The conjugacy problem for some extensions of F_n, $\mathbb{Z}^m$, B_n and F.

Enric Ventura

Departament de Matemàtica Aplicada III
Universitat Politècnica de Catalunya

and

CRM-Montreal

Urbana Group Theory Seminar

October 28th, 2010.

- O. Bogopolski, A. Martino, O. Maslakova, E. Ventura, Free-by-cyclic groups have solvable conjugacy problem, *Bulletin of the London Mathematical Society*, **38**(5) (2006), 787–794.

- O. Bogopolski, A. Martino, E. Ventura, Orbit decidability and the conjugacy problem for some extensions of groups, *Transactions of the American Mathematical Society* **362** (2010), 2003–2036.

- V. Romanko'v, E. Ventura, Twisted conjugacy problem for endomorphisms of metabelian groups, *Algebra and Logic* **48**(2) (2009), 89–98.

- J. González-Meneses, E. Ventura, Twisted conjugacy in the braid group, *preprint*.

- J. Burillo, F. Matucci, E. Ventura, The conjugacy problem for extensions of Thompson's group, *preprint*.

- Z. Šunić, E. Ventura, The conjugacy problem in self-similar groups, *preprint*, arXiv:1010.1993v1, Oct. 2010.

Outline

1. The conjugacy problem for free-by-cyclic groups
2. The main theorem
3. The conjugacy problem for free-by-free groups
4. The conjugacy problem for (free abelian)-by-free groups
5. The conjugacy problem for Braid-by-free groups
6. The conjugacy problem for Thompson-by-free groups
7. The conjugacy problem for automata

Outline

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\varphi\colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta\colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u\colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\varphi\colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta\colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u\colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\varphi\colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta\colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u\colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\varphi\colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta\colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u\colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\varphi\colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta\colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u\colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\varphi \colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta \colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u \colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\varphi \colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta \colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u \colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\varphi \colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta \colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u \colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Step 1:

Find a problem you like

Conjugacy problem for free-by-cyclic groups

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \ \rangle$ be a free group on $\{x_1, \ldots, x_n\}$ ($n \geq 2$), and let $\varphi \in Aut(F_n)$. The free-by-cyclic group $F_n \rtimes_\varphi \mathbb{Z}$ is defined as

$$F_n \rtimes_\varphi \mathbb{Z} \quad = \langle x_1, \ldots, x_n, \ t \mid t^{-1} x_i t = x_i \varphi \rangle.$$

With $x_i t = t(x_i \varphi)$ and $x_i t^{-1} = t^{-1}(x_i \varphi^{-1})$, we can move all t's to the left and get the usual normal form for elements in $F_n \rtimes_\varphi \mathbb{Z}$:

$$t^r w, \quad \text{with } r \in \mathbb{Z}, w \in F_n.$$

Problem

Solve the conjugacy problem in $F_n \rtimes_\varphi \mathbb{Z}$.

Conjugacy problem for free-by-cyclic groups

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \, \rangle$ be a free group on $\{x_1, \ldots, x_n\}$ ($n \geq 2$), and let $\varphi \in Aut(F_n)$. The free-by-cyclic group $F_n \rtimes_\varphi \mathbb{Z}$ is defined as

$$F_n \rtimes_\varphi \mathbb{Z} \quad = \langle x_1, \ldots, x_n, \, t \mid t^{-1} x_i t = x_i \varphi \rangle.$$

With $x_i t = t(x_i \varphi)$ and $x_i t^{-1} = t^{-1}(x_i \varphi^{-1})$, we can move all t's to the left and get the usual normal form for elements in $F_n \rtimes_\varphi \mathbb{Z}$:

$$t^r w, \text{ with } r \in \mathbb{Z}, w \in F_n.$$

Problem

Solve the conjugacy problem in $F_n \rtimes_\varphi \mathbb{Z}$.

Conjugacy problem for free-by-cyclic groups

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \ \rangle$ be a free group on $\{x_1, \ldots, x_n\}$ ($n \geq 2$), and let $\varphi \in Aut(F_n)$. The free-by-cyclic group $F_n \rtimes_\varphi \mathbb{Z}$ is defined as

$$F_n \rtimes_\varphi \mathbb{Z} \quad = \langle x_1, \ldots, x_n, \, t \mid t^{-1} x_i t = x_i \varphi \rangle.$$

With $x_i t = t(x_i \varphi)$ and $x_i t^{-1} = t^{-1}(x_i \varphi^{-1})$, we can move all t's to the left and get the usual normal form for elements in $F_n \rtimes_\varphi \mathbb{Z}$:

$$t^r w, \text{ with } r \in \mathbb{Z}, w \in F_n.$$

Problem

Solve the conjugacy problem in $F_n \rtimes_\varphi \mathbb{Z}$.

Step 2:

Push the problem into your favorite territory

Converting it into a free group problem

Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$. Then,

$$\begin{aligned}(g^{-1}t^{-k})(t^r u)(t^k g) &= g^{-1}t^r(u\varphi^k)g \\ &= t^r(g\varphi^r)^{-1}(u\varphi^k)g.\end{aligned}$$

$t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.

Definition

For $\varphi \in End(G)$, two elements $u, v \in G$ are said to be *φ-twisted conjugated*, denoted $u \sim_\varphi v$, if $v = (g\varphi)^{-1}ug$ for some $g \in G$.

Definition

The *twisted conjugacy problem* for G, denoted $TCP(G)$: "Given $\varphi \in Aut(G)$ and $u, v \in G$ decide whether $u \sim_\varphi v$".

Converting it into a free group problem

Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$. Then,

$$\begin{aligned}(g^{-1}t^{-k})(t^r u)(t^k g) &= g^{-1}t^r(u\varphi^k)g \\ &= t^r(g\varphi^r)^{-1}(u\varphi^k)g.\end{aligned}$$

$t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.

Definition

For $\varphi \in End(G)$, two elements $u, v \in G$ are said to be *φ-twisted conjugated*, denoted $u \sim_\varphi v$, if $v = (g\varphi)^{-1}ug$ for some $g \in G$.

