Algebraic and algorithmic aspects of $\mathbb{Z}^{\mathbf{m}} \times \mathbf{F_n}$: fixed subgroups and quantification of inertia

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To an unfinished poetry of a budding mathematician,

Évariste Galois.

1 Abstract

This work is based on the family of groups $\mathbb{Z}^m \times F_n$, namely free-abelian times free groups, direct products of finitely many copies of $\mathbb{Z}$ and a finitely generated free group F_n. These are a special type of right-angled Artin groups (RAAGs). In this work we use general combinatorics, one dimensional geometry and algebraic techniques to play with elements of $\mathbb{Z}^m \times F_n$ and to solve some algorithmic problems concerning ranks of the subgroups, automorphisms and its fixed point subgroups, subgroup intersection problem heading towards a cryptography application using the group $\mathbb{Z}^m \times F_n$.

The core methodology of this work involves the use of Stallings graph to work with subgroups of F_n and to deal with the abelian part we use linear algebra, systems of equations, Smith normal form of integral matrices, etc.

This thesis is to study algorithmic problems of $\mathbb{Z}^m \times F_n$, a natural extension of free groups and a first step towards further generalization into another two main directions: semi-direct products, and partially commutative groups (PC-groups).

The three principal projects of this thesis are the following:

(1) "Degrees of compression and inertia for free-abelian times free groups",[35]: In the lattice of subgroups of a free group, the rank function is not monotone with respect to inclusion (i.e., $H \leqslant K$ does not imply $r(H) \leqslant r(K)$). This makes it interesting to define and study relaxed versions for this monotonic property. Based on the classical definitions of compressed and inert subgroups, we introduce the concepts of *degree of compression* and *degree of inertia* of a finitely generated subgroup H of a given group G, as an attempt to quantify how close (or far) is H from being compressed and inert and

so, from satisfying the above mentioned monotonic property. In the case of $\mathbb{Z}^m \times F_n$, we show that the degree of compression is algorithmically computable, we give an upper bound for the degree of inertia, and relate both degrees with those of the projection $H\pi$ to the free part. With some extra assumptions over the supremum involved in the definition of *degree of inertia*, we define another notion called *restricted degree of inertia*. In the case of $\mathbb{Z}^m \times F_n$, beyond giving the upper bound, we are able to give an equality formula for the *restricted degree of inertia*.

(2) "Fixed subgroups and computation of auto-fixed closures in free-abelian times free groups", [36]: The automorphisms of $\mathbb{Z}^m$ are just matrices from $\mathrm{GL}_m(\mathbb{Z})$, and so their fixed elements are just the eigenspace of eigenvalue 1 of the corresponding matrix. On the other hand, the study of the properties of fixed point subgroups by automorphisms of free groups F_n is much more complicated but they are well studied in the literature. The classical result by Dyer–Scott about fixed subgroups of finite order automorphisms of F_n being free factors of F_n is no longer true in $\mathbb{Z}^m \times F_n$ (with the natural adaptation of the concept of "factor" in our groups), gives us the feeling that fixed point subgroups in $\mathbb{Z}^m \times F_n$ have more degenerated behaviour than in the free group. Within this more general context, we prove a relaxed version in the spirit of Bestvina–Handel Theorem: the rank of fixed subgroups of finite order automorphisms is uniformly bounded in terms of n and m. We also study *periodic points* of endomorphisms of $\mathbb{Z}^m \times F_n$. For any given automorphism it is a very natural question to ask for the elements which are fixed by that automorphism. The dual problem to this is, for a given finitely generated subgroup H, to ask whether there exists a finite collection of automorphisms which fix exactly that particular subgroup H point-wise. In this text we also solve this dual problem and give an algorithm to compute auto-fixed closures (roughly speaking, for a given subgroup H of G, the *auto-fixed closure* is the fixed subgroup of the *point-wise stabilizer* of H with respect to the automorphisms of G) of finitely generated subgroups of $\mathbb{Z}^m \times F_n$.

(3) “Computing intersections of subgroups in free-abelian times free groups, and an application to secret sharing”, [11]: In this project we develop a secret-sharing scheme taking advantage of the fact that $\mathbb{Z}^m \times F_n$ does not satisfy the Howson property, i.e., it contains finitely generated subgroups whose intersection is not finitely generated. Concretely, the shares for the k players are going to be k finitely generated subgroups $H_1, \ldots, H_k$ of $\mathbb{Z}^m \times F_n$ such that every intersection of shares is not finitely generated, except for the total one $\cap_{i=1}^{k} H_i$, which is taken as the secret. This way we significantly increase the difficulty for an illegal coalition of players to extract any practical additional information about the secret (since the intersection of their shares illegitimately shared is not finitely generated so they can only hope to compute a finite truncation of it). We prove that, for any integer k, one can effectively built such a family of subgroups of $\mathbb{Z}^m \times F_n$, and to give an effective algorithm to compute the secret, i.e., the intersection $\bigcap_{i=1}^{k} H_i$ without having the computation of smaller intersections (which are not finitely generated).

> “Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality.”
>
> *Richard Courant*

2 Introduction

My dissertation deals with algorithmic problems in the field of “Combinatorial and Geometric Group Theory”. Algorithmic problems regarding free groups are among the most studied (over a century) results of combinatorial and geometric group theory. A natural extension of this line of research is to solve problems of the same type in more general contexts such as right-angled Artin groups (RAAGs), hyperbolic groups or automatic groups. Among these classes of groups, to deal with hyperbolic groups, topological and geometric tools are more needed. But there are two lines of research within RAAGs, one is more geometric and another one is more combinatorial.

Many of the algorithmic problems (which are decidable in free groups) turn out to be more complicated, or even undecidable in RAAGs, a family which tends to behave rather perversely in many issues. Thus a good deal of attention has been devoted to particular sub-classes of RAAGs such as Droms groups or direct products of free abelian and free groups. Finitely generated direct products of free abelian and free groups, $\mathbb{Z}^m \times F_n$ is the ambient group throughout my dissertation.

Algorithmic problems of $\mathbb{Z}^m \times F_n$ constitutes not only a natural and interesting starting point by itself, but more importantly a fruitful source of ideas for further generalization to, for example, semi-direct product of finitely many copies of $\mathbb{Z}$ and a finitely generated free group, $\mathbb{Z}^m \rtimes F_n$ and Droms groups. Our main focus will be on existence of such algorithms (computability), rather than on their efficiency (complexity).

For a group G, we write $r(G)$ to denote the rank of G, i.e., the minimum cardinal of a generating set for G. The rank function plays an important role as in our most of the algorithms aimed to solve if a given subgroup is finitely generated. In the commutative realm, the rank function is increasing in the sense that $H \leqslant K \leqslant G$ implies $\mathrm{r}(H) \leqslant \mathrm{r}(K)$. This is far from true in general, and the main expression of this phenomena can be found in the context of free groups F_n, where the free group of countably infinite rank easily embeds into the free group of rank 2, $F_{\aleph_0} \leqslant F_2$. However, when restricting ourselves to certain families of groups and subgroups, the rank function tends to behave less wildly and somehow closer to the commutative behaviour. An example of this situation is again in finitely generated free groups, but restricting our attention to subgroups fixed by automorphisms or endomorphisms: the story began in [16], where Dyer–Scott showed that $\mathrm{Fix}(\varphi)$ is a free factor of F_n for every finite order automorphism $\varphi \in \mathrm{Aut}(F_n)$, and conjectured that $\mathrm{r}(\mathrm{Fix}(\varphi)) \leqslant n$, in general. This was proved later by Bestvina–Handel [3], and extended several times in subsequent papers, all of them pointing to the direction that the rank function, when restricted to subgroups fixed by endomorphisms, tends to behave similarly to the abelian case. In 1989, Imrich–Turner [19] improved the result for the case of endomorphisms, i.e., if $\phi \in \mathrm{End}(F_n)$, $\mathrm{r}(\mathrm{Fix}\,\phi) \leqslant n$. Then Dicks–Ventura [14] introduced the notions of *inertia* and *compression*. For monomorphisms (injective endomorphisms), these notions were used from more general point of view (see (i) of Theorem 2.2.4).

