

# On the difficulty of inverting automorphisms of free groups

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Groups and Geometry in the South East

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# Outline

- 1 Motivation
- 2 Main definition
- 3 Free groups
- 4 Lower bounds: a good enough example
- 5 Upper bounds: outer space
- 6 The special case of rank 2

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# Motivation

(Joint work with P. Silva and M. Ladra.)

Find a group  $G$  where  $\cdot$  is “easy” but  $()^{-1}$  is “difficult”.

Natural candidate:  $\text{Aut}(F_n)$ , where  $F_r = \langle a_1, \dots, a_r \mid \rangle$ .

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# Motivation

(composing)

$$F_3 = \langle a, b, c \mid \rangle.$$

$$\begin{array}{lcl} \phi: F_3 & \rightarrow & F_3 \\ a & \mapsto & ab \\ b & \mapsto & ab^2c \\ c & \mapsto & bc^2 \end{array} \qquad \begin{array}{lcl} \psi: F_3 & \rightarrow & F_3 \\ a & \mapsto & bc^{-1} \\ b & \mapsto & a^{-1}bc \\ c & \mapsto & c^{-1}. \end{array}$$

$$\begin{array}{lcl} \phi\psi: F_3 & \rightarrow & F_3 \\ a & \mapsto & bc^{-1}a^{-1}bc \\ b & \mapsto & bc^{-1}a^{-1}bca^{-1}b \\ c & \mapsto & a^{-1}bc^{-1}. \end{array}$$

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(inverting)

$$F_3 = \langle a, b, c \mid \rangle.$$

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$$F_5 = \langle a, b, c, d, e \mid \rangle.$$

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In this talk...

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- we see that inverting in  $\text{Aut}(F_r)$  is not that bad (only “polynomially hard”).
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# Main definition

## Definition

Let  $G$  be a group with a finite set of generators  $A = \{a_1, \dots, a_r\}$ . We have the *word metric*: for  $g \in G$ ,

$$|g| = \min\{n \mid g = a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n}\}.$$

## Definition

For  $\theta \in \text{Aut}(G)$ , note  $\theta$  is determined by  $a_1\theta, \dots, a_r\theta$  and define

$$\|\theta\|_1 = |a_1\theta| + \cdots + |a_r\theta|,$$

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## Observation

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$$\alpha_A(n) = \max\{\|\theta^{-1}\|_1 \mid \theta \in \text{Aut}(G), \|\theta\|_1 \leq n\}.$$

Clearly,  $\alpha_A(n) \leq \alpha_A(n+1)$ .

*The bigger is  $\alpha_A$ , the more "difficult" will be to invert automorphisms of  $G$  (with respect to the given set of generators  $A$ ).*

## Question

*Determine the asymptotic growth of the function  $\alpha_G$ .*

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# Independence from A

## Proposition

Let  $G$  be a group and  $A = \{a_1, \dots, a_r\}$  and  $B = \{b_1, \dots, b_s\}$  be two finite sets of generators. Then,  $\exists C > 0$  s. t.  $\forall \theta \in \text{Aut}(G)$

$$\frac{1}{C} \|\theta\|_B \leq \|\theta\|_A \leq C \|\theta\|_B$$

*Proof.* Take  $|b_i|_A \leq M$ ,  $|a_j|_B \leq N$  and let  $C = MNrs$ .

$$\begin{aligned} \|\theta\|_B &= |b_1\theta|_B + \dots + |b_s\theta|_B \\ &\leq |b_1\theta|_A N + \dots + |b_s\theta|_A N \\ &\leq N(|b_1|_A \|\theta\|_A + \dots + |b_s|_A \|\theta\|_A) \\ &\leq NMm \|\theta\|_A \leq C \|\theta\|_A. \end{aligned}$$

By symmetry,  $\|\theta\|_A \leq C \|\theta\|_B$ , so  $\frac{1}{C} \|\theta\|_B \leq \|\theta\|_A$ .

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# Independence from $A$

## Corollary

$$\frac{1}{C} \cdot \alpha_B\left(\frac{n}{C}\right) \leq \alpha_A(n) \leq C \cdot \alpha_B(Cn).$$

### *Proof.*

$$\begin{aligned} \alpha_A(n) &= \max\{\|\theta^{-1}\|_A \mid \theta \in \text{Aut}(G), \|\theta\|_A \leq n\} \\ &\leq \max\{\|\theta^{-1}\|_A \mid \theta \in \text{Aut}(G), \|\theta\|_B \leq Cn\} \\ &\leq \max\{C\|\theta^{-1}\|_B \mid \theta \in \text{Aut}(G), \|\theta\|_B \leq Cn\} \\ &= C \cdot \max\{\|\theta^{-1}\|_B \mid \theta \in \text{Aut}(G), \|\theta\|_B \leq Cn\} \\ &= C \cdot \alpha_B(Cn). \end{aligned}$$

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# Independence from $A$

*Hence,  $\alpha_A(n)$  is independent from  $A$  (up to a multiplicative constant in the domain and in the range).*

*Denote it by  $\alpha_G(n)$ .*

## Question

*Are there groups  $G$  with  $\alpha_G(n)$  linear ? quadratic? ... exponential?*

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## Definition

For  $\Theta \in \text{Out}(G)$ , define

$$\|\Theta\|_1 = \min\{\|\theta\|_1 \mid \theta \in \Theta\},$$

$$\|\Theta\|_\infty = \min\{\|\theta\|_\infty \mid \theta \in \Theta\},$$

## Definition

For a finitely generated group  $G$ ,

$$\beta(n) = \max\{\|\Theta^{-1}\|_1 \mid \Theta \in \text{Out}(G), \|\Theta\|_1 \leq n\}.$$

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# Free group case

For the rest of the talk,  $G = F_r = \langle a_1, \dots, a_r \mid \rangle$ .