Definition

The *twisted conjugacy problem* for G, denoted $TCP(G)$: "Given $\varphi \in Aut(G)$ and $u, v \in G$ decide whether $u \sim_\varphi v$".

Converting it into a free group problem

Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$. Then,

$$\begin{aligned}(g^{-1}t^{-k})(t^r u)(t^k g) &= g^{-1}t^r(u\varphi^k)g \\ &= t^r(g\varphi^r)^{-1}(u\varphi^k)g.\end{aligned}$$

$t^r u$ and $t^s v$ conj. in M_φ $\iff$ *$r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.*

Definition

For $\varphi \in End(G)$, two elements $u, v \in G$ are said to be φ-twisted conjugated, denoted $u \sim_\varphi v$, if $v = (g\varphi)^{-1}ug$ for some $g \in G$.

Definition

The twisted conjugacy problem for G, denoted $TCP(G)$: "Given $\varphi \in Aut(G)$ and $u, v \in G$ decide whether $u \sim_\varphi v$".

Converting it into a free group problem

Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$. Then,

$$\begin{aligned}(g^{-1}t^{-k})(t^r u)(t^k g) &= g^{-1}t^r(u\varphi^k)g \\ &= t^r(g\varphi^r)^{-1}(u\varphi^k)g.\end{aligned}$$

$t^r u$ and $t^s v$ conj. in M_φ $\iff$ *$r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.*

Definition

For $\varphi \in End(G)$, two elements $u, v \in G$ are said to be φ-twisted conjugated, denoted $u \sim_\varphi v$, if $v = (g\varphi)^{-1}ug$ for some $g \in G$.

Definition

The twisted conjugacy problem for G, denoted $TCP(G)$: "Given $\varphi \in Aut(G)$ and $u, v \in G$ decide whether $u \sim_\varphi v$".

Converting it into a free group problem

Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$. Then,

$$\begin{aligned}(g^{-1}t^{-k})(t^r u)(t^k g) &= g^{-1}t^r(u\varphi^k)g \\ &= t^r(g\varphi^r)^{-1}(u\varphi^k)g.\end{aligned}$$

$t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, *$v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.*

Definition

For $\varphi \in End(G)$, two elements $u, v \in G$ are said to be φ-twisted conjugated, denoted $u \sim_\varphi v$, if $v = (g\varphi)^{-1}ug$ for some $g \in G$.

Definition

The twisted conjugacy problem for G, denoted $TCP(G)$: "Given $\varphi \in Aut(G)$ and $u, v \in G$ decide whether $u \sim_\varphi v$".

Step 3:

We solved it

$TCP(F_n)$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ for automorphisms is solvable.

Proof. Given $\varphi \in \mathrm{Aut}(F_n)$ and $u, v \in F_n$:

1. Extend to $F_n * \langle z \rangle$ and $\hat{\varphi}\colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.

- **Claim:** for $g \in F_n$, $v = (g\varphi)^{-1}ug \;\Leftrightarrow\; g^{-1}zg \in \mathrm{Fix}(\hat{\varphi}\gamma_v)$.

2. Compute a basis for $\mathrm{Fix}(\hat{\varphi}\gamma_v)$.

3. Check whether $\mathrm{Fix}(\hat{\varphi}\gamma_v)$ contains $g^{-1}zg$ for some $g \in F_n$, using Stallings' automata. □

Theorem (Maslakova)

Fixed subgroups of automorphisms of free groups are computable.

$TCP(F_n)$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ for automorphisms is solvable.

Proof. Given $\varphi \in \operatorname{Aut}(F_n)$ and $u, v \in F_n$:

1. Extend to $F_n * \langle z \rangle$ and $\hat{\varphi} \colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.

- **Claim:** for $g \in F_n$, $v = (g\varphi)^{-1} ug \;\Leftrightarrow\; g^{-1} zg \in \operatorname{Fix}(\hat{\varphi}\gamma_v)$.

2. Compute a basis for $\operatorname{Fix}(\hat{\varphi}\gamma_v)$.

3. Check whether $\operatorname{Fix}(\hat{\varphi}\gamma_v)$ contains $g^{-1}zg$ for some $g \in F_n$, using Stallings' automata. □

Theorem (Maslakova)

Fixed subgroups of automorphisms of free groups are computable.

$TCP(F_n)$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ for automorphisms is solvable.

Proof. Given $\varphi \in \text{Aut}(F_n)$ and $u, v \in F_n$:

1 Extend to $F_n * \langle z \rangle$ and $\hat{\varphi} \colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.

- **Claim:** for $g \in F_n$, $v = (g\varphi)^{-1}ug \;\Leftrightarrow\; g^{-1}zg \in \text{Fix}(\hat{\varphi}\gamma_v)$.

2 Compute a basis for $\text{Fix}(\hat{\varphi}\gamma_v)$.

3 Check whether $\text{Fix}(\hat{\varphi}\gamma_v)$ contains $g^{-1}zg$ for some $g \in F_n$, using Stallings' automata. □

Theorem (Maslakova)

Fixed subgroups of automorphisms of free groups are computable.

$TCP(F_n)$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ for automorphisms is solvable.

Proof. Given $\varphi \in \operatorname{Aut}(F_n)$ and $u, v \in F_n$:

1 Extend to $F_n * \langle z \rangle$ and $\hat{\varphi}\colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.

- **Claim:** for $g \in F_n$, $v = (g\varphi)^{-1}ug \Leftrightarrow g^{-1}zg \in \operatorname{Fix}(\hat{\varphi}\gamma_v)$.

2 Compute a basis for $\operatorname{Fix}(\hat{\varphi}\gamma_v)$.

3 Check whether $\operatorname{Fix}(\hat{\varphi}\gamma_v)$ contains $g^{-1}zg$ for some $g \in F_n$, using Stallings' automata. □

Theorem (Maslakova)

Fixed subgroups of automorphisms of free groups are computable.

$TCP(F_n)$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ for automorphisms is solvable.

Proof. Given $\varphi \in \mathrm{Aut}(F_n)$ and $u, v \in F_n$:

1. Extend to $F_n * \langle z \rangle$ and $\hat{\varphi} \colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.

- **Claim:** for $g \in F_n$, $v = (g\varphi)^{-1}ug \;\Leftrightarrow\; g^{-1}zg \in \mathrm{Fix}(\hat{\varphi}\gamma_v)$.