On the contrary when we try to investigate the properties of fixed point subgroups of endomorphisms (and automorphisms) of finitely generated direct products of free-abelian and free groups, $\mathbb{Z}^m \times F_n$, the fixed point subgroups behave in a more degenerated way. Because the lattice of subgroups of these groups is quite different from that of free groups, since $\mathbb{Z}^m \times F_n$ is not Howson (i.e., the intersection of two finitely generated subgroups is not necessarily finite generated) as soon as $m \geqslant 1$ and $n \geqslant 2$. This affects seriously the behaviour of the rank function, forcing many situations to degenerate with respect to what happens in free groups. However, there are still several surviving governing rules; we concentrate on some of them, specially about those concerning subgroups fixed by automorphisms of $\mathbb{Z}^m \times F_n$ to compute periodic points (also for endomorphisms) and finite presentation of the auto-fixed subgroups.

The fact that the groups $\mathbb{Z}^m \times F_n$ contain finitely generated subgroups whose intersection is not finitely generated (i.e., $\mathbb{Z}^m \times F_n$ is not Howson) opens up the following new window towards an application to cryptography. The classical secrete sharing scheme among k players uses affine varieties and consists of choosing a point $p \in \mathbb{R}^k$ as a secret, and k affine linearly independent hyperplanes in $\mathbb{R}^k$ whose intersection is precisely the point p; then give an hyperplane to each player as a share. If all the players put in common their shares they can compute the intersections of the k hyperplanes and get the secret p, while any team of $k' < k$ players do not have access to the secrete because the intersection of their k' hyperplanes is an affine variety of dimension $k - k' > 0$, giving them only the information that the secret is one of the infinitely many points in there.

We propose a variation of this secrete sharing scheme where the shares are finitely generated subgroups $H_1, \ldots, H_k$ of $\mathbb{Z}^m \times F_n$ and the secret is their intersection $H_1 \cap \cdots \cap H_k$, being again finitely generated. We will organize the shares in such a way that all the intersections of less than k of those subgroups are never finitely generated; this introduces an extra difficulty for an illegitimate set of players (namely any set of less than k) because they cannot even compute the intersection of their shares to get closer to the secret (the best they can do is to compute a finite truncation of this not finitely generated intersection, having then the uncertainty whether it contains the secret or not). To achieve this goal we have to solve the following two technical problems: (1) for any $k \geqslant 3$, find an effective way to construct such shares $H_1, \ldots, H_k$ all of them being finitely generated, and such that for every subset $I \subseteq \{1, \ldots, k\}$, $\cap_{i \in I} H_i$ being finitely generated if and only if $I = \{1, k\}$; and (2) find an algorithm to compute $H_1 \cap \cdots \cap H_k$ effectively from the independent H_i's. In [10], Delgado–Ventura developed an algorithm to decide if the intersection of two finitely generated subgroups is finitely generated or not and, in case it is, compute it; but note this is not useful in our situation because we cannot compute $H_1 \cap \cdots \cap H_k$ intersection them two by two since all the smaller intersections are not finitely generated by construction.

2.1 Free-abelian times free groups

Throughout the document, we fix an alphabet $Z = \{z_1, \ldots, z_n\}$ of n letters, and consider the free group on it, $F(Z)$, also denoted by F_n. Any direct product of a free-abelian group, $\mathbb{Z}^m$, $m \geqslant 0$, and a free group, F_n, $n \geqslant 0$, will be called, for short, a *free-abelian times free* group, $G = \mathbb{Z}^m \times F_n$. We will work in G with multiplicative notation (as it is a non-abelian group as soon as $n \geqslant 2$) but want to refer to its subgroup $\mathbb{Z}^m \leqslant G$ with the standard additive notation (elements thought as row vectors with addition). To make these compatible, consider the standard presentations $\mathbb{Z}^m = \langle t_1, \ldots, t_m \mid [t_i, t_j],\ \ i, j = 1, \ldots, m \rangle$ and $F_n = \langle z_1, \ldots, z_n \mid \, \rangle$, and the standard normal form for elements from G with the t_i's on the left and in increasing order, namely $t_1^{a_1} \cdots t_m^{a_m} w(z_1, \ldots, z_n)$, where $a_1, \ldots, a_m \in \mathbb{Z}$ and $w \in F_n$ is a reduced word on the alphabet $Z = \{z_1, \ldots, z_n\}$; then, let us abbreviate this in the form

$$t_1^{a_1} \cdots t_m^{a_m} w(z_1, \ldots, z_n) = t^{(a_1, \ldots, a_m)} w(z_1, \ldots, z_n) = t^a w(z_1, \ldots, z_n),$$

where $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ is the row vector made with the integers a_i's, and t is a meaningless symbol serving only as a pillar for holding the vector $a = (a_1, \ldots, a_m)$ up in the exponent. This way, the operation in G is given by $(t^a u)(t^b v) = t^a t^b uv = t^{a+b} uv$ in multiplicative notation, while the abelian part works additively, as usual, up in the exponent. We denote by π the natural projection to the free part, $\pi \colon \mathbb{Z}^m \times F_n \twoheadrightarrow F_n$, $t^a u \mapsto u$.

Structure of subgroups of $\mathbb{Z}^{\mathbf{m}} \times \mathbf{F_n}$

The natural decomposition of $\mathbb{Z}^m \times F_n$ gives a short exact sequence, namely

$$1 \longrightarrow \mathbb{Z}^m \xrightarrow{\iota} \mathbb{Z}^m \times F_n \xrightarrow{\pi} F_n \longrightarrow 1, \tag{2.1}$$

where ι is the inclusion map, π is the natural projection $t^a u \mapsto u$, and therefore $\operatorname{Ker}(\pi) = \mathbb{Z}^m = \operatorname{Im}(\iota)$. Restricting this short exact sequence to any subgroup $H \leqslant \mathbb{Z}^m \times F_n$ we get,

$$1 \longrightarrow \operatorname{Ker}(\pi_{|H}) \xrightarrow{\iota} H \xrightarrow{\pi_{|H}} H\pi \longrightarrow 1. \tag{2.2}$$

where $1 \leqslant \operatorname{Ker}(\pi_{|H}) = H \cap \operatorname{Ker}(\pi) = H \cap \mathbb{Z}^m = L_H \leqslant \mathbb{Z}^m$ and $1 \leqslant H\pi \leqslant F_n$. Therefore, L_H is a free-abelian group and $H\pi$ is a free group. Since $H\pi$ is free, $\pi_{|H}$ has a splitting,

$$H \xleftarrow{f} H\pi \tag{2.3}$$

sending back each element of a chosen free basis for $H\pi$ to an arbitrary pre-image. Hence, f is injective and $H\pi f \leqslant H$ is isomorphic to $H\pi$. Thus, $H \simeq \operatorname{Ker}(\pi_{|H}) \times H\pi f$ is *free-abelian times free*.