For every  $w \in F_r$ ,  $|w|$  is its free length.

$$|vw| \leq |v| + |w|,$$

$$|w^n| \leq |n||w|.$$

For  $\theta \in \text{Aut}(F_r)$  and  $\Theta \in \text{Out}(F_r)$ ,

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For  $\theta \in \text{Aut}(F_r)$  and  $\Theta \in \text{Out}(F_r)$ ,

$$\|\theta\|_1 = |a_1\theta| + \dots + |a_r\theta|,$$

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# Free group case

For the rest of the talk,  $G = F_r = \langle a_1, \dots, a_r \mid \rangle$ .

For every  $w \in F_r$ ,  $|w|$  is its *free length*.

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# Main results

## Theorem

*For rank  $r = 2$  we have*

- (i) *for  $n \geq 4$ ,  $\alpha_2(n) \leq \frac{(n-1)^2}{2}$ ,*
- (ii) *for  $n \geq n_0$ ,  $\frac{n^2}{16} \leq \alpha_2(n)$ ,*
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*For  $r \geq 3$  there exist  $K = K(r)$ ,  $K' = K'(r)$ , and  $M = M(r)$  such that, for  $n \geq 1$ ,*

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# A lower bound for $\beta_r$

## Theorem

For  $r \geq 2$ , and  $n \geq n_0$ , we have  $\frac{1}{2^{r-1}} n^{r-1} \leq \beta_r(n)$ .

**Proof:** For  $r \geq 2$  and  $n \geq 1$ , consider

$$\begin{array}{ll}
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A straightforward calculation shows that  $\|\psi_{r,n}\|_1 = (r-1)n + r$ , and  $\|\psi_{r,n}^{-1}\|_1 = n^{r-1} + 2n^{r-2} + \dots + (r-1)n + r \geq n^{r-1}$ .

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Hence, for  $n \geq r$ ,

$$\beta_r(n) \geq \beta_r((r-1)n+r) \geq n^{r-1}.$$

Now, for  $n$  big enough, take the closest multiple of  $r$  below,

$$n \geq rm > n - r,$$

and

$$\beta_r(n) \geq \beta_r(rm) \geq m^{r-1} > \left(\frac{n-r}{r}\right)^{r-1} = \left(\frac{n}{r} - 1\right)^{r-1} \geq \frac{1}{2r^{r-1}} n^{r-1}. \quad \square$$

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For  $r \geq 2$ , and  $n \geq n_0$ , we have  $\frac{(r-1)^{r-1}}{2r^{2r-1}} n^r \leq \alpha_r(n)$ .

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# Outer space

To prove the upper bound

$$(ii) \beta_r(n) \leq Kn^M,$$

we'll need to use the recently discovered **metric** in the **outer space**  $\mathcal{X}_r$ .

## Definition

- By **graf**  $\Gamma$  we mean a finite, connected graph of rank  $r$ , with no vertices of degree 1 or 2.
- A **metric** on  $\Gamma$  is a map  $\ell: E\Gamma \rightarrow [0, 1]$  such that  $\sum_{e \in E\Gamma} \ell(e) = 1$ , and  $\{e \in E\Gamma \mid \ell(e) = 0\}$  is a forest.
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The *outer space*  $\mathcal{X}_r$  is

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There is a natural action of  $\text{Aut}(F_r)$  on  $\mathcal{X}_r$ , given by

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# Metric on $\mathcal{X}_r$

## Definition

Let  $x, x' \in \mathcal{X}_r$ ,  $x = (\Gamma, f, \ell)$ ,  $x' = (\Gamma', f', \ell')$ . A *difference of markings* is a map  $\alpha: \Gamma \rightarrow \Gamma'$ , which is *linear over edges* and  $f\alpha \simeq f'$ .

For such an  $\alpha$ , define  $\sigma(\alpha)$  to be its *maximum slope over edges*.

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$\mathcal{X}_r$  admits the following “metric”:

$$d(x, x') = \min\{\log(\sigma(\alpha)) \mid \alpha \text{ diff. markings}\}.$$

*This minimum is achieved by Arzela-Ascoli's theorem.*

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Let  $x, x' \in \mathcal{X}_r$ ,  $x = (\Gamma, f, \ell)$ ,  $x' = (\Gamma', f', \ell')$ . A *difference of markings* is a map  $\alpha: \Gamma \rightarrow \Gamma'$ , which is *linear over edges* and  $f\alpha \simeq f'$ .

For such an  $\alpha$ , define  $\sigma(\alpha)$  to be its *maximum slope over edges*.

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$\mathcal{X}_r$  admits the following “metric”:

$$d(x, x') = \min\{\log(\sigma(\alpha)) \mid \alpha \text{ diff. markings}\}.$$

This minimum is achieved by Arzela-Ascoli's theorem.

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- (i)  $d(x, y) \geq 0$ , and  $= 0 \Leftrightarrow x = y$ .
- (ii)  $d(x, z) \leq d(x, y) + d(y, z)$ .
- (iii)  $Out(F_r)$  acts by isometries, i.e.  $d(\phi \cdot x, \phi \cdot y) = d(x, y)$ .
- (iv) But...  $d(x, y) \neq d(y, x)$  in general.

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For  $\epsilon > 0$ , the  $\epsilon$ -thick part of  $\mathcal{X}_r$  is

$$\mathcal{X}_r(\epsilon) = \{(\Gamma, f, \ell) \in \mathcal{X}_r \mid \ell(p) \geq \epsilon \ \forall \text{ closed path } p \neq 1\}$$

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## Theorem (Bestvina-AlgomKfir)

For any  $\epsilon > 0$  there is constant  $M = M(r, \epsilon)$  such that for all  $x, y \in \mathcal{X}_r(\epsilon)$ ,

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## Corollary

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# Proof

*Remind*  $\beta_r(n) = \max\{\|\Theta^{-1}\|_1 \mid \theta \in \text{Aut } F_r, \|\Theta\|_1 \leq n\}$ .

**Proof.** Given  $\theta \in \Theta \in \text{Out}(F_r)$ , consider  $x = (R_r, \text{id}, \ell_0) \in \mathcal{X}_r$ , and  $\theta \cdot x = (R_r, \theta, \ell_0) \in \mathcal{X}_r$ , where  $\ell_0$  is the uniform metric.

$$\begin{aligned}
 d(x, \theta \cdot x) &= \min\{\log(\sigma(\alpha)) \mid \alpha \text{ diff. markings}\} \\
 &= \log\left(\min\{\sigma(\theta\gamma_w\gamma_p) \mid w \in F_r, p = \text{"half petal"}\}\right) \\
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Now, using Bestvina-Algom-Kfir theorem,

$$\log(\|\Theta^{-1}\|_1) \sim d(x, \theta^{-1} \cdot x) = d(\theta \cdot x, x) \leq Md(x, \theta \cdot x) \sim M \log(\|\Theta\|_1).$$

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# Outline

- 1 Motivation
- 2 Main definition
- 3 Free groups
- 4 Lower bounds: a good enough example
- 5 Upper bounds: outer space
- 6 The special case of rank 2**

# The rank 2 case

These functions for  $\text{Aut}(F_2)$  are much easier to understand due to the following technical lemmas.

## Lemma

*Let  $\varphi \in \text{Aut}(F_2)$  be positive. Then  $\varphi^{-1}$  is cyclically reduced and  $\|\varphi^{-1}\|_1 = \|\varphi\|_1$ .*

## Lemma

*For every  $\theta \in \text{Aut}(F_2)$ , there exist two letter permuting autos  $\psi_1, \psi_2 \in \text{Aut}(F_2)$ , a positive one  $\varphi \in \text{Aut}^+(F_2)$ , and an element  $g \in F_2$ , such that  $\theta = \psi_1 \varphi \psi_2 \lambda_g$  and  $\|\varphi\|_1 + 2|g| \leq \|\theta\|_1$ .*

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# The rank 2 case: $\beta_2$

## Theorem

*For every  $\theta \in \text{Aut}(F_2)$ ,  $||[\theta^{-1}]||_1 = ||[\theta]||_1$ . Hence,  $\beta_2(n) = n$ .*

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On the other hand,

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*For rank  $r = 2$  we have*

- (i) *for  $n \geq 4$ ,  $\alpha_2(n) \leq \frac{(n-1)^2}{2}$ ,*
- (ii) *for  $n \geq n_0$ ,  $\frac{n^2}{16} \leq \alpha_2(n)$ ,*
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*For  $r \geq 3$  there exist  $K = K(r)$ ,  $K' = K'(r)$ , and  $M = M(r)$  such that, for  $n \geq 1$ ,*

- (i)  *$Kn^r \leq \alpha_r(n)$ ,*
- (ii)  *$Kn^{r-1} \leq \beta_r(n) \leq K'n^M$ .*

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- (iii) for  $n \geq 1$ ,  $\beta_2(n) = n$ .

## Theorem

For  $r \geq 3$  there exist  $K = K(r)$ ,  $K' = K'(r)$ , and  $M = M(r)$  such that, for  $n \geq 1$ ,

- (i)  $Kn^r \leq \alpha_r(n)$ ,
- (ii)  $Kn^{r-1} \leq \beta_r(n) \leq K'n^M$ .

1. Motivation  
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2. Main definition  
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3. Free groups  
○○

3. Lower bounds  
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4. Upper bounds  
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5. The special case of rank 2  
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# THANKS