2. Compute a basis for $\mathrm{Fix}(\hat{\varphi}\gamma_v)$.

3. Check whether $\mathrm{Fix}(\hat{\varphi}\gamma_v)$ contains $g^{-1}zg$ for some $g \in F_n$, using Stallings' automata. □

Theorem (Maslakova)

Fixed subgroups of automorphisms of free groups are computable.

$TCP(F_n)$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ for automorphisms is solvable.

Proof. Given $\varphi \in \mathrm{Aut}\,(F_n)$ and $u, v \in F_n$:

1 Extend to $F_n * \langle z \rangle$ and $\hat{\varphi} \colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.

- **Claim:** for $g \in F_n$, $v = (g\varphi)^{-1}ug \;\Leftrightarrow\; g^{-1}zg \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v)$.

2 Compute a basis for $\mathrm{Fix}\,(\hat{\varphi}\gamma_v)$.

3 Check whether $\mathrm{Fix}\,(\hat{\varphi}\gamma_v)$ contains $g^{-1}zg$ for some $g \in F_n$, using Stallings' automata. □

Theorem (Maslakova)

Fixed subgroups of automorphisms of free groups are computable.

$TCP(F_n)$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ for automorphisms is solvable.

Proof. Given $\varphi \in \mathrm{Aut}(F_n)$ and $u, v \in F_n$:

1. Extend to $F_n * \langle z \rangle$ and $\hat{\varphi} \colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.

- **Claim:** for $g \in F_n$, $v = (g\varphi)^{-1} ug \;\Leftrightarrow\; g^{-1}zg \in \mathrm{Fix}(\hat{\varphi}\gamma_v)$.

2. Compute a basis for $\mathrm{Fix}(\hat{\varphi}\gamma_v)$.

3. Check whether $\mathrm{Fix}(\hat{\varphi}\gamma_v)$ contains $g^{-1}zg$ for some $g \in F_n$, using Stallings' automata. □

Theorem (Maslakova)

Fixed subgroups of automorphisms of free groups are computable.

$CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Given $t^r u,\ t^s v \in M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.
- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k \pm \lambda r}$; hence,

  $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

$CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Given $t^r u,\ t^s v \in M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.
- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k \pm \lambda r}$; hence,

  $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

$CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Given $t^r u,\ t^s v \in M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.
- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

  $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.
- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

$CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Given $t^r u,\ t^s v \in M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.
- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

  $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

$CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Given $t^r u,\ t^s v \in M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.
- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

$t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

$CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Given $t^r u,\ t^s v \in M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.
- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

  $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

Step 4:

A mistake was found

$CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $(g^{-1}t^{-k})(t^r u)(t^k g) = g^{-1}t^r(u\varphi^k)g = t^r(g\varphi^r)^{-1}(u\varphi^k)g.$
- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.

- **Case 1: $r \neq 0$**

- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

  $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

$CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable

- **Case 2: $r = 0$**
- Still infinitely many k's:

u and v conj. in M_φ $\iff$ $v \sim u\varphi^k$ for some $k \in \mathbb{Z}$.

- This is precisely Brinkmann's result:

Theorem (Brinkmann, 2006)

Given an automorphism $\phi \colon F_n \to F_n$ and $u, v \in F_n$, it is decidable whether $v \sim u\phi^k$ for some $k \in \mathbb{Z}$.

- Hence, $CP(M_\varphi)$ is solvable. □

$CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable

- **Case 2: $r = 0$**
- Still infinitely many k's:

$$u \text{ and } v \text{ conj. in } M_\varphi \iff v \sim u\varphi^k \text{ for some } k \in \mathbb{Z}.$$

- This is precisely Brinkmann's result:

Theorem (Brinkmann, 2006)

Given an automorphism $\phi\colon F_n \to F_n$ and $u, v \in F_n$, it is decidable whether $v \sim u\phi^k$ for some $k \in \mathbb{Z}$.

- Hence, $CP(M_\varphi)$ is solvable. □

$CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable

- **Case 2: $r = 0$**
- Still infinitely many k's:

$$u \text{ and } v \text{ conj. in } M_\varphi \iff v \sim u\varphi^k \text{ for some } k \in \mathbb{Z}.$$

- This is precisely Brinkmann's result:

Theorem (Brinkmann, 2006)

Given an automorphism $\phi\colon F_n \to F_n$ and $u, v \in F_n$, it is decidable whether $v \sim u\phi^k$ for some $k \in \mathbb{Z}$.

- Hence, $CP(M_\varphi)$ is solvable. □

$CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable

- **Case 2:** $r = 0$
- Still infinitely many k's:

$$u \text{ and } v \text{ conj. in } M_\varphi \iff v \sim u\varphi^k \text{ for some } k \in \mathbb{Z}.$$

- This is precisely Brinkmann's result:

Theorem (Brinkmann, 2006)

Given an automorphism $\phi\colon F_n \to F_n$ and $u, v \in F_n$, it is decidable whether $v \sim u\phi^k$ for some $k \in \mathbb{Z}$.

- Hence, $CP(M_\varphi)$ is solvable. $\square$

Step 5:

Intuition always ahead

A crucial comment

A. Martino: *"The whole argument essentially works the same way in presence of more stable letters, i.e. for free-by-free groups"*

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \ \rangle$ be the free group on $\{x_1, \ldots, x_n\}$ ($n \geq 2$), and let $\varphi_1, \ldots, \varphi_m \in Aut(F_n)$. The free-by-free group $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ is

$$F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m = \langle x_1, \ldots, x_n,\ t_1, \ldots, t_m \mid t_j^{-1} x_i t_j = x_i \varphi_j \rangle.$$

But this looks wrong, taking into account Miller's examples...

A crucial comment

A. Martino: *"The whole argument essentially works the same way in presence of more stable letters, i.e. for free-by-free groups"*

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \rangle$ be the free group on $\{x_1, \ldots, x_n\}$ ($n \geq 2$), and let $\varphi_1, \ldots, \varphi_m \in Aut(F_n)$. The free-by-free group $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ is

$$F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m = \langle x_1, \ldots, x_n,\ t_1, \ldots, t_m \mid t_j^{-1} x_i t_j = x_i\varphi_j \rangle.$$

But this looks wrong, taking into account Miller's examples...