An easy observation is that H is finitely generated if and only if $H\pi$ is also finitely generated. Furthermore, as $H \leqslant \mathbb{Z}^m \times F_n$, $n \geqslant 2$, is again free-abelian times free, $H \simeq \mathbb{Z}^{m'} \times F_{n'}$, for some $0 \leqslant m' \leqslant m$ and some $0 \leqslant n' \leqslant \infty$. According to Delgado–Ventura [10, Def. 1.3], a *basis* of a finitely generated subgroup $H \leqslant_{fg} G$ is a set of generators for H of the form $\{t^{a_1}u_1, \ldots, t^{a_r}u_r, t^{b_1}, \ldots, t^{b_s}\}$, where $a_1, \ldots, a_r \in \mathbb{Z}^m$, $\{u_1, \ldots, u_r\}$ is a free-basis of $H\pi \leqslant F_n$, and $\{b_1, \ldots, b_s\}$ is an abelian-basis of $L_H = H \cap \mathbb{Z}^m \leqslant \mathbb{Z}^m$. (Note that, to avoid confusions, we reserve the word *basis* for G, in contrast with *abelian-basis* and *free-basis* for the corresponding concepts in $\mathbb{Z}^m$ and F_n, respectively.) It was showed in [10] that every such subgroup $H \leqslant_{fg} G$ admits a basis, algorithmically computable from any given set of generators.

2.2 Compression, inertia and parametrization of the rank function

In the spirit of Bestvina–Handel [3] result, the following notions were first introduced by Dicks–Ventura [14] and turned out to be quite relevant in the subsequent literature:

Definition 2.2.1. Let G be a group. A finitely generated subgroup $H \leqslant_{fg} G$ is said to be *compressed* in G if $\mathrm{r}(H) \leqslant \mathrm{r}(K)$, for every $H \leqslant K \leqslant G$. And H is said to be *inert* in G if $\mathrm{r}(H \cap K) \leqslant \mathrm{r}(K)$, for every $K \leqslant G$. (Note that, equivalently, in both definitions one can restrict the attention to those subgroups K's being finitely generated, denoted by $K \leqslant_{fg} G$.)

Inert subgroups are closed under finite intersections if our ambient group G satisfies Howson property. Because, let H_1, H_2 are inert, then $\mathrm{r}(H_1 \cap K), \mathrm{r}(H_2 \cap K) \leqslant \mathrm{r}(K)$, for every $K \leqslant G$. Hence we have $\mathrm{r}(H_1 \cap H_2 \cap K) \leqslant \mathrm{r}(H_2 \cap K) \leqslant \mathrm{r}(K)$, for every $K \leqslant G$ (notice that first inequality holds as H_1 is inert and the second inequality holds as H_2 is inert), i.e., $(H_1 \cap H_2)$ is inert, and so using induction we can prove that any finite intersection of inert subgroups is again inert.

If $H \leqslant_{fg} G$ is inert, $\mathrm{r}(H \cap K) \leqslant \mathrm{r}(K)$, for every $K \leqslant G$ and so $\mathrm{r}(H) \leqslant \mathrm{r}(K)$, for every $H \leqslant K \leqslant G$, in other words, H is compressed. Thus inert subgroups are compressed while the other implication is not true in general; an example was given in [44] as an application of the following result:

Theorem 2.2.2. *(Wu–Ventura–Zhang, [44]) Let* $G = \langle a, b \mid bab^{-1}a \rangle^l \times \mathbb{Z}^p$*,* $l \geqslant 1, p \geqslant 0$*. For every* $\phi \in \mathrm{End}(G)$*,* $\mathrm{Fix}\,\phi$ *is compressed in* G*.*

Example 2.2.3. Consider $G = \langle a, b \mid bab^{-1}a \rangle \times \langle c \mid \rangle$, the direct product of the Klein bottle group with the group of integers. And let $\phi\colon G \to G$, $a \mapsto a$, $b \mapsto ba$, $c \mapsto c$; straightforward calculations show that it is a well-defined automorphism $\phi \in \operatorname{Aut}(G)$, and $\operatorname{Fix}(\phi) = \langle a, b^2, c \rangle \simeq \mathbb{Z}^3$. By aforementioned theorem [44], $\operatorname{Fix}(\phi)$ is compressed but not inert in G. Because $\operatorname{Fix}(\phi) \cap \langle ac, b \rangle = \langle ac, a^2, b^2 \rangle \simeq \mathbb{Z}^3$

Several important known results involving these concepts include the following:

Theorem 2.2.4. *(i) (Dicks–Ventura, [14]): Arbitrary intersections of fixed subgroups of injective endomorphisms of F_n are inert in F_n;*

(ii) (Martino–Ventura, [28]): arbitrary intersections of fixed subgroups of endomorphisms of F_n are compressed in F_n;

(iii) (Wu–Zhang, [45]): arbitrary intersections of fixed subgroups of automorphisms of closed surface groups G with negative Euler characteristic are inert in G;

(iv) (Wu–Ventura–Zhang, [44]): arbitrary intersections of fixed subgroups of endomorphisms of surface groups G are compressed in G.

Also, in [44] and [46], Wu–Ventura–Zhang and Zhang–Ventura–Wu studied similar questions within the family of finite direct products of free and surface groups, where more interesting phenomena show up.

We introduce a quantification for these notions of compression and inertia and study it within the families of free groups, and free-abelian times free groups. For technical reasons it is better to work with the so-called *reduced rank* of a group G, defined as $\tilde{\mathrm{r}}(G) = \max\{0, \mathrm{r}(G) - 1\}$, i.e., one unit less than the rank except for the trivial group for which we take zero (note that then, $\tilde{\mathrm{r}}(1) = \tilde{\mathrm{r}}(\mathbb{Z}) = 0$ while $0 = \mathrm{r}(1) \neq \mathrm{r}(\mathbb{Z}) = 1$). Observe that $H \leqslant G$ is *compressed* in G if and only if $\tilde{\mathrm{r}}(H)/\tilde{\mathrm{r}}(K) \leqslant 1$ for every $H \leqslant K \leqslant_{fg} G$; and that $H \leqslant G$ is *inert* in G if and only if $\tilde{\mathrm{r}}(H \cap K)/\tilde{\mathrm{r}}(K) \leqslant 1$ for every $K \leqslant_{fg} G$

(understanding in both cases that $0/0 = 1$). This motivates the following quantitative definitions:

Definition 2.2.5. Let G be a group and $H \leqslant_{fg} G$. The *degree of compression* of H in G is $\mathrm{dc}_G(H) = \sup_K\{\tilde{\mathrm{r}}(H)/\,\tilde{\mathrm{r}}(K)\}$, where the supremum is taken over all subgroups $H \leqslant K \leqslant_{fg} G$. Similarly, the *degree of inertia* of H in G is $\mathrm{di}_G(H) = \sup_K\{\tilde{\mathrm{r}}(H \cap K)/\,\tilde{\mathrm{r}}(K)\}$, where the supremum is taken over all $K \leqslant_{fg} G$ satisfying $H \cap K \leqslant_{fg} G$ (in both cases, $0/0$ is understood to be 1).

Note that, taking $K = H$, we get $\mathrm{dc}_G(H) \geqslant 1$ and $\mathrm{di}_G(H) \geqslant 1$. So, the possibility of K being cyclic (which leads in both cases to $0/0 = 1$) is irrelevant in both definitions and we can restrict the two supremums to non-cyclic K's without changing their final values. Along Chapter 3, when working with these two concepts we will implicitly assume, without explicit mentioning, that the working subgroups K are non-cyclic when necessary.

