrow (I)\chi = 0$. Note that $H_{I \cup J} = H_I \cap H_J$.

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A group G is intersection-saturated if every k-configuration is realizable in G.

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Outline

Free-times-free-abelian groups

$\mathbb{G} = \mathbb{F}_n \times \mathbb{Z}^m = \langle x_1, \ldots, x_n, t_1, \ldots, t_m \mid [x_i, t_j] = 1, [t_i, t_k] = 1 \rangle.$

Normal form: $\forall g \in \mathbb{G}$, $g = w(x_1, \ldots, x_n) t_1^{a_1} \cdots t_m^{a_m} = wt^{\mathbf{a}}$, *where* $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m$. *This way,* $(ut^{\mathbf{a}})(vt) = uvt^{\mathbf{a}+\mathbf{b}}$.

Observation

These groups sit in a split short exact sequence; and, for $H \leqslant \mathbb{G}$,

$$1 \to \mathbb{Z}^m \stackrel{\iota}{\hookrightarrow} \mathbb{G} \stackrel{\pi}{\twoheadrightarrow} \mathbb{F}_n \to 1,$$

$$1 \to L_H = H \cap \mathbb{Z}^m \hookrightarrow H \twoheadrightarrow H\pi \to 1.$$

Moreover, H *is finitely generated* $\Leftrightarrow$ $H\pi$ *is so.*

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Free-times-free-abelian groups

Proposition (Delgado–V. '13)

Every subgroup $H \leqslant \mathbb{G}$ admits a (computable) basis

$$H = \langle u_1 t^{\mathbf{a_1}}, u_2 t^{\mathbf{a_2}}, \ldots, u_r t^{\mathbf{a_r}}; t^{\mathbf{b_1}}, \ldots, t^{\mathbf{b_s}} \rangle,$$

where $\{u_1, \ldots, u_r\}$ is a free-basis for $H\pi$, $\mathbf{a_1}, \ldots, \mathbf{a_r} \in \mathbb{Z}^m$, $0 \leq r \leq \infty$, $\mathbf{b_1}, \ldots, \mathbf{b_s} \in \mathbb{Z}^m$ is an abelian-basis for $L_H = H \cap \mathbb{Z}^m$, and $0 \leq s \leq m$.

Proposition (Moldavanski)

The groups $F_n \times \mathbb{Z}^m$, $n \geq 2$, $m \geq 1$, are not Howson.

Question

Are them intersection-saturated?... ... no... but collectively yes ...

Theorem (Delgado–Roy–V. '22)

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The groups $F_n \times \mathbb{Z}^m$, $n \geq 2$, $m \geq 1$, are not Howson.

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Are them intersection-saturated?... ... no... but collectively yes ...

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- *The set of configs realizable in $\mathbb{F}_n \times \mathbb{Z}^m$ increases strictly with m;*
- *Every configuration is realizable in $\mathbb{F}_n \times \mathbb{Z}^m$ for $m \gg 0$.*

Free-times-free-abelian groups

Proposition (Delgado–V. '13)

Every subgroup $H \leqslant \mathbb{G}$ admits a (computable) basis

$$H = \langle u_1 t^{\mathbf{a_1}}, u_2 t^{\mathbf{a_2}}, \ldots, u_r t^{\mathbf{a_r}}; t^{\mathbf{b_1}}, \ldots, t^{\mathbf{b_s}} \rangle,$$

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Theorem (Delgado–V. '13)

There is an algorithm which, on input (a set of generators for) $H, K \leqslant_{fg} \mathbb{G}$, decides whether $H \cap K$ is f.g. and, if so, computes a basis for it.

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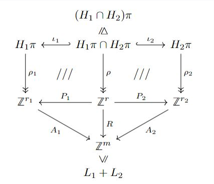
Given (basis for) subgroups $H_1, H_2 \leqslant_{fg} \mathbb{G} = \mathbb{F}_n \times \mathbb{Z}^m$, consider



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So, $H_1 \cap H_2$ is f.g. $\Leftrightarrow \begin{cases} r = 0, 1 \text{ or} \\ r \geq 2 \text{ and } (H_1 \cap H_2)\pi \leqslant_{fi} H_1\pi \cap H_2\pi. \end{cases}$ □

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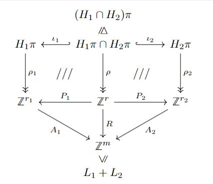

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Example

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Again, not true without the hypothesis $\min(r', r'') \neq 1$.