The comment was right...

He was right... the whole argument essentially works the same way except that in the second case, a much stronger problem arises:

u and v conj. in $M_{\varphi_1,\ldots,\varphi_m}$ $\iff$ $v \sim u\varphi$ for some $\varphi \in \langle \varphi_1, \ldots, \varphi_m \rangle \leqslant Aut(F_n)$.

Theorem

$CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$ is solvable if and only if $\langle \varphi_1, \ldots, \varphi_m \rangle$ is orbit decidable.

Definition

A subgroup $A \leqslant Aut(F)$ is said to be orbit decidable (O.D.) if $\exists$ an algorithm s.t., given $u, v \in F$ decides whether $v \sim u\alpha$ for some $\alpha \in A$.

The comment was right...

He was right... the whole argument essentially works the same way except that in the second case, a much stronger problem arises:

u and v conj. in $M_{\varphi_1,\ldots,\varphi_m}$ $\iff$ *$v \sim u\varphi$ for some $\varphi \in \langle \varphi_1, \ldots, \varphi_m \rangle \leqslant Aut(F_n)$.*

Theorem

$CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$ is solvable if and only if $\langle \varphi_1, \ldots, \varphi_m \rangle$ is orbit decidable.

Definition

A subgroup $A \leqslant Aut(F)$ is said to be orbit decidable (O.D.) if $\exists$ an algorithm s.t., given $u, v \in F$ decides whether $v \sim u\alpha$ for some $\alpha \in A$.

The comment was right...

He was right... the whole argument essentially works the same way except that in the second case, a much stronger problem arises:

u and v conj. in $M_{\varphi_1,\ldots,\varphi_m}$ $\iff$ $v \sim u\varphi$ for some $\varphi \in \langle \varphi_1, \ldots, \varphi_m \rangle \leqslant Aut(F_n)$.

Theorem

$CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$ is solvable if and only if $\langle \varphi_1, \ldots, \varphi_m \rangle$ is orbit decidable.

Definition

A subgroup $A \leqslant Aut(F)$ is said to be orbit decidable (O.D.) if $\exists$ an algorithm s.t., given $u, v \in F$ decides whether $v \sim u\alpha$ for some $\alpha \in A$.

The comment was right...

He was right... the whole argument essentially works the same way except that in the second case, a much stronger problem arises:

u and v conj. in $M_{\varphi_1,\ldots,\varphi_m}$ $\iff$ $v \sim u\varphi$ *for some* $\varphi \in \langle \varphi_1, \ldots, \varphi_m \rangle \leqslant Aut(F_n)$.

Theorem

$CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$ is solvable if and only if $\langle \varphi_1, \ldots, \varphi_m \rangle$ is orbit decidable.

Definition

A subgroup $A \leqslant Aut(F)$ is said to be orbit decidable (O.D.) if $\exists$ an algorithm s.t., given $u, v \in F$ decides whether $v \sim u\alpha$ for some $\alpha \in A$.

Step 6:

Extend as much as possible

Outline

The main result

Theorem (Bogopolski-Martino-V., 2008)

Let

$$1 \longrightarrow F \stackrel{\alpha}{\longrightarrow} G \stackrel{\beta}{\longrightarrow} H \longrightarrow 1$$

be an algorithmic short exact sequence of groups such that

(i) *$TCP(F)$ is solvable,*

(ii) *$CP(H)$ is solvable,*

(iii) *there is an algorithm which, given an input $1 \neq h \in H$, computes a finite set of elements $z_{h,1}, \ldots, z_{h,t_h} \in H$ such that*

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

Then,

$$CP(G) \text{ is solvable} \iff A_G = \left\{ \begin{array}{rcl} \gamma_g : F & \to & F \\ x & \mapsto & g^{-1}xg \end{array} \middle| \, g \in G \right\}$$

$$\leqslant Aut(F) \text{ is orbit decidable.}$$

The main result

Theorem (Bogopolski-Martino-V., 2008)

Let

$$1 \longrightarrow F \stackrel{\alpha}{\longrightarrow} G \stackrel{\beta}{\longrightarrow} H \longrightarrow 1$$

be an algorithmic short exact sequence of groups such that

(i) *$TCP(F)$ is solvable,*

(ii) *$CP(H)$ is solvable,*

(iii) *there is an algorithm which, given an input $1 \neq h \in H$, computes a finite set of elements $z_{h,1}, \ldots, z_{h,t_h} \in H$ such that*

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

Then,

$$CP(G) \text{ is solvable } \iff A_G = \left\{ \begin{array}{rcl} \gamma_g : F & \to & F \\ x & \mapsto & g^{-1}xg \end{array} \,\middle|\, g \in G \right\}$$

$$\leqslant Aut(F) \text{ is orbit decidable.}$$

The main result

Theorem (Bogopolski-Martino-V., 2008)

Let

$$1 \longrightarrow F \stackrel{\alpha}{\longrightarrow} G \stackrel{\beta}{\longrightarrow} H \longrightarrow 1$$

be an algorithmic short exact sequence of groups such that

(i) *$TCP(F)$ is solvable,*

(ii) *$CP(H)$ is solvable,*

(iii) *there is an algorithm which, given an input $1 \neq h \in H$, computes a finite set of elements $z_{h,1}, \ldots, z_{h,t_h} \in H$ such that*

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

Then,

$$CP(G) \text{ is solvable} \iff A_G = \left\{ \begin{array}{rcl} \gamma_g : F & \to & F \\ x & \mapsto & g^{-1}xg \end{array} \,\middle|\, g \in G \right\}$$

$$\leqslant Aut(F) \text{ is orbit decidable.}$$

The main result

Theorem (Bogopolski-Martino-V., 2008)

Let

$$1 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$$

be an algorithmic short exact sequence of groups such that

(i) *$TCP(F)$ is solvable,*

(ii) *$CP(H)$ is solvable,*

(iii) *there is an algorithm which, given an input $1 \neq h \in H$, computes a finite set of elements $z_{h,1}, \ldots, z_{h,t_h} \in H$ such that*