Note also that the supremum in the definition of degree of compression is always a maximum, since the numerator has a fixed value and the denominator takes only natural values. Although we do not have any particular example, the supremum in the definition of degree of inertia could (in principle) not be attained at any particular subgroup K. In this sense, the following is an intriguing question for which, at the time of writing, we have no idea how to answer:

Question 2.2.6. *Is there a (finitely generated) group G and a subgroup $H \leqslant_{fg} G$ such that $\mathrm{di}_G(H)$ is irrational? Or such that the supremum in $\mathrm{di}_G(H)$ is not a maximum?*

Observe that in the definition of degree of inertia, we take the supremum only over those subgroups $K \leqslant_{fg} G$ whose intersection with H is again finitely generated. In groups G with the Howson property (the intersection of any two finitely generated subgroups is again finitely generated), like free groups, this is no restriction at all and that supremum is over all finitely generated K's. Otherwise, if G is not Howson, we are eliminating, on purpose, those possible finitely generated K's having non-finitely generated intersection with H (which would force $\mathrm{di}_G(H)$ to be automatically infinite). However observe that, even with the present definition, $\mathrm{di}_G(H)$ may be infinite (see (ii) of Theorem 3.4.2).

We adapt the definition of inertia to the non-Howson environments by saying that a subgroup $H \leqslant G$ is *finitary inert* in G if $\mathrm{r}(H \cap K) \leqslant \mathrm{r}(K)$ for every $K \leqslant_{fg} G$ such that $H \cap K \leqslant_{fg} G$. The following observation then follows directly from the definitions and presents the values of $\mathrm{dc}_G(H)$ and $\mathrm{di}_G(H)$ as a quantification of how far is the subgroup $H \leqslant_{fg} G$ from being compressed and being finitary inert in G, respectively:

Observation 2.2.7. *Let G be a group and $H \leqslant_{fg} G$.*

(i) $1 \leqslant \mathrm{dc}_G(H) \leqslant \mathrm{di}_G(H)$;

(ii) $\mathrm{dc}_G(H) = 1$ *if and only if H is compressed in G;*

(iii) $\mathrm{di}_G(H) = 1$ *if and only if H is finitary inert in G.*

The following intriguing question is open, as far as we know:

Question 2.2.8. *Is there a (finitely generated) group G with a subgroup $H \leqslant_{fg} G$ being finitary inert but not inert? (i.e., satisfying $\tilde{\mathrm{r}}(H \cap K) \leqslant \tilde{\mathrm{r}}(K)$ for every $K \leqslant_{fg} G$ with $H \cap K \leqslant_{fg} G$, but simultaneously admitting some $K_0 \leqslant_{fg} G$ with $\tilde{\mathrm{r}}(H \cap K_0) = \infty$?).*

We state now a couple of elementary properties of these concepts for later use. To work with group morphisms, we use the convention of writing arguments on the left, i.e., $\phi \colon G_1 \to G_2$, $g \mapsto g\phi$; and so, compositions as written: $g\phi\psi = (g\phi)\psi$. Accordingly, we write conjugations on the right, $H^g = g^{-1}Hg$, and commutators in the form $[a, b] = a^{-1}b^{-1}ab$.

Lemma 2.2.9. *Let $\phi \colon G_1 \to G_2$ be an isomorphism of groups. For every $H \leqslant_{fg} G_1$,* *(i)* $\mathrm{dc}_{G_2}(H\phi) = \mathrm{dc}_{G_1}(H)$*;*

(ii) $\mathrm{di}_{G_2}(H\phi) = \mathrm{di}_{G_1}(H)$*.*

Proof. For every $K \leqslant_{fg} G_1$ with $H \leqslant K$, we have $K\phi \leqslant_{fg} G_2$ and $H\phi \leqslant K\phi$ so, $\tilde{\mathrm{r}}(H) = \tilde{\mathrm{r}}(H\phi) \leqslant \mathrm{dc}_{G_2}(H\phi) \cdot \tilde{\mathrm{r}}(K\phi) = \mathrm{dc}_{G_2}(H\phi) \cdot \tilde{\mathrm{r}}(K)$. Therefore, $\mathrm{dc}_{G_1}(H) \leqslant \mathrm{dc}_{G_2}(H\phi)$. By symmetry, we get (i).

Similarly, for every $K \leqslant_{fg} G_1$ with $H \cap K \leqslant_{fg} G_1$, we have $K\phi \leqslant_{fg} G_2$ and $H\phi \cap K\phi = (H \cap K)\phi \leqslant_{fg} G_2$ so, $\tilde{\mathrm{r}}(H \cap K) = \tilde{\mathrm{r}}((H \cap K)\phi) = \tilde{\mathrm{r}}(H\phi \cap K\phi) \leqslant \mathrm{di}_{G_2}(H\phi) \cdot \tilde{\mathrm{r}}(K\phi) = \mathrm{di}_{G_2}(H\phi) \cdot \tilde{\mathrm{r}}(K)$. Therefore, $\mathrm{di}_{G_1}(H) \leqslant \mathrm{di}_{G_2}(H\phi)$. By symmetry, we deduce (ii). $\square$

Corollary 2.2.10. *Let G be a group. For every $H \leqslant_{fg} G$ and every $g \in G$, $\mathrm{dc}_G(H^g) = \mathrm{dc}_G(H)$ and $\mathrm{di}_G(H^g) = \mathrm{di}_G(H)$.* $\square$

We study these notions for the case of the free group and obtain the following result in Section 3.2 of Chapter 3:

Theorem. *(3.2.8) For any finitely generated free group $G = F_n$, the function dc_{F_n} is computable; more precisely, there is an algorithm which, on input $h_1, \ldots, h_r \in F_n$, it computes the value of $\mathrm{dc}_G(\langle h_1, \ldots, h_r \rangle)$ and outputs a free basis of a subgroup $K \leqslant_{fg} F_n$ where it is attained.*

The question whether di_{F_n} is computable (related to the question whether the corresponding supremum is a maximum or not) in free groups seems to be much more delicate. In Section 3.2 we refer to a quite similar question, which was successfully solved recently by S. Ivanov in [20]. However, at the time of writing, we do not know how to use this result to eventually compute di_{F_n}.

Then, we concentrate in free-abelian times free groups, $G = \mathbb{Z}^m \times F_n$, where the situation is richer and trickier because, for $m \geqslant 1$, $n \geqslant 2$, G is known to be non-Howson. In

Sections 3.3 and 3.4 we study the degree of compression and the degree of inertia for these groups, respectively, and prove the main results of Chapter 3:

Theorem. *(3.3.3) For any given $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$, any basis for it $\{t^{a_1}u_1, \ldots, t^{a_r}u_r, t^{b_1}, \ldots, t^{b_s}\}$, and using the notation from Section 3.3, we have*

$$\mathrm{dc}_G(H) = \tilde{\mathrm{r}}(H) \Big/ \min_{J \in \mathcal{AE}_{F_n}(H\pi)} \{\tilde{\mathrm{r}}(J) + d(A, B, U_J)\}.$$

Moreover, $\mathrm{dc}_G(H)$ is algorithmically computable.

