Characterizing solvability of the conjugacy problem for free-by-free and [free abelian]-by-free groups

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Outline

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Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\varphi\colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta\colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u\colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

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For $\varphi \in End(G)$, two elements $u, v \in G$ are said to be φ-twisted conjugated, denoted $u \sim_\varphi v$, if $v = (g\varphi)^{-1}ug$ for some $g \in G$.

Definition

The twisted conjugacy problem for G, denoted $TCP(G)$: "Given $\varphi \in Aut(G)$ and $u, v \in G$ decide whether $u \sim_\varphi v$".

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1. Extend to $F_n * \langle z \rangle$ and $\hat{\varphi}\colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.

- **Claim:** for $g \in F_n$, we have $v = (g\varphi)^{-1} ug \;\Leftrightarrow\; z\gamma_g \in \mathrm{Fix}(\hat{\varphi}\gamma_v)$.

2. Compute a basis for $\mathrm{Fix}(\hat{\varphi}\gamma_v)$.

3. Check whether $\mathrm{Fix}(\hat{\varphi}\gamma_v)$ contains $z\gamma_g$ for some $g \in F_n$, using Stallings' automata. □

Theorem (Maslakova)

Fixed subgroups of automorphisms of free groups are computable.

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Let $F_n = \langle x_1, \ldots, x_n \mid \; \rangle$ be a free group on $\{x_1, \ldots, x_n\}$ ($n \geq 2$), and let $\varphi \in Aut(F_n)$. The free-by-cyclic group $F_n \rtimes_\varphi \mathbb{Z}$ is defined as

$$F_n \rtimes_\varphi \mathbb{Z} \quad = \langle x_1, \ldots, x_n, \; t \mid t^{-1} x_i t = x_i \varphi \rangle.$$

With $x_i t = t(x_i \varphi)$ and $x_i t^{-1} = t^{-1}(x_i \varphi^{-1})$, we can move all t's to the left and get the usual normal form for elements in $F_n \rtimes_\varphi \mathbb{Z}$:

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Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

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▶ **Case 1:** $r \neq 0$

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- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

- **Case 2: $r = 0$**
- Still infinitely many k's:

$$u \text{ and } v \text{ conj. in } M_\varphi \iff v \sim u\varphi^k \text{ for some } k \in \mathbb{Z}.$$

- This is precisely Brinkmann's result:

Theorem (Brinkmann, 2006)

Given an automorphism $\phi\colon F_n \to F_n$ and $u, v \in F_n$, it is decidable whether $v \sim u\phi^k$ for some $k \in \mathbb{Z}$.

- Hence, $CP(M_\varphi)$ is solvable. □

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- **Case 2: $r = 0$**
- Still infinitely many k's:

$$u \text{ and } v \text{ conj. in } M_\varphi \iff v \sim u\varphi^k \text{ for some } k \in \mathbb{Z}.$$

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The whole argument "works" very similarly with several stable letters, i.e. for free-by-free groups

Definition

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Outline

Theorem (Bogopolski-Martino-V., 2008)

Let

$$1 \longrightarrow F \stackrel{\alpha}{\longrightarrow} G \stackrel{\beta}{\longrightarrow} H \longrightarrow 1$$

be an algorithmic short exact sequence of groups such that

(i) *$TCP(F)$ is solvable,*

(ii) *$CP(H)$ is solvable,*

(iii) *there is an algorithm which, given an input $1 \neq h \in H$, computes a finite set of elements $z_{h,1}, \ldots, z_{h,t_h} \in H$ such that*

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

Then,

$$CP(G) \text{ is solvable} \iff A_G = \left\{ \begin{array}{rcl} \gamma_g : F & \to & F \\ x & \mapsto & g^{-1}xg \end{array} \;\middle|\; g \in G \right\}$$

$\leqslant Aut(F)$ is orbit decidable.

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Previous results in this language:

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Every finitely generated, virtually

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Theorem (work in progress)

- *(w. J.Burillo & F.Matucci) Thomson's group has solvable TCP,*
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Theorem (Bogopolski-Martino-Maslakova-V., 2005)

Let G be a group (given as a finite presentation) and $K \leqslant G$ a finite index subgroup (given by generators). Then,

- *if K is characteristic and $TCP(K)$ is solvable, then $TCP(G)$ is solvable,*
- *if K is normal and $TCP(K)$ is solvable, then $CP(G)$ is solvable.*

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

There exists a group G with $CP(G)$ solvable but $TCP(G)$ unsolvable.

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Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

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Free-by-cyclic groups have solvable conjugacy problem.

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The full $Aut(F_n)$ is O.D.

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Theorem (Bogopolski-Martino-V., 2008)

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There are free-by-free groups with unsolvable conjugacy problem.

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There exist 14 automorphisms $\varphi_1, \ldots, \varphi_{14} \in Aut(F_3)$ such that $\langle \varphi_1, \ldots, \varphi_{14} \rangle \leqslant Aut(F_3)$ is orbit undecidable.

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Theorem (linear algebra)

Cyclic subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

$\mathbb{Z}^n$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

Theorem (elementary)

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If $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $GL_2(\mathbb{Z})$ is O.D.

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Every $\mathbb{Z}^2$-by-free group has solvable conjugacy problem.

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What about negative results in the [free abelian]-by-free case ?

Proposition (Bogopolski-Martino-V., 2008)

Let F be a group, and let $A \leqslant B \leqslant Aut(F)$ and $w \in F$ be such that $B \cap Stab^*(w) = 1$. Then,

$$OD(A) \text{ solvable} \quad \Rightarrow \quad MP(A, B) \text{ solvable.}$$

Corollary

Let F be a group, and let $A \leqslant B \leqslant Aut(F)$ and $w \in F$ be such that $B \cap Stab^*(w) = 1$. If $B \simeq F_2 \times F_2$ and A is the Mihailova subgroup corresponding to a group with unsolvable word problem then, $A \leqslant Aut(F)$ is orbit undecidable.

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With the following embedding (and $w = qaqbq$)

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we obtain an alternative proof for unsolvability of the conjugacy problem in Miller's examples.

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And $F_2 \times F_2 \leqslant GL_2(\mathbb{Z}) \times GL_2(\mathbb{Z}) \leqslant GL_4(\mathbb{Z})$. So...

Theorem (Bogopolski-Martino-V., 2008)

There exist 14 matrices $M_1, \ldots, M_{14} \in GL_n(\mathbb{Z})$, for $n \geqslant 4$, such that $\langle M_1, \ldots, M_{14} \rangle \leqslant GL_n(\mathbb{Z})$ is orbit undecidable.

Corollary

There exists a $\mathbb{Z}^4$-by-F_{14} group with unsolvable conjugacy problem.

Question

Does $GL_3(\mathbb{Z})$ contain orbit undecidable subgroups ?

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THANKS