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

Then,

$$CP(G) \text{ is solvable} \iff A_G = \left\{ \begin{array}{rccc} \gamma_g : & F & \to & F \\ & x & \mapsto & g^{-1}xg \end{array} \,\middle|\, g \in G \right\}$$

$$\leqslant Aut(F) \text{ is orbit decidable.}$$

The main result

Theorem (Bogopolski-Martino-V., 2008)

Let

$$1 \longrightarrow F \stackrel{\alpha}{\longrightarrow} G \stackrel{\beta}{\longrightarrow} H \longrightarrow 1$$

be an algorithmic short exact sequence of groups such that

(i) *$TCP(F)$ is solvable,*

(ii) *$CP(H)$ is solvable,*

(iii) *there is an algorithm which, given an input $1 \neq h \in H$, computes a finite set of elements $z_{h,1}, \ldots, z_{h,t_h} \in H$ such that*

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

Then,

$$CP(G) \text{ is solvable} \iff A_G = \left\{ \begin{array}{rccc} \gamma_g : F & \to & F \\ x & \mapsto & g^{-1}xg \end{array} \,\middle|\, g \in G \right\}$$

$\leqslant Aut(F)$ *is orbit decidable.*

The previous result

Previous result in this language:

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

And Miller's examples must correspond to orbit undecidable subgroups of $Aut(F_n)$...

The previous result

Previous result in this language:

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

And Miller's examples must correspond to orbit undecidable subgroups of $Aut(F_n)$...

The previous result

Previous result in this language:

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

And Miller's examples must correspond to orbit undecidable subgroups of $Aut(F_n)$...

Negative results

Proposition (Bogopolski-Martino-V., 2008)

Let F be a group, and let $A \leqslant B \leqslant Aut(F)$ and $w \in F$ be such that $B \cap Stab^*(w) = 1$. Then,

$$OD(A) \text{ solvable} \quad \Rightarrow \quad MP(A, B) \text{ solvable.}$$

Corollary

Let F be a group, and let $A \leqslant B \leqslant Aut(F)$ and $w \in F$ be such that $B \cap Stab^*(w) = 1$. If $B \simeq F_2 \times F_2$ and A is the Mihailova subgroup corresponding to a group with unsolvable word problem then, $A \leqslant Aut(F)$ is orbit undecidable.

Negative results

Proposition (Bogopolski-Martino-V., 2008)

Let F be a group, and let $A \leqslant B \leqslant Aut(F)$ and $w \in F$ be such that $B \cap Stab^*(w) = 1$. Then,

$$OD(A) \text{ solvable} \quad \Rightarrow \quad MP(A, B) \text{ solvable.}$$

Corollary

Let F be a group, and let $A \leqslant B \leqslant Aut(F)$ and $w \in F$ be such that $B \cap Stab^*(w) = 1$. If $B \simeq F_2 \times F_2$ and A is the Mihailova subgroup corresponding to a group with unsolvable word problem then, $A \leqslant Aut(F)$ is orbit undecidable.

Negative results

With the following embedding (and $w = qaqbq$)

$$\begin{array}{rcl} F_2 \times F_2 & \longrightarrow & \mathrm{Aut}\,(F_3) \\ (u, v) & \mapsto & {}_u\theta_v \colon \begin{array}{rcl} F_3 & \to & F_3 \\ q & \mapsto & u^{-1}qv \\ a & \mapsto & a \\ b & \mapsto & b \end{array} \end{array}$$

we obtain an alternative proof for unsolvability of the conjugacy problem in Miller's examples.

And any other way of embedding $F_2 \times F_2$ in $\mathrm{Aut}\,(F_3)$ will provide new examples.

Negative results

With the following embedding (and $w = qaqbq$)

$$\begin{array}{ccc} F_2 \times F_2 & \longrightarrow & \mathrm{Aut}(F_3) \\ (u,v) & \mapsto & {}_u\theta_v \colon \begin{array}{ccc} F_3 & \to & F_3 \\ q & \mapsto & u^{-1}qv \\ a & \mapsto & a \\ b & \mapsto & b \end{array} \end{array}$$

we obtain an alternative proof for unsolvability of the conjugacy problem in Miller's examples.

And any other way of embedding $F_2 \times F_2$ in $\mathrm{Aut}(F_3)$ will provide new examples.

More generally ...

A similar programme can be done for every extension $F \rtimes H$

$$1 \to F \to F \rtimes H \to H \to 1$$

satisfying

(i) $TCP(F)$ is solvable,

(ii) $CP(H)$ is solvable,

(iii) H has small and computable centralizers

So,

For any group F where you can solve $TCP(F)$, you are in a perfect situation to study the conjugacy problem in the family of free (or torsion-free hyperbolic) extensions of F.

More generally ...

A similar programme can be done for every extension $F \rtimes H$

$$1 \to F \to F \rtimes H \to H \to 1$$

satisfying

(i) $TCP(F)$ is solvable,

(ii) $CP(H)$ is solvable,

(iii) H has small and computable centralizers

So,

For any group F where you can solve $TCP(F)$, you are in a perfect situation to study the conjugacy problem in the family of free (or torsion-free hyperbolic) extensions of F.

Outline

1. The conjugacy problem for free-by-cyclic groups
2. The main theorem
3. The conjugacy problem for free-by-free groups
4. The conjugacy problem for (free abelian)-by-free groups
5. The conjugacy problem for Braid-by-free groups
6. The conjugacy problem for Thompson-by-free groups
7. The conjugacy problem for automata

Free-by-free groups

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead)

The full $Aut(F_n)$ is O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle = Aut(F_n)$ then $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

Free-by-free groups

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead)

The full $Aut(F_n)$ is O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle = Aut(F_n)$ then $F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m$ has solvable conjugacy problem.

Free-by-free groups

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead)

The full $Aut(F_n)$ is O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle = Aut(F_n)$ then $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

Free-by-free groups

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead)

The full $Aut(F_n)$ is O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle = Aut(F_n)$ then $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

Free-by-free groups

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $Aut(F_n)$ are O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle$ is of finite index in $Aut(F_n)$ then $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $Aut(F_2)$ is O.D.

Corollary

Every F_2-by-free group has solvable conjugacy problem.

Free-by-free groups

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $Aut(F_n)$ are O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle$ is of finite index in $Aut(F_n)$ then $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $Aut(F_2)$ is O.D.

Corollary

Every F_2-by-free group has solvable conjugacy problem.

Free-by-free groups

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $Aut(F_n)$ are O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle$ is of finite index in $Aut(F_n)$ then $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $Aut(F_2)$ is O.D.

Corollary

Every F_2-by-free group has solvable conjugacy problem.

Free-by-free groups

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $Aut(F_n)$ are O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle$ is of finite index in $Aut(F_n)$ then $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $Aut(F_2)$ is O.D.

Corollary

Every F_2-by-free group has solvable conjugacy problem.

Free-by-free groups

But...

Theorem (Miller, 70's)

There are free-by-free groups with unsolvable conjugacy problem.

Corollary

There exist 14 automorphisms $\varphi_1, \ldots, \varphi_{14} \in Aut(F_3)$ such that $\langle \varphi_1, \ldots, \varphi_{14} \rangle \leqslant Aut(F_3)$ is orbit undecidable.

Free-by-free groups

But...

Theorem (Miller, 70's)

There are free-by-free groups with unsolvable conjugacy problem.

Corollary

There exist 14 automorphisms $\varphi_1, \ldots, \varphi_{14} \in Aut(F_3)$ such that $\langle \varphi_1, \ldots, \varphi_{14} \rangle \leqslant Aut(F_3)$ is orbit undecidable.

Free-by-free groups

But...

Theorem (Miller, 70's)

There are free-by-free groups with unsolvable conjugacy problem.

Corollary

There exist 14 automorphisms $\varphi_1, \ldots, \varphi_{14} \in Aut(F_3)$ such that $\langle \varphi_1, \ldots, \varphi_{14} \rangle \leqslant Aut(F_3)$ is orbit undecidable.

Outline

(Free abelian)-by-free groups

Definition

Let $\mathbb{Z}^n = \langle x_1, \ldots, x_n \mid [x_i, x_j] \rangle$ be the free abelian group of rank $n \geq 2$, and let $M_1, \ldots, M_m \in Aut(\mathbb{Z}^n) = GL_n(\mathbb{Z})$. The (free abelian)-by-free group $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ is defined as

$$F_n \rtimes_{M_1,\ldots,M_m} F_m = \langle x_1, \ldots, x_n,\ t_1, \ldots, t_m \mid t_j^{-1} x_i t_j = x_i M_j,\ [x_i, x_j] = 1 \rangle.$$

The sequence

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m \longrightarrow F_m \longrightarrow 1$$

again satisfies (i), (ii) and (iii). So,

$CP(\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m)$ is solvable $\Leftrightarrow \langle M_1, \ldots, M_m \rangle \leqslant GL_n(\mathbb{Z})$ is O.D.

(Free abelian)-by-free groups

Definition

Let $\mathbb{Z}^n = \langle x_1, \ldots, x_n \mid [x_i, x_j] \rangle$ be the free abelian group of rank $n \geq 2$, and let $M_1, \ldots, M_m \in Aut(\mathbb{Z}^n) = GL_n(\mathbb{Z})$. The (free abelian)-by-free group $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ is defined as

$$F_n \rtimes_{M_1,\ldots,M_m} F_m = \langle x_1, \ldots, x_n,\ t_1, \ldots, t_m \mid t_j^{-1} x_i t_j = x_i M_j,\ [x_i, x_j] = 1 \rangle.$$

The sequence

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m \longrightarrow F_m \longrightarrow 1$$

again satisfies (i), (ii) and (iii). So,

$CP(\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m)$ is solvable $\Leftrightarrow \langle M_1, \ldots, M_m \rangle \leqslant GL_n(\mathbb{Z})$ is O.D.

(Free abelian)-by-free groups

Definition

Let $\mathbb{Z}^n = \langle x_1, \ldots, x_n \mid [x_i, x_j] \rangle$ be the free abelian group of rank $n \geq 2$, and let $M_1, \ldots, M_m \in Aut(\mathbb{Z}^n) = GL_n(\mathbb{Z})$. The (free abelian)-by-free group $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ is defined as

$$F_n \rtimes_{M_1,\ldots,M_m} F_m = \langle x_1, \ldots, x_n,\ t_1, \ldots, t_m \mid t_j^{-1} x_i t_j = x_i M_j,\ \ [x_i, x_j] = 1 \rangle.$$

The sequence

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m \longrightarrow F_m \longrightarrow 1$$

again satisfies (i), (ii) and (iii). So,

$CP(\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m)$ is solvable $\Leftrightarrow \langle M_1, \ldots, M_m \rangle \leqslant GL_n(\mathbb{Z})$ is O.D.

(Free abelian)-by-free groups

Theorem (linear algebra)

Cyclic subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

$\mathbb{Z}^n$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

Theorem (elementary)

The full $GL_n(\mathbb{Z})$ is O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

(Free abelian)-by-free groups

Theorem (linear algebra)

Cyclic subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

$\mathbb{Z}^n$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

Theorem (elementary)

The full $GL_n(\mathbb{Z})$ is O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

(Free abelian)-by-free groups

Theorem (linear algebra)

Cyclic subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

$\mathbb{Z}^n$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

Theorem (elementary)

The full $GL_n(\mathbb{Z})$ is O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

(Free abelian)-by-free groups

Theorem (linear algebra)

Cyclic subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

$\mathbb{Z}^n$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

Theorem (elementary)

The full $GL_n(\mathbb{Z})$ is O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

(Free abelian)-by-free groups

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $GL_2(\mathbb{Z})$ is O.D.

Corollary

Every $\mathbb{Z}^2$-by-free group has solvable conjugacy problem.

(Free abelian)-by-free groups

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $GL_2(\mathbb{Z})$ is O.D.

Corollary

Every $\mathbb{Z}^2$-by-free group has solvable conjugacy problem.

(Free abelian)-by-free groups

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $GL_2(\mathbb{Z})$ is O.D.