Theorem. *(3.4.2) Let $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$, and let $L_H = H \cap \mathbb{Z}^m$.*

(i) If $\mathrm{r}(H\pi) \leqslant 1$ then $\mathrm{di}_G(H) = 1$;

(ii) if $\mathrm{r}(H\pi) \geqslant 2$ and $[\mathbb{Z}^m : L_H] = \infty$ then $\mathrm{di}_G(H) = \infty$;

(iii) if $\mathrm{r}(H\pi) \geqslant 2$ and $[\mathbb{Z}^m : L_H] = l < \infty$ then $\mathrm{di}_G(H) \leqslant l\,\mathrm{di}_{F_n}(H\pi)$.

To have a proper equality formula, we modify the definition of degree of inertia with some extra technical assumptions and define *restricted degree of inertia*.

Definition 2.2.11. Let G be a group, $\pi\colon G \to G/Z(G)$ where $Z(G)$ is the center of the group G. Let $H \leqslant_{fg} G$ such that $H\pi$ is not virtually cyclic and $H\pi \nleqslant [G\pi, G\pi]$. The *restricted degree of inertia* of H in G is $\mathrm{di}'_G(H) = sup_K\{\tilde{\mathrm{r}}(H \cap K)/\tilde{\mathrm{r}}(K)\}$, where the supremum is taken over all $K \leqslant_{fg} G$ satisfying $H \cap K \leqslant_{fg} G$, $[H\pi : H\pi \cap K\pi] = \infty$ and $H\pi \cap K\pi \nleqslant [G\pi, G\pi]$ and here $0/0$ is understood to be 1.

For restricted degree of inertia we prove that:

Theorem. *(3.5.14) Let $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$, such that $H\pi$ is not cyclic and $H\pi \nleqslant [F_n, F_n]$ and let $L_H = H \cap \mathbb{Z}^m$;*

(i) if $[\mathbb{Z}^m : L_H] = \infty$ then $\mathrm{di}'_G(H) = \infty$;

(ii) if $[\mathbb{Z}^m : L_H] = l$ then $\mathrm{di}'_G(H) = l\,\mathrm{di}'_{F_n}(H\pi)$.

2.3 Endomorphisms and automorphisms of $\mathbb{Z}^{\mathbf{m}} \times \mathbf{F_n}$

We shall use lowercase Greek letters for endomorphisms of free groups, $\phi\colon F_n \mapsto F_n$ and uppercase Greek letters for endomorphisms of free-abelian times free groups, $\Psi\colon \mathbb{Z}^m \times F_n \mapsto \mathbb{Z}^m \times F_n$. In particular, $\Gamma_{t^a u} = \Gamma_u \in \mathrm{Inn}(G)$ is the right conjugation by $t^a u$ (or, equivalently, by u).

Delgado–Ventura [10, Props. 5.1] gave a classification of all endomorphisms of $G = \mathbb{Z}^m \times F_n$, $n \geqslant 2$, in two types,

(I) $\Psi_{\phi,Q,P} = t^a u \mapsto t^{aQ+u^{ab}P}(u\phi)$, where $\phi \in \mathrm{End}(F_n)$, $Q \in M_{m\times m}(\mathbb{Z})$, $P \in M_{n\times m}(\mathbb{Z})$, and $u^{ab} \in \mathbb{Z}^n$ is the abelianization of $u \in F_n$.

(II) $\Psi_{z,l,h,Q,P} = t^a u \mapsto t^{aQ+u^{ab}P} z^{al^T+u^{ab}h^T}$, where $1 \neq z \in F_n$ is not a proper power, $Q \in M_{m\times m}(\mathbb{Z})$, $P \in M_{n\times m}(\mathbb{Z})$, $0 \neq l \in \mathbb{Z}^m$, and $h \in \mathbb{Z}^n$.

In the same paper Delgado–Ventura [10, Props. 5.1, 5.2(iii)] also showed that every automorphism Ψ of the group $G = \mathbb{Z}^m \times F_n$, $n \geqslant 2$, is of type (I) with $\phi \in \mathrm{Aut}(F_n)$ and $Q \in \mathrm{GL}_m(\mathbb{Z})$. Furthermore, the composition and inversion of automorphisms work like this:

$$\Psi_{\phi,Q,P}\Psi_{\phi',Q',P'} = \Psi_{\phi\phi',QQ',PQ'+AP'}, \qquad (\Psi_{\phi,Q,P})^{-1} = \Psi_{\phi^{-1},Q^{-1},-A^{-1}PQ^{-1}}, \tag{2.4}$$

where $A \in M_n(\mathbb{Z})$ is the matrix of the abelianization of ϕ; see [10, Lem. 5.4].

Fixed subgroups by morphisms in $\mathbb{Z}^{\mathbf{m}} \times \mathbf{F_n}$

As we have the structure of the endomorphisms and automorphisms, we try to investigate more relevant results (which are already done in free groups F_n and have a deep impact in this line of research) about the fixed point subgroups by automorphisms (and endomorphisms).

Given a set $S \subseteq \operatorname{End}(G)$, let $\operatorname{Fix}(S)$ denote the subgroup of G consisting of those $g \in G$ which are fixed by every element of S, $\operatorname{Fix}(S) = \{g \in G \mid g\phi = g, \forall \phi \in S\} = \cap_{\phi \in S} \operatorname{Fix}(\phi)$, called the *fixed subgroup* of S (read $\operatorname{Fix}(\emptyset) = G$). For simplicity, we write $\operatorname{Fix}\phi = \operatorname{Fix}(\{\phi\})$.

Definition 2.3.1. For any group G and an endomorphism $\phi \in \operatorname{End}(G)$, define its *periodic subgroup* as $\operatorname{Per}\psi = \cup_{p=1}^{\infty} \operatorname{Fix}\psi^p$

Note that this is always a subgroup since $x \in \operatorname{Fix}\psi^p$ and $y \in \operatorname{Fix}\psi^q$ imply $xy \in \operatorname{Fix}\psi^{pq}$. And also observe that $\operatorname{Per}\psi$ contains the lattice of subgroups given by $\operatorname{Fix}\psi^p$, $p \in \mathbb{N}$, with inclusions among them according to divisibility among the exponents: if $r|s$ then $\operatorname{Fix}\phi^r \leqslant \operatorname{Fix}\phi^s$; and also, if $\operatorname{Fix}\phi^r \leqslant \operatorname{Fix}\phi^s$ and $d = \gcd(r,s) = \alpha r + \beta s$, $\alpha, \beta \in \mathbb{Z}$, then $\operatorname{Fix}\phi^r = \operatorname{Fix}\phi^d$ and $d|s$.

Restricting ourselves to the case of finite order automorphisms of $\mathbb{Z}^m \times F_n$, we first uniformly bound the order of automorphisms in terms of the ambient ranks n, m. Then we prove that the rank of the fixed point subgroup is also bounded by some constant (depending upon only n, m) which works for any arbitrary finite order automorphism.

Theorem. *(4.3.1) Let $G = \mathbb{Z}^m \times F_n$, $m, n \geqslant 0$.*

(i) There exists a computable constant $C_1 = C_1(m,n)$ such that, for every $\Psi \in \operatorname{Aut}(G)$ of finite order, $\operatorname{ord}(\Psi) \leqslant C_1$.

(ii) There exists a computable constant $C_2 = C_2(m,n)$ such that, for every $\Psi \in \operatorname{Aut}(G)$ of finite order, $\operatorname{r}(\operatorname{Fix}\Psi) \leqslant C_2$.

We deduce the periodicity formula for endomorphisms of $\mathbb{Z}^m \times F_n$.