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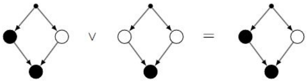
Outline

Positive results

Definition

Define the join of two k-configurations χ and χ' as

$$\begin{aligned} \chi \vee \chi' : \mathcal{P}([k]) \setminus \{\varnothing\} &\to \{0, 1\} \\ I &\mapsto \begin{cases} 0 & \text{if } (I)\chi = (I)\chi' = 0, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$



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Let χ' (resp. χ'') be k-config. realized by $H_1', \ldots, H_k' \leqslant \mathbb{G}' = \mathbb{F}_{n'} \times \mathbb{Z}^{m'}$ (resp. $H_1'', \ldots, H_k'' \leqslant \mathbb{G}'' = \mathbb{F}_{n''} \times \mathbb{Z}^{m''}$) with $r_I' = \operatorname{rk}\left(\bigcap_{i \in I} H_i' \pi\right) \neq 1$ (resp. $r_I'' \neq 1$) $\forall\, I \subseteq [k]$ with $|I| \geq 2$. Then, $\chi' \vee \chi''$ is realizable in $\mathbb{G}' \circledast \mathbb{G}'' = \mathbb{F}_{n'+n''} \times \mathbb{Z}^{m'+m''}$ by $H_1 = \langle H_1', H_1'' \rangle, \ldots, H_k = \langle H_k', H_k'' \rangle$, again satisfying $r_I \neq 1$ $\forall\, I \subseteq [k]$ with $|I| \geq 2$.

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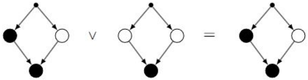

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Positive results

Proposition

The k-config. $\chi_{[k]}$ is realizable in $\mathbb{F}_n \times \mathbb{Z}^{k-1}$.

(Sketch of proof)

$$
\begin{aligned}
H_1 &= \langle x, y; t^{e_2}, \ldots, t^{e_{k-1}} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^{k-1}, \\
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\end{aligned}
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Corollary

Any almost-zero k-config. χ_{I_0} is realizable in $\mathbb{F}_n \times \mathbb{Z}^{|I_0|-1}$ by subgroups $H_1, \ldots, H_k$ further satisfying $\operatorname{rk}\left(\bigcap_{i \in I} H_i \pi\right) \neq 1$, for every $\emptyset \neq I \subseteq [k]$.

Positive results

Proposition

The k-config. $\chi_{[k]}$ is realizable in $\mathbb{F}_n \times \mathbb{Z}^{k-1}$.

(Sketch of proof)

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Theorem (Delgado–Roy–V. '22)

Every k-configuration $\chi_{\mathcal{I}}$ is realizable in $\mathbb{F}_n \times \mathbb{Z}^m$, for $n \geq 2$ and $m \geq \sum_{I \in \mathcal{I}} (|I| - 1)$.

(proof)

- *Decompose* $\chi_{\mathcal{I}} = \chi_{I_1} \vee \cdots \vee \chi_{I_r}$, *where* $\mathcal{I} = \{I_1, \ldots, I_r\}$;
- *realize each* χ_{I_j} *in* $\mathbb{F}_2 \times \mathbb{Z}^{|I_j|-1}$, $j = 1, \ldots, r$;
- *put together in a strongly complementary way.* □

Example

Consider $\chi = \chi_{\mathcal{I}}$, *where* $\mathcal{I} = \{\{1\}, \{2,3\}, \{1,3,4\}, \{2,3,4\}\}$. *Let us realize it in* $\mathbb{F}_2 \times \mathbb{Z}^m$ *for* $m = 0 + 1 + 2 + 2 = 5$. *Decomposing* χ, *we have*

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Positive results

Example (cont.)

In $\mathbb{F}_2 = \langle x, y \mid - \rangle$ take the freely independent words $u_j = y^{-j}xy^j \in \mathbb{F}_2$, $j \in \mathbb{Z}$. Let $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}, \mathbf{e_5}\}$ be the canonical basis for $\mathbb{Z}^5$. Realize:

- $\chi_{\{1\}}$ as $H_1' = \langle \ldots, u_{-2}, u_{-1} \rangle$, $H_2' = \{1\}$, $H_3' = \{1\}$, $H_4' = \{1\}$, all inside $G' = \langle \ldots, u_{-2}, u_{-1}; - \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^5$;
- $\chi_{\{2,3\}}$ as $H_1'' = \{1\}$, $H_2'' = \langle u_0, u_1 \rangle$, $H_3'' = \langle u_0, u_1 t^{\mathbf{e_1}} \rangle$, $H_4'' = \{1\}$, all inside $G'' = \langle u_0, u_1; t^{\mathbf{e_1}} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^5$;
- $\chi_{\{1,3,4\}}$ as $H_1''' = \langle u_2, u_3; t^{\mathbf{e_3}} \rangle$, $H_2''' = \{1\}$, $H_3''' = \langle u_2, u_3; t^{\mathbf{e_2}} \rangle$, $H_4''' = \langle u_2, u_3 t^{\mathbf{e_2}}; t^{\mathbf{e_3} - \mathbf{e_2}} \rangle$, all inside $G''' = \langle u_2, u_3; t^{\mathbf{e_2}}, t^{\mathbf{e_3}} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^5$;
- $\chi_{\{2,3,4\}}$ as $H_1'''' = \{1\}$, $H_2'''' = \langle u_4, u_5; t^{\mathbf{e_5}} \rangle$, $H_3'''' = \langle u_4, u_5; t^{\mathbf{e_4}} \rangle$, $H_4'''' = \langle u_4, u_5 t^{\mathbf{e_4}}; t^{\mathbf{e_5} - \mathbf{e_4}} \rangle$, all inside $G'''' = \langle u_4, u_5; t^{\mathbf{e_4}}, t^{\mathbf{e_5}} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^5$.

And note that $\operatorname{rk}\left(\bigcap_{i \in I} H_i' \pi\right) \neq 1$, $\operatorname{rk}\left(\bigcap_{i \in I} H_i'' \pi\right) \neq 1$, $\operatorname{rk}\left(\bigcap_{i \in I} H_i''' \pi\right) \neq 1$, *and* $\operatorname{rk}\left(\bigcap_{i \in I} H_i'''' \pi\right) \neq 1$. *Therefore, we can realize χ by the following subgroups*

Positive results

Example (cont.)

In $\mathbb{F}_2 = \langle x, y \mid - \rangle$ *take the freely independent words* $u_j = y^{-j} x y^j \in \mathbb{F}_2$, $j \in \mathbb{Z}$. *Let* $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}, \mathbf{e_5}\}$ *be the canonical basis for* $\mathbb{Z}^5$. *Realize:*

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And note that $\operatorname{rk}\left(\bigcap_{i\in I} H_i'\pi\right) \neq 1$, $\operatorname{rk}\left(\bigcap_{i\in I} H_i''\pi\right) \neq 1$, $\operatorname{rk}\left(\bigcap_{i\in I} H_i'''\pi\right) \neq 1$, *and* $\operatorname{rk}\left(\bigcap_{i\in I} H_i''''\pi\right) \neq 1$. *Therefore, we can realize* χ *by the following subgroups*

Positive results

Example (cont.)

$$\begin{aligned} H_1 &= \langle \ldots, u_{-2}, u_{-1}, u_2, u_3; t^{\mathbf{e}_3} \rangle, \\ H_2 &= \langle u_0, u_1, u_4, u_5; t^{\mathbf{e}_5} \rangle, \\ H_3 &= \langle u_0, u_1 t^{\mathbf{e}_1}, u_2, u_3, u_4, u_5; t^{\mathbf{e}_2}, t^{\mathbf{e}_4} \rangle, \\ H_4 &= \langle u_2, u_3 t^{\mathbf{e}_2}, u_4, u_5 t^{\mathbf{e}_4}; t^{\mathbf{e}_3 - \mathbf{e}_2}, t^{\mathbf{e}_5 - \mathbf{e}_4} \rangle. \end{aligned}$$

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Corollary

$\mathbb{F}_2 \times \left(\bigoplus_{\aleph_0} \mathbb{Z} \right)$ *is intersection-saturated.*

Theorem (Delgado–Roy–V. '22)

There exist finitely presented intersection-saturated groups.