Corollary

Every $\mathbb{Z}^2$-by-free group has solvable conjugacy problem.

(Free abelian)-by-free groups

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $GL_2(\mathbb{Z})$ is O.D.

Corollary

Every $\mathbb{Z}^2$-by-free group has solvable conjugacy problem.

(Free abelian)-by-free groups

$F_2 \times F_2 \leqslant GL_2(\mathbb{Z}) \times GL_2(\mathbb{Z}) \leqslant GL_4(\mathbb{Z})$. So...

Theorem (Bogopolski-Martino-V., 2008)

There exist 14 matrices $M_1, \ldots, M_{14} \in GL_n(\mathbb{Z})$, for $n \geqslant 4$, such that $\langle M_1, \ldots, M_{14} \rangle \leqslant GL_n(\mathbb{Z})$ is orbit undecidable.

Corollary

There exists a $\mathbb{Z}^4$-by-F_{14} group with unsolvable conjugacy problem.

Question

Does $GL_3(\mathbb{Z})$ contain orbit undecidable subgroups ?

Question

Does there exist $\mathbb{Z}^3$-by-free groups with unsolvable conjugacy problem ?

(Free abelian)-by-free groups

$F_2 \times F_2 \leqslant GL_2(\mathbb{Z}) \times GL_2(\mathbb{Z}) \leqslant GL_4(\mathbb{Z})$. So...

Theorem (Bogopolski-Martino-V., 2008)

There exist 14 matrices $M_1, \ldots, M_{14} \in GL_n(\mathbb{Z})$, for $n \geqslant 4$, such that $\langle M_1, \ldots, M_{14} \rangle \leqslant GL_n(\mathbb{Z})$ is orbit undecidable.

Corollary

There exists a $\mathbb{Z}^4$-by-F_{14} group with unsolvable conjugacy problem.

Question

Does $GL_3(\mathbb{Z})$ contain orbit undecidable subgroups ?

Question

Does there exist $\mathbb{Z}^3$-by-free groups with unsolvable conjugacy problem ?

(Free abelian)-by-free groups

$F_2 \times F_2 \leqslant GL_2(\mathbb{Z}) \times GL_2(\mathbb{Z}) \leqslant GL_4(\mathbb{Z})$. So...

Theorem (Bogopolski-Martino-V., 2008)

There exist 14 matrices $M_1, \ldots, M_{14} \in GL_n(\mathbb{Z})$, for $n \geqslant 4$, such that $\langle M_1, \ldots, M_{14} \rangle \leqslant GL_n(\mathbb{Z})$ is orbit undecidable.

Corollary

There exists a $\mathbb{Z}^4$-by-F_{14} group with unsolvable conjugacy problem.

Question

Does $GL_3(\mathbb{Z})$ contain orbit undecidable subgroups ?

Question

Does there exist $\mathbb{Z}^3$-by-free groups with unsolvable conjugacy problem ?

(Free abelian)-by-free groups

$F_2 \times F_2 \leqslant GL_2(\mathbb{Z}) \times GL_2(\mathbb{Z}) \leqslant GL_4(\mathbb{Z})$. So...

Theorem (Bogopolski-Martino-V., 2008)

There exist 14 matrices $M_1, \ldots, M_{14} \in GL_n(\mathbb{Z})$, for $n \geqslant 4$, such that $\langle M_1, \ldots, M_{14} \rangle \leqslant GL_n(\mathbb{Z})$ is orbit undecidable.

Corollary

There exists a $\mathbb{Z}^4$-by-F_{14} group with unsolvable conjugacy problem.

Question

Does $GL_3(\mathbb{Z})$ contain orbit undecidable subgroups ?

Question

Does there exist $\mathbb{Z}^3$-by-free groups with unsolvable conjugacy problem ?

(Free abelian)-by-free groups

$F_2 \times F_2 \leqslant GL_2(\mathbb{Z}) \times GL_2(\mathbb{Z}) \leqslant GL_4(\mathbb{Z})$. So...

Theorem (Bogopolski-Martino-V., 2008)

There exist 14 matrices $M_1, \ldots, M_{14} \in GL_n(\mathbb{Z})$, for $n \geqslant 4$, such that $\langle M_1, \ldots, M_{14} \rangle \leqslant GL_n(\mathbb{Z})$ is orbit undecidable.

Corollary

There exists a $\mathbb{Z}^4$-by-F_{14} group with unsolvable conjugacy problem.

Question

Does $GL_3(\mathbb{Z})$ contain orbit undecidable subgroups ?

Question

Does there exist $\mathbb{Z}^3$-by-free groups with unsolvable conjugacy problem ?

Outline

Braid-by-free groups

Consider the braid group on n strands, given by the classical presentation

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| \geqslant 2) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (1 \leqslant i \leqslant n-2) \end{array} \right\rangle.$$

Theorem

The conjugacy problem is solvable in B_n.

And the automorphism group is easy:

Theorem (Dyer, Grossman)

$|Out(B_n)| = 2$. More precisely, $Aut(B_n) = Inn(B_n) \sqcup Inn(B_n) \cdot \varepsilon$, where $\varepsilon \colon B_n \to B_n$ is the automorphism which inverts all generators, $\sigma_i \mapsto \sigma_i^{-1}$.

Braid-by-free groups

Consider the braid group on n strands, given by the classical presentation

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| \geqslant 2) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (1 \leqslant i \leqslant n-2) \end{array} \right\rangle .$$

Theorem

The conjugacy problem is solvable in B_n.

And the automorphism group is easy:

Theorem (Dyer, Grossman)

$|Out(B_n)| = 2$. More precisely, $Aut(B_n) = Inn(B_n) \sqcup Inn(B_n) \cdot \varepsilon$, where $\varepsilon \colon B_n \to B_n$ is the automorphism which inverts all generators, $\sigma_i \mapsto \sigma_i^{-1}$.

Braid-by-free groups

Consider the braid group on n strands, given by the classical presentation

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| \geqslant 2) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (1 \leqslant i \leqslant n-2) \end{array} \right\rangle.$$

Theorem

The conjugacy problem is solvable in B_n.

And the automorphism group is easy:

Theorem (Dyer, Grossman)

$|Out(B_n)| = 2$. More precisely, $Aut(B_n) = Inn(B_n) \sqcup Inn(B_n) \cdot \varepsilon$, where $\varepsilon \colon B_n \to B_n$ is the automorphism which inverts all generators, $\sigma_i \mapsto \sigma_i^{-1}$.

Braid-by-free groups

Theorem (González-Meneses, V. 2009)

$TCP(B_n)$ is solvable.

Observation

Every subgroup $A \leqslant Aut(B_n)$ is orbit decidable.

Corollary (González-Meneses, V. 2009)

Every extension of B_n by a free group (or torsion-free hyperbolic) has solvable conjugacy problem.

Braid-by-free groups

Theorem (González-Meneses, V. 2009)

$TCP(B_n)$ is solvable.

Observation

Every subgroup $A \leqslant Aut(B_n)$ is orbit decidable.

Corollary (González-Meneses, V. 2009)

Every extension of B_n by a free group (or torsion-free hyperbolic) has solvable conjugacy problem.

Outline

Thompson-by-free groups

Consider Thompson's group F (piecewise linear increasing maps $[0,1] \to [0,1]$ with diadic breakpoints, and slopes being powers of 2).

Theorem

The conjugacy problem is solvable in B_n.

And the automorphism group is big, but easy:

Theorem (Brin)

For every $\varphi \in Aut(F)$, there exists $\tau \in EP_2$ such that $\varphi(g) = \tau^{-1} g \tau$, for every $g \in F$.

Thompson-by-free groups

Consider Thompson's group F (piecewise linear increasing maps $[0,1] \to [0,1]$ with diadic breakpoints, and slopes being powers of 2).

Theorem

The conjugacy problem is solvable in B_n.

And the automorphism group is big, but easy:

Theorem (Brin)

For every $\varphi \in Aut(F)$, there exists $\tau \in EP_2$ such that $\varphi(g) = \tau^{-1} g \tau$, for every $g \in F$.

Thompson-by-free groups

Consider Thompson's group F (piecewise linear increasing maps $[0,1] \to [0,1]$ with diadic breakpoints, and slopes being powers of 2).

Theorem

The conjugacy problem is solvable in B_n.

And the automorphism group is big, but easy:

Theorem (Brin)

For every $\varphi \in Aut(F)$, there exists $\tau \in EP_2$ such that $\varphi(g) = \tau^{-1} g \tau$, for every $g \in F$.

Thompson-by-free groups

Theorem (Burillo-Matucci-V. 2010)

$TCP(F)$ is solvable.

But...

Observation

$F_2 \times F_2$ embeds in $Aut(F)$.

Corollary (Burillo-Matucci-V. 2010)

There are extensions of Thompson's group F by a free group, $F \rtimes F_m$, with unsolvable conjugacy problem.

Thompson-by-free groups

Theorem (Burillo-Matucci-V. 2010)

$TCP(F)$ is solvable.

But...

Observation

$F_2 \times F_2$ embeds in $Aut(F)$.

Corollary (Burillo-Matucci-V. 2010)

There are extensions of Thompson's group F by a free group, $F \rtimes F_m$, with unsolvable conjugacy problem.

Outline

1. The conjugacy problem for free-by-cyclic groups
2. The main theorem
3. The conjugacy problem for free-by-free groups
4. The conjugacy problem for (free abelian)-by-free groups
5. The conjugacy problem for Braid-by-free groups
6. The conjugacy problem for Thompson-by-free groups
7. The conjugacy problem for automata

Automata groups

Proposition (Šunić-V., 2010)

For $d \geqslant 6$, the group $GL_d(\mathbb{Z})$ contains orbit undecidable, free subgroups.

So, for $d \geqslant 6$, there exists a group of the form

$$\Gamma = \mathbb{Z}^d \rtimes F_m \leqslant \mathbb{Z}^d \rtimes GL_d(\mathbb{Z})$$

with unsolvable conjugacy problem.

Theorem (Šunić-V., 2010)

Such a group $\Gamma = \mathbb{Z}^d \rtimes F_m$ can be realized as an automaton group.

Corollary

There exists automaton groups with unsolvable conjugacy problem.

Automata groups

Proposition (Šunić-V., 2010)

For $d \geqslant 6$, the group $GL_d(\mathbb{Z})$ contains orbit undecidable, free subgroups.

So, for $d \geqslant 6$, there exists a group of the form

$$\Gamma = \mathbb{Z}^d \rtimes F_m \leqslant \mathbb{Z}^d \rtimes GL_d(\mathbb{Z})$$

with unsolvable conjugacy problem.

Theorem (Šunić-V., 2010)

Such a group $\Gamma = \mathbb{Z}^d \rtimes F_m$ can be realized as an automaton group.

Corollary

There exists automaton groups with unsolvable conjugacy problem.

Automata groups

Proposition (Šunić-V., 2010)

For $d \geqslant 6$, the group $GL_d(\mathbb{Z})$ contains orbit undecidable, free subgroups.

So, for $d \geqslant 6$, there exists a group of the form

$$\Gamma = \mathbb{Z}^d \rtimes F_m \leqslant \mathbb{Z}^d \rtimes GL_d(\mathbb{Z})$$

with unsolvable conjugacy problem.

Theorem (Šunić-V., 2010)

Such a group $\Gamma = \mathbb{Z}^d \rtimes F_m$ can be realized as an automaton group.

Corollary

There exists automaton groups with unsolvable conjugacy problem.

Automata groups

Proposition (Šunić-V., 2010)

For $d \geqslant 6$, the group $GL_d(\mathbb{Z})$ contains orbit undecidable, free subgroups.

So, for $d \geqslant 6$, there exists a group of the form

$$\Gamma = \mathbb{Z}^d \rtimes F_m \leqslant \mathbb{Z}^d \rtimes GL_d(\mathbb{Z})$$

with unsolvable conjugacy problem.

Theorem (Šunić-V., 2010)

Such a group $\Gamma = \mathbb{Z}^d \rtimes F_m$ can be realized as an automaton group.

Corollary

There exists automaton groups with unsolvable conjugacy problem.

THANKS