Theorem. *(4.4.3) There exists a computable constant $C_3 = C_3(m,n)$ such that $\operatorname{Per}\Psi = \operatorname{Fix}\Psi^{C_3}$, for every $\Psi \in \operatorname{End}(\mathbb{Z}^m \times F_n)$.*

We also prove that the point-wise stabilizer (with respect to automorphisms), $\mathrm{Aut}_H(G)$ (the set of automorphisms of G which fix H point-wise) of a finitely generated subgroup H of $\mathbb{Z}^m \times F_n$ is finitely presented.

Theorem. *(4.5.14) Let $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$, given by a finite set of generators. Then the stabilizer, $\mathrm{Aut}_H(G)$, of H is finitely presented, and a finite set of generators and relations is algorithmically computable.*

We also give an algorithmic computation for the auto-fixed closure (the set of elements fixed by every automorphism fixing H point-wise) of a finitely generated subgroup H of $\mathbb{Z}^m \times F_n$, deciding whether H is auto-fixed or not.

Theorem. *(4.5.18) Let $G = \mathbb{Z}^m \times F_n$. There is an algorithm which, given a finite set of generators for a subgroup $H \leqslant_{fg} G$, outputs a finite set of automorphisms $\Psi_1, \ldots, \Psi_k \in \mathrm{Aut}(G)$ such that $\text{a-Cl}_G(H) = \mathrm{Fix}\,\Psi_1 \cap \cdots \cap \mathrm{Fix}\,\Psi_k$, decides whether this is finitely generated or not and, in case it is, computes a basis for it.*

Corollary. *(4.5.19) One can algorithmically decide whether a given $H \leqslant_{fg} G$ is auto-fixed or not, and in case it is, compute a finite set of automorphisms $\Psi_1, \ldots, \Psi_k \in \mathrm{Aut}(G)$ such that $H = \mathrm{Fix}\,\Psi_1 \cap \cdots \cap \mathrm{Fix}\,\Psi_k$.*

2.4 Finite intersection of subgroups in $\mathbb{Z}^m \times \mathbf{F_n}$

In [10] Delgado–Ventura gave an algorithm which decides if the intersection of two finitely generated subgroups is finitely generated or not and in the affirmative case, they also computed a set of generators of the intersection. In this document, we generalize this result from two to any finite family. In other words, we decide if a finite intersection of finitely generated subgroups is again finitely generated or not and in case it is, compute a set of generators for this intersection.

Theorem. *([5.1.2]) Let $H_1, \ldots, H_k \leqslant_{fg} \mathbb{Z}^m \times F_n$, where k is finite and each H_i is given by a finite set of generators. Then it is algorithmically decidable if $\bigcap_{i=1}^k H_i$ is finitely generated or not and in the affirmative case we can compute generators for $\bigcap_{i=1}^k H_i$.*

At a first glance, the reader may think that this can be done just by induction procedure of two-by-two intersections. But this is not true for $\mathbb{Z}^m \times F_n$. In this context we have this following theorem and based on this theorem we develop a secret sharing scheme.

Theorem. *([5.2.2]) For every $k \geqslant 3$ we can always build a family $\mathcal{F} = \{H_1, H_2, \ldots, H_k\}$ of finitely generated subgroups of $\mathbb{Z}^m \times F_n$ such that for each nonempty subfamily $\mathcal{S} \subseteq \mathcal{F}$:*

$$\bigcap_{i \in \mathcal{S}} H_i \text{ is finitely generated} \Leftrightarrow \#\mathcal{S} \in \{1, k\}. \tag{2.5}$$

The main three projects of this dissertation are explained in chapters 3, 4, 5 and in the following three paragraphs, I try to give a brief overview how the sections are organised in each chapter.

Chapter 3. In Section 3.1, we give a very brief overview of Stallings graph which is extremely used throughout this chapter. In Section 3.2, we connect our definition of degree of inertia with the *Walter Neumann coefficient* of $H \leqslant_{fg} F_n$ as $\sigma(H) := \sup_{K \leqslant_{fg} F_n} \tilde{\mathrm{r}}(H,K)/\tilde{\mathrm{r}}(H)\,\tilde{\mathrm{r}}(K)$, where $\tilde{\mathrm{r}}(H,K) = \sum_{s \in H \backslash F_n / K} \tilde{\mathrm{r}}(H \cap K^s)$ defined by S. Ivanov [20]. We give an effective computation of $\mathrm{dc}_{F_n}(H)$ and a free basis of a subgroup K where the maximum is attained. In Section 3.3, we algorithmically compute the degree of compression $\mathrm{dc}_G(H)$, for any finitely generated subgroup $H \leqslant_{fg} G$, where $G = \mathbb{Z}^m \times F_n$ and in the way of computation we proof some lemmas which reflect the fact that solving problems in $\mathbb{Z}^m \times F_n$ are not always reducible to the corresponding problems in $\mathbb{Z}^m$ and F_n. In Section 3.4, we study the degree of inertia for subgroups H of $G = \mathbb{Z}^m \times F_n$ and relate it to the corresponding degree of inertia of $H\pi$ in F_n; it turns out that the index of $H \cap \mathbb{Z}^m$ in $\mathbb{Z}^m$ (whether finite or infinite) is closely related to the degree of inertia of H. Unfortunately, the situation here is more complicated and we can only prove an upper bound for di_G in terms of di_{F_n} and the previously mentioned index of $H \cap \mathbb{Z}^m$ in $\mathbb{Z}^m$; the computability of this function remains open, as in the free case. In Section 3.5, we add some extra technical conditions to define *restricted degree of inertia*, di'_G, analyze this new notion and succeed in the project of computing $\mathrm{di}'_G(H)$ in terms of $\mathrm{di}'_{F_n}(H\pi)$ and the index $[\mathbb{Z}^m : H \cap \mathbb{Z}^m]$ to provide an equality formula for di'_G.

Chapter 4. In Section 4.1, we collect several folklore facts about $\mathrm{GL}_m(\mathbb{Z})$ for later use; for completeness, we provide proofs highlighting several technical subtleties coming from the fact that $\mathbb{Z}$ is not a field, but just an integral domain. In Section 4.2, we introduce the notion of *factor* for $\mathbb{Z}^m \times F_n$. This notion can be considered as an analogue to the concepts of *direct summand* in $\mathbb{Z}^m$ and *free factor* in F_n. We also prove Takahashi's theorem for $\mathbb{Z}^m \times F_n$ (see Theorem 4.2.5). In Section 4.3, we concentrate on finite order automorphisms of $\mathbb{Z}^m \times F_n$ and show that their fixed subgroups are always finitely generated, with rank globally bounded by a computable constant depending only on the ambient ranks m, n (and not depending on the specific automorphism in use); see Theorem 4.3.1. In Section 4.4, we turn to study periodic points and we manage to extend to free-abelian times free groups, a result known to hold both in free-abelian groups and in free groups: the periodic subgroup of an endomorphism equals the fixed subgroup of a high enough power and, furthermore, this exponent can be taken uniform for all endomorphisms, depending only on the ambient ranks m, n; see Theorem 4.4.3. In Section 4.5, we consider the auto-fixed closure of a finitely generated subgroup H; we prove that it always equals a finite intersection of fixed subgroups, we compute the candidate automorphisms, we decide whether it is finitely generated or not, and in case it is, we effectively compute a basis for it; see Theorem 4.5.18. As a consequence, we obtain an algorithm to decide whether a given finitely generated subgroup $H \leqslant \mathbb{Z}^m \times F_n$

is auto-fixed or not; see Corollary 4.5.19. To achieve this goal, we make use of a recent result by M. Day about stabilizers of tuples of conjugacy classes in right angled Artin groups being finitely presented, and we prove the analogous version for tuples of exact elements in $\mathbb{Z}^m \times F_n$. In fact, we only need finite generation and computability of these stabilizers; however, for completeness, we also prove its finite presentability.

Chapter 5. In Section 5.1, we give an algorithm to decide if any finite intersection of finitely generated subgroups of $\mathbb{Z}^m \times F_n$ is finitely generated or not and in affirmative case, this algorithm also computes a generating set for the finite intersection. In Section 5.2, we develop a secret sharing scheme based on the existence of a finite family of finitely generated subgroups of $\mathbb{Z}^m \times F_n$, whose any intermediate intersection is not finitely generated but the whole intersection is finitely generated.

3 Degrees of compression and inertia for free-abelian times free groups

"Pure mathematics is, in its way, the poetry of logical ideas."

— **Albert Einstein**

In this chapter we introduce the concepts of degree of inertia, $\mathrm{di}(H)$, and degree of compression, $\mathrm{dc}(H)$, of a finitely generated subgroup H of a given group G. For the case of direct products of free-abelian and free groups, we compute the degree of compression algorithmically and give an upper bound for the degree of inertia. Also with some technical assumptions, we produce an equality formula for restricted degree of inertia connecting it with $H\pi$, the projection of H into F_n.

In this aspect of research, the story began with **Scott Conjecture** [16], for every $\phi \in \mathrm{Aut}(F_n), \mathrm{r}(\mathrm{Fix}(\phi)) \leqslant n$ and they showed that $\mathrm{Fix}(\phi)$ is a free factor of F_n for every finite order automorphism $\phi \in \mathrm{Aut}(F_n)$. In 1988, Bestvina and Handel proved this conjecture. In their very influential paper [3] published in the Annals of Mathematics, Bestvina and Handel proved that every irreducible automorphism of F_n has a train-track [3] representative. In the same paper they introduced the notion of a relative train-track and applied train-track methods to solve the Scott Conjecture; it is the main result in this line of research. In 1989 Imrich-Turner [19] extended the result to endos, i.e., for $\phi \in \mathrm{End}(F_n), \mathrm{r}(\mathrm{Fix}(\phi)) \leqslant n$. Then in the paper [14] Dicks–Ventura introduced the concepts of compression and inertia (see the definition 2.2.1). In this context the most important results are depicted together in Theorem 2.2.4.

We quantify these concepts by introducing degree of compression and degree of inertia (see the definition 2.2.5). Both in F_n and $G = \mathbb{Z}^m \times F_n$, for given subgroup H with a set of generators, we are able to compute $\mathrm{dc}_{F_n}(H)$ and $\mathrm{dc}_G(H)$ algorithmically. And for degree inertia, we deduce formulae for $\mathrm{di}_G(H)$ and $\mathrm{di}'_G(H)$ in terms of $\mathrm{di}_{F_n}(H\pi)$.

3.1 Stallings graph for free groups

In classical fashion, algebra was used for fruitful resolution of geometric problems. But during second half of the past century, researchers like Artin, Gromov, Stallings, Tits and Thurston, among others, created geometric methods and useful tools for the study of algebraic objects. Stallings automata for free groups is one of these tools. In an influential paper J. R. Stallings [37] gave a bijection between finitely generated subgroups of free group and Stallings automata. In the subsequent literature this automaton became crucial for the modern understanding of the lattice of subgroups of free group. A *Stallings automata* is a finite Z-labeled oriented graph $(\Gamma_Z, \odot)$ with a distinguished vertex $\odot$ called base point such that,

(1) Γ_Z is connected,

(2) no vertex of degree 1 except possibly $\odot$ (Γ_Z is a core-graph),

(3) no two edges with the same label go out of (or in to) the same vertex.

To any given Stallings automaton Γ_Z, we can associate its fundamental group:

$$\pi(\Gamma_Z) = \{ \text{ labels of closed paths at } \odot\} \leqslant F(Z),$$

clearly, a subgroup of $F(Z)$, the free group on the alphabet set Z, also denoted by F_n. In any automaton if we have the situation like Fig 3.1, we can fold and identify vertices. This operation is called Stallings folding. The fundamental group of the Stallings automaton

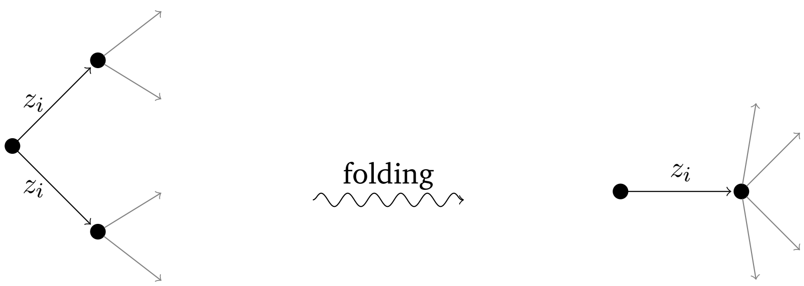

Figure 3.1: Stallings folding

remains unchanged under this operation. The Stallings automata is a very useful tool to solve membership problem, subgroup intersection problem and finite index problem for free groups.

The first picture of Fig 3.2 depicts the flower automaton for $H = \langle baba^{-1}, aba^{-1}, aba^2 \rangle$. After doing consecutive foldings, as a final picture, we get the Stallings automata for H. From Stallings Lemma, $\pi(\Gamma(H), \odot) = \langle b, aba^{-1}, a^3 \rangle$ which is the same as the subgroup $H = \langle baba^{-1}, aba^{-1}, aba^2 \rangle$.

3.2 Degrees of compression and inertia for the free group

In the present section we study the degrees of compression and inertia in the context of the free group, i.e., the functions dc_{F_n} and di_{F_n}.

Hanna Neumann proved in [33] that $\tilde{\mathrm{r}}(H \cap K) \leqslant 2\,\tilde{\mathrm{r}}(H)\,\tilde{\mathrm{r}}(K)$, for every $H, K \leqslant_{fg} F_n$. And the same assertion removing the factor "2" became soon known as the **Hanna Neumann conjecture**. This has been a major problem in Geometric Group Theory, with lots of partial results and improvements appearing in the literature since then.

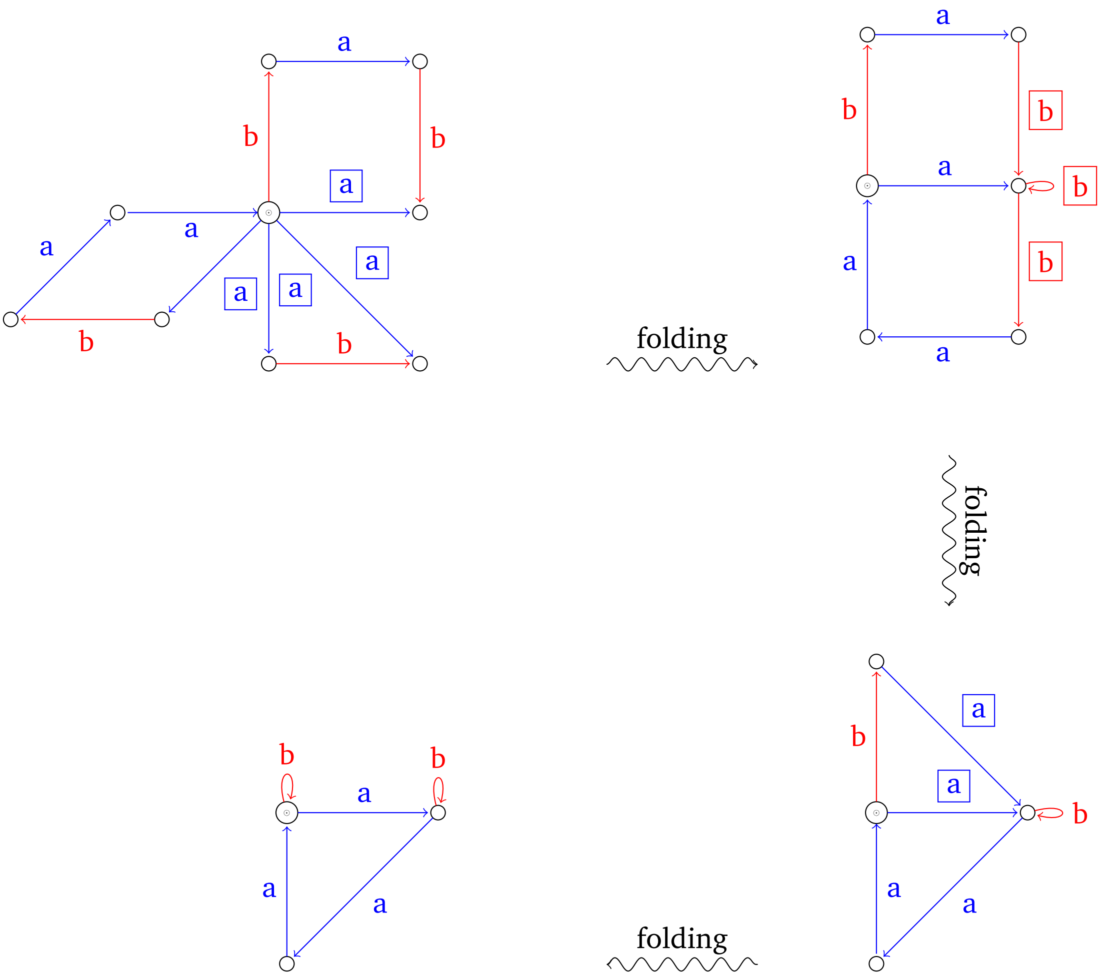

Figure 3.2: Stallings automata

An interesting one was done by W. Neumann in [34], who proved the stronger fact $\sum_{s\in S} \tilde{\mathrm{r}}(H \cap K^s) \leqslant 2\,\tilde{\mathrm{r}}(H)\,\tilde{\mathrm{r}}(K)$ (known as the strengthen Hanna Neumann inequality), where S is any set of double coset representatives of F_n modulo H on the left and K on the right (i.e., $S \subseteq F_n$ contains one and only one element in each double coset $H\backslash F_n/K$); in particular, this implies that, for all $H, K \leqslant_{fg} F_n$, *all except finitely many* of the intersections $H \cap K^s$ are trivial or cyclic. Few years ago the Hanna Neumann conjecture, even in its strengthen version, has been completely resolved in the positive, independently by J. Friedman [18] and by I. Mineyev [32] (see also W. Dicks [13]). This can be interpreted as the following upper bound for $\mathrm{dc}_{F_n}(H)$ and $\mathrm{di}_{F_n}(H)$ in terms of the subgroup $H \leqslant_{fg} F_n$:

Observation 3.2.1. *For $H \leqslant_{fg} F_n$, we have $1 \leqslant \mathrm{dc}_{F_n}(H) \leqslant \mathrm{di}_{F_n}(H) \leqslant \tilde{\mathrm{r}}(H)$.*

Friedman–Mineyev's inequality is easily seen to be tight (consider, for example, the subgroups $H = \langle a, b^{-1}ab\rangle$ and $K = \langle b, a^2, aba\rangle$ of F_2, and its intersection $H \cap K = \langle a^2, b^{-1}a^2b, b^{-1}aba\rangle$); therefore, it can be interpreted in the following way: "the smallest possible multiplicative constant $\alpha \in \mathbb{R}$ satisfying $\tilde{\mathrm{r}}(H \cap K) \leqslant \alpha\,\tilde{\mathrm{r}}(H)\,\tilde{\mathrm{r}}(K)$, for every $H, K \leqslant_{fg} F_n$, is $\alpha = 1$". Now fix the subgroup H: by definition, the smallest possible constant $\alpha \in \mathbb{R}$ satisfying $\tilde{\mathrm{r}}(H \cap K) \leqslant \alpha\,\tilde{\mathrm{r}}(H)\,\tilde{\mathrm{r}}(K)$, for every $K \leqslant_{fg} F_n$, is $\alpha = \frac{\mathrm{di}_{F_n}(H)}{\tilde{\mathrm{r}}(H)}$.

S. Ivanov [20] already considered and studied the strengthened version of what we call here the degree of inertia. He defined the *Walter Neumann coefficient* of $H \leqslant_{fg} F_n$ as $\sigma(H) := \sup_{K\leqslant_{fg} F_n}\{\tilde{\mathrm{r}}(H,K)/\,\tilde{\mathrm{r}}(H)\,\tilde{\mathrm{r}}(K)\}$, where $\tilde{\mathrm{r}}(H,K) = \sum_{s\in H\backslash F_n/K} \tilde{\mathrm{r}}(H \cap K^s)$ (understanding $0/0 = 1$). In other words, $\sigma(H)$ is the smallest possible constant $\alpha \in \mathbb{R}$ such that $\tilde{\mathrm{r}}(H,K) \leqslant \alpha\,\tilde{\mathrm{r}}(H)\,\tilde{\mathrm{r}}(K)$, for every $K \leqslant_{fg} F_n$. Using linear programming techniques, Ivanov was able to prove the following remarkable result:

Theorem 3.2.2 (Ivanov, [20])**.** *For any finitely generated free group F_n, the function σ is computable and the supremum is a maximum; more precisely, there is an algorithm which,* *on input $h_1, \ldots, h_r \in F_n$, it computes the value of $\sigma(\langle h_1, \ldots, h_r \rangle)$ and outputs a free basis of a subgroup $K \leqslant_{fg} F_n$ where that supremum is attained.*

Ivanov's proof is involved and technical. Although it looks quite similar, we have been unable to adapt Ivanov's arguments to answer any of the following questions which, as far as we know, remain open:

Question 3.2.3. *Is the function di_{F_n} computable? Is that supremum always a maximum? More precisely, is there and algorithm which, on input $h_1, \ldots, h_r \in F_n$, it computes the value of $\mathrm{di}_{F_n}(\langle h_1, \ldots, h_r \rangle)$? Or even more, it outputs a free basis of a subgroup $K \leqslant_{fg} F_n$ where it is attained?*

The corresponding questions for the degree of compression are much easier and can be established with the use of Stallings graphs, algebraic extensions, and Takahasi's Theorem.

Definition 3.2.4. Let $H \leqslant_{fg} K \leqslant_{fg} F_n$. If H is a free factor of K we write $H \leqslant_{ff} K$. On the other extreme, the extension $H \leqslant K$ is said to be *algebraic extension*, denoted as $H \leqslant_{alg} K$, if H is not contained in any proper free factor of K, i.e., if $H \leqslant A \leