

Computing fixed closures in free groups

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Outline

- 1 Some history
- 2 Algorithmic results
- 3 Needed tools
- 4 The proof

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- 2 Algorithmic results
- 3 Needed tools
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Notation

- $A = \{a_1, \dots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \dots, a_n, a_n^{-1}\}$.
- F_n is the free group on A .
- $\text{Aut}(F_n) \subseteq \text{Mono}(F_n) \subseteq \text{End}(F_n)$.
- I let endomorphisms $\phi: F_n \rightarrow F_n$ act on the right, $x \mapsto x\phi$.
- $\text{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leq F_n$.
- If $S \subseteq \text{End}(F_n)$ then

$$\text{Fix}(S) = \{x \in F_n \mid x\phi = x \forall \phi \in S\} = \bigcap_{\phi \in S} \text{Fix}(\phi) \leq F_n.$$

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Fixed subgroups are complicated

$$\begin{aligned}\phi: F_3 &\rightarrow F_3 \\ a &\mapsto a \\ b &\mapsto ba \\ c &\mapsto ca^2\end{aligned}$$

$$\text{Fix } \phi = \langle a, bab^{-1}, cac^{-1} \rangle$$

$$\begin{aligned}\varphi: F_4 &\rightarrow F_4 \\ a &\mapsto dac \\ b &\mapsto c^{-1}a^{-1}d^{-1}ac \\ c &\mapsto c^{-1}a^{-1}b^{-1}ac \\ d &\mapsto c^{-1}a^{-1}bc\end{aligned}$$

$$\text{Fix } \varphi = \langle w \rangle, \text{ where...}$$

$$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdada c d c d b c d a^{-1}a^{-1}d^{-1}a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}.$$

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What is known about fixed subgroups ?

Theorem (Dyer-Scott, 75)

Let $G \leq \text{Aut}(F_n)$ be a finite group of automorphisms of F_n . Then, $\text{Fix}(G) \leq_{\text{ff}} F_n$; in particular, $r(\text{Fix}(G)) \leq n$.

Conjecture (Scott)

For every $\phi \in \text{Aut}(F_n)$, $r(\text{Fix}(\phi)) \leq n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in \text{Aut}(F_n)$. Then $r(\text{Fix}(\phi)) < \infty$.

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Train-tracks

Main result in this story:

Theorem (Bestvina-Handel, 88 (published 92))

Let $\phi \in \text{Aut}(F_n)$. Then $r(\text{Fix}(\phi)) \leq n$.

introducing the theory of train-tracks for graphs.

After Bestvina-Handel, live continues ...

Theorem (Imrich-Turner, 89)

Let $\phi \in \text{End}(F_n)$. Then $r(\text{Fix}(\phi)) \leq n$.

Theorem (Turner, 96)

Let $\phi \in \text{End}(F_n)$. If ϕ is not bijective then $r(\text{Fix}(\phi)) \leq n - 1$.

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Inertia

Definition

A subgroup $H \leq F_n$ is called *inert* if $r(H \cap K) \leq r(K)$ for every $K \leq F_n$.

Theorem (Dicks-V, 96)

Let $G \subseteq \text{Mon}(F_n)$ be an arbitrary set of monomorphisms of F_n . Then, $\text{Fix}(G)$ is inert; in particular, $r(\text{Fix}(G)) \leq n$.

Theorem (Bergman, 99)

Let $G \subseteq \text{End}(F_n)$ be an arbitrary set of endomorphisms of F_n . Then, $r(\text{Fix}(G)) \leq n$.

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The four families

Definition

A subgroup $H \leq F_n$ is said to be

- **1-auto-fixed** if $H = \text{Fix}(\phi)$ for some $\phi \in \text{Aut}(F_n)$,
- 1-endo-fixed if $H = \text{Fix}(\phi)$ for some $\phi \in \text{End}(F_n)$,
- auto-fixed if $H = \text{Fix}(S)$ for some $S \subseteq \text{Aut}(F_n)$,
- endo-fixed if $H = \text{Fix}(S)$ for some $S \subseteq \text{End}(F_n)$,

Easy to see that 1-mono-fixed = 1-auto-fixed.

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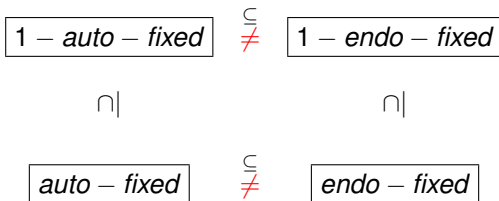
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Relations between them

$$\boxed{1 - \text{auto} - \text{fixed}} \subseteq \boxed{1 - \text{endo} - \text{fixed}}$$
$$\cap | \qquad \qquad \qquad \cap |$$
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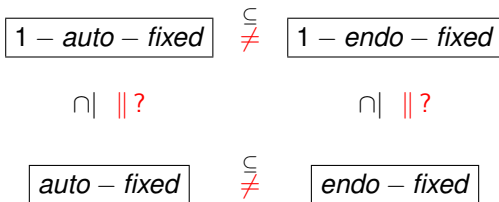
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Example (Martino-V., 03; Ciobanu-Dicks, 06)

Let $F_3 = \langle a, b, c \rangle$ and $H = \langle b, \text{cabcab}^{-1}c^{-1} \rangle \leq F_3$. Then, $H = \text{Fix}(a \mapsto 1, b \mapsto b, c \mapsto \text{cabcab}^{-1}c^{-1})$, but H is **NOT** the fixed subgroup of any set of automorphism of F_3 .

Relations between them



Theorem (Martino-V., 00)

Let $S \subseteq \text{End}(F_n)$. Then, $\exists \phi \in \langle S \rangle$ such that $\text{Fix}(S) \leq_{\text{ff}} \text{Fix}(\phi)$.

But... free factors of 1-endo-fixed (1-auto-fixed) subgroups need not be even endo-fixed (auto-fixed).

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Computing fixed subgroups

Proposition (Turner, 86)

*There exists a **pseudo-algorithm** to compute fix of an endo.*

Easy but is **not** an algorithm...

Theorem (Maslakova, 03)

Fixed subgroups of automorphisms of F_n are computable.

Difficult but it **is** an algorithm!

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Deciding fixedness

In this talk, I'll solve the two dual problems:

Theorem

Given $H \leq_{\text{fg}} F_n$, one can algorithmically decide whether

- i) H is auto-fixed or not,*
- ii) H is endo-fixed or not,*

and in the affirmative case, find a finite family, $S = \{\phi_1, \dots, \phi_m\}$, of automorphisms (endomorphisms) of F_n such that $\text{Fix}(S) = H$.

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Given $H \leq_{\text{fg}} F_n$, one can algorithmically decide whether

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Definition

Given $H \leq_{\text{fg}} F_n$, we define the (*auto-* and *endo-*) *stabilizer of H* , respectively, as

$$\text{Aut}_H(F_n) = \{\phi \in \text{Aut}(F_n) \mid H \leq \text{Fix}(\phi)\} \leq \text{Aut}(F_n)$$

and

$$\text{End}_H(F_n) = \{\phi \in \text{End}(F_n) \mid H \leq \text{Fix}(\phi)\} \leq \text{End}(F_n)$$

Definition

Given $H \leq F_n$, we define the *auto-closure* and *endo-closure* of H as

$$a\text{-Cl}(H) = \text{Fix}(\text{Aut}_H(F_n)) \geq H$$

and

$$e\text{-Cl}(H) = \text{Fix}(\text{End}_H(F_n)) \geq H$$

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$$\text{Aut}_H(F_n) = \{\phi \in \text{Aut}(F_n) \mid H \leq \text{Fix}(\phi)\} \leq \text{Aut}(F_n)$$

and

$$\text{End}_H(F_n) = \{\phi \in \text{End}(F_n) \mid H \leq \text{Fix}(\phi)\} \leq \text{End}(F_n)$$

Definition

Given $H \leq F_n$, we define the *auto-closure* and *endo-closure* of H as

$$\text{a-Cl}(H) = \text{Fix}(\text{Aut}_H(F_n)) \geq H$$

and

$$\text{e-Cl}(H) = \text{Fix}(\text{End}_H(F_n)) \geq H$$

Main result

Theorem

For every $H \leq_{\text{fg}} F_n$, $a\text{-Cl}(H)$ and $e\text{-Cl}(H)$ are finitely generated and one can algorithmically compute bases for them.

Corollary

Auto-fixedness and endo-fixedness are decidable.

Observe that $e\text{-Cl}(H) \leq a\text{-Cl}(H)$ but, in general, they are not equal.

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Retracts

Definition

A subgroup $H \leq F_n$ is a **retract** if there exists a **retraction**, i.e. a morphism $\rho: F_n \rightarrow H$ which restricts to the identity of H .

Free factors are retracts, but there are more.

Observation

If $H \leq F_n$ is a retract then $r(H) \leq n$ (and, $r(H) = n \Leftrightarrow H = F_n$).

Observation (Turner)

It is algorithmically decidable whether a given $H \leq F_n$ is a retract or not.

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The stable image

Definition

Let $\phi \in \text{End}(F_n)$. The *stable image* of ϕ is $F_n\phi^\infty = \bigcap_{i=1}^{\infty} F_n\phi^i$.

Theorem (Imrich-Turner, 89)

For every endomorphism $\phi: F_n \rightarrow F_n$,

- i) $F_n\phi^\infty$ is ϕ -invariant,
- ii) the restriction $\phi: F_n\phi^\infty \rightarrow F_n\phi^\infty$ is an isomorphism,
- iii) $F_n\phi^\infty$ is a retract.

Example: For $\phi: F_2 \rightarrow F_2$, $a \mapsto a$, $b \mapsto b^2$, we have $F_2\phi = \langle a, b^2 \rangle$,
 $F_2\phi^2 = \langle a, b^4 \rangle$, $F_2\phi^3 = \langle a, b^8 \rangle$, ... So, $F_2\phi^\infty = \langle a \rangle \leq_{\text{ff}} F_2$.

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Stallings' graphs and intersections

Theorem (Stallings, 83)

For any free group $F_n = F(A)$, there is an effectively computable bijection

$$\{\text{f.g. subgroups of } F_n\} \longleftrightarrow \{\text{finite } A\text{-labeled core graphs}\}$$

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Algebraic extensions

Definition

An extension of subgroups $H \leq K \leq F_n$ is called *algebraic*, denoted $H \leq_{\text{alg}} K$, if H is not contained in any proper free factor of K . Write

$$\mathcal{AE}(H) = \{K \leq F_n \mid H \leq_{\text{alg}} K\}.$$

Theorem (Takahasi, 51)

If $H \leq_{\text{fg}} F_n$ then $\mathcal{AE}(H)$ is finite and computable (i.e. H has finitely many algebraic extensions, all of them are finitely generated, and bases are computable from H).

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Outline

- 1 Some history
- 2 Algorithmic results
- 3 Needed tools
- 4 The proof**

The automorphism case

Theorem (McCool)

Let $H \leq_{\text{fg}} F_n$. Then $\text{Aut}_H(F_n)$ is finitely generated (in fact, finitely presented) and a finite set of generators (and relations) is algorithmically computable from H .

Theorem

For every $H \leq_{\text{fg}} F_n$, $a\text{-Cl}(H)$ is finitely generated and algorithmically computable.

Proof. $a\text{-Cl}(H) = \text{Fix}(\text{Aut}_H(F_n))$
 $= \text{Fix}(\langle \phi_1, \dots, \phi_m \rangle)$
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A similar approach **does not** work because:

- we **don't know** how to compute fix subgroups of endomorphisms
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Example

Consider $F_3 = \langle a, b, c \rangle$, the element $d = ba[c^2, b]a^{-1}$, and the subgroup $H = \langle a, d \rangle \leq F_3$. Clearly, the morphisms

$$\begin{array}{ccc}
 \psi: F_3 & \rightarrow & F_3 & \quad & \phi: F_3 & \rightarrow & F_3 & \quad & \phi^n \psi: F_3 & \rightarrow & F_3 \\
 a & \mapsto & a & & a & \mapsto & a & & a & \mapsto & a \\
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satisfy $H \leq \text{Fix}(\phi^n \psi)$ for every $n \in \mathbb{Z}$.

With some computations, Ciobanu-Dicks-06 show that

$$\text{End}_H(F_3) = \{\text{Id}, \phi^n \psi \mid n \in \mathbb{Z}\}.$$

But, $\phi^m \psi \cdot \phi^n \psi = \phi^m \psi$. Hence, $\text{End}_H(F_3)$ is not finitely generated.

Furthermore, $a\text{-Cl}(H) = \text{Fix}(\text{Id}) = F_3$ and $e\text{-Cl}(H) = \text{Fix}(\psi) = H$.

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Theorem

For every $H \leq_{\text{fg}} F_n$, $e\text{-Cl}(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \dots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{\text{ret}}(H) = \{H_1, \dots, H_r\}$ ($1 \leq r \leq q$).
- Write the generators of H as words on the generators of each one of these H_i 's, $i = 1, \dots, r$.
- Compute bases for $a\text{-Cl}_{H_1}(H), \dots, a\text{-Cl}_{H_r}(H)$.
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$$a\text{-Cl}_{H_1}(H) \cap \cdots \cap a\text{-Cl}_{H_r}(H) = e\text{-Cl}(H).$$

Proof. Let us see that

$$\bigcap_{i=1}^r \bigcap_{\substack{\alpha \in \text{Aut}(H_i) \\ H \leq \text{Fix}(\alpha)}} \text{Fix}(\alpha) = \bigcap_{\substack{\beta \in \text{End}(F_n) \\ H \leq \text{Fix}(\beta)}} \text{Fix}(\beta).$$

- Take $\beta \in \text{End}(F_n)$ with $H \leq \text{Fix}(\beta)$.
- $\exists i = 1, \dots, r$ such that $H \leq_{\text{alg}} H_i \leq_{\text{ff}} F\beta^\infty \leq F$.
- Now, β restricts to an automorphism $\alpha: H_i \rightarrow H_i$.
- And, clearly, $H \leq \text{Fix}(\alpha) \leq \text{Fix}(\beta)$.
- Hence, we have " \leq ".

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- Take $H_i \in \mathcal{AE}_{ret}(H)$, and $\alpha \in \text{Aut}(H_i)$ with $H \leq \text{Fix}(\alpha)$.
- Let $\rho: F \rightarrow H_i$ be a retraction, and consider the endomorphism,
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THANKS