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(Proof 1)

- *Consider Thomson's group F;*
- *it is well know to be finitely presented and to contain $\oplus_{\aleph_0} \mathbb{Z}$;*
- *therefore, $\mathbb{F}_2 \times F$ is intersection-saturated.* □
- *(Need to take $\mathbb{F}_2 \times$ because F does not contain $\mathbb{F}_2$.)*

(Proof 2)

- *Consider $G = \left(\oplus_{\aleph_0} \mathbb{Z} \right) \rtimes_\alpha \mathbb{Z}$, where α is the automorphism given by right translation of generators;*
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An obstruction

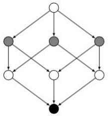
Lemma

Let $H_1, \ldots, H_k \leqslant \mathbb{G} = \mathbb{F}_n \times \mathbb{Z}^m$. Suppose that, for $\emptyset \neq I, J \subseteq [k]$, H_I and H_J are f.g. whereas $H_{I \cup J} = H_I \cap H_J$ is not. Then, $\exists i \in I$, $\exists j \in J$ s.t. $L_i = H_i \cap \mathbb{Z}^m$ and $L_j = H_j \cap \mathbb{Z}^m$ both have rank strictly smaller than m.

Proposition

Let χ be a k-config. and $\emptyset \neq I_1, \ldots, I_r \subseteq [k]$ be $r \geq 2$ subsets s.t. $\forall j \in [r]$, $(I_1 \cup \cdots \cup \widehat{I_j} \cup \cdots \cup I_r)\chi = \mathbf{0}$, but $(I_1 \cup \cdots \cup I_r)\chi = \mathbf{1}$. Then χ is not realizable in $\mathbb{F}_n \times \mathbb{Z}^{r-2}$.

Corollary

The 3-configurations  *are not realizable in $\mathbb{F}_n \times \mathbb{Z}$.*

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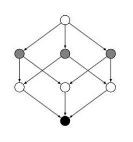
The 3-configurations  *are not realizable in $\mathbb{F}_n \times \mathbb{Z}$.*

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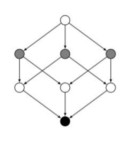
The 3-configurations  *are not realizable in $\mathbb{F}_n \times \mathbb{Z}$.*

An obstruction

Proposition

The k-configuration $\chi_{[k]}$ is realizable in $\mathbb{F}_n \times \mathbb{Z}^{k-1}$, but not in $\mathbb{F}_n \times \mathbb{Z}^{k-2}$.
Hence, the set of configurations realizable in $\mathbb{F}_n \times \mathbb{Z}^m$ increases strictly with m.

Outline

More on configurations

Definition

Let χ be a k-config. and let $i \in [k]$. Its restriction to $\widehat{i} = [k] \setminus \{i\}$ is the $(k-1)$-configuration

$$\begin{aligned} \chi_{|\widehat{i}} \colon \mathcal{P}([k] \setminus \{i\}) \setminus \{\varnothing\} &\to \{0, 1\} \\ I &\mapsto (I)\chi\,. \end{aligned}$$

Definition

Given two k-configurations χ, χ' and $\delta \in \{0, 1\}$, we define

$$\begin{aligned} \chi \boxplus_\delta \chi' \colon \mathcal{P}([k+1]) \setminus \{\varnothing\} &\to \{0, 1\} \\ I &\mapsto \begin{cases} (I)\chi & \text{if } k+1 \notin I, \\ (I \setminus \{k+1\})\chi' & \text{if } \{k+1\} \subsetneq I, \\ \delta & \text{if } \{k+1\} = I, \end{cases} \end{aligned}$$

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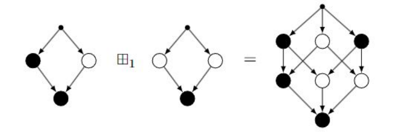
More on cofigurations

Example



Definition

Let χ be a k-configuration, and $i \in [k]$. The index i is said to be 0-monochromatic (in χ) if $(I)\chi = 0$ $\forall I \subseteq [k]$ containing i; i.e., if $\chi = \chi_{|\hat{i}} \boxplus_0 \mathbf{0}$. Similarly, the index i is said to be 1-monochromatic (in χ) if $\chi = \chi_{|\hat{i}} \boxplus_1 \mathbf{1}$.

Lemma

If a k-configuration χ is realizable in $\mathbb{F}_n$ with $n \geq 2$, then the $(k+1)$-configurations $\chi \boxplus_0 \mathbf{0}$, $\chi \boxplus_1 \mathbf{1}$, $\chi \boxplus_0 \chi$, and $\chi \boxplus_1 \chi$ are also realizable in $\mathbb{F}_n$.

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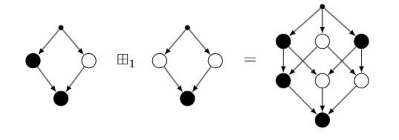

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If a k-configuration χ is realizable in $\mathbb{F}_n$ with $n \geq 2$, then the $(k+1)$-configurations $\chi \boxplus_0 \mathbf{0}$, $\chi \boxplus_1 \mathbf{1}$, $\chi \boxplus_0 \chi$, and $\chi \boxplus_1 \chi$ are also realizable in $\mathbb{F}_n$.

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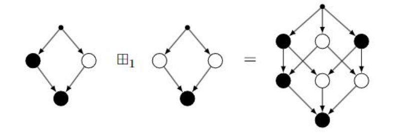

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If a k-configuration χ is realizable in $\mathbb{F}_n$ with $n \geq 2$, then the $(k+1)$-configurations $\chi \boxplus_0 \mathbf{0}$, $\chi \boxplus_1 \mathbf{1}$, $\chi \boxplus_0 \chi$, and $\chi \boxplus_1 \chi$ are also realizable in $\mathbb{F}_n$.

Characterization for the free case

(Proof)

Let $\mathbb{F}_2 * \mathbb{F}_{\aleph_0} \simeq W * U = \langle w_1, w_2, \ldots \rangle * \langle u, v \rangle \leqslant \mathbb{F}_n$, and take $H_1, \ldots, H_k \leqslant W \leqslant \mathbb{F}_n$ realizing χ. Now, in order to realize:

- $\chi \boxplus_0 \mathbf{0}$, take $\widetilde{H}_1 = H_1, \ldots, \widetilde{H}_k = H_k$, and $\widetilde{H}_{k+1} = \{1\}$;
- $\chi \boxplus_1 \mathbf{1}$, take $\widetilde{H}_1 = H_1 * \langle u, v \rangle, \ldots, \widetilde{H}_k = H_k * \langle u, v \rangle$ and $\widetilde{H}_{k+1} = \ll u \gg_U$: $\widetilde{H}_1, \ldots, \widetilde{H}_k$ realize $\chi \vee \mathbf{0} = \chi$ and, for every $i \neq k+1$, $\widetilde{H}_{k+1} \cap \widetilde{H}_i = \widetilde{H}_{k+1}$ which is non-f.g.;
- $\chi \boxplus_0 \chi$, take $\widetilde{H}_1 = H_1, \ldots, \widetilde{H}_k = H_k$, and $\widetilde{H}_{k+1} = \mathbb{F}_n$;
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Definition

A k-configuration χ is said to be Howson if, for every $\emptyset \neq I, J \subseteq [k]$, $(I)\chi = (J)\chi = 0 \;\Rightarrow\; (I \cup J)\chi = 0$.

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A k-configuration χ *is said to be Howson if, for every* $\emptyset \neq I, J \subseteq [k]$, $(I)\chi = (J)\chi = 0 \Rightarrow (I \cup J)\chi = 0$.

Characterization for the free case

(Proof)

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Theorem (Delgado–Roy–V., '22)

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For $\Leftarrow$, we will do induction on the cardinal of the support of χ, say s (regardless of its size k).

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Is there an algorithm which, on input m and χ, decides whether χ is realizable in $\mathbb{F}_n \times \mathbb{Z}^m$ (and, in the affirmative case, computes such a realization)?

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GRÀCIES

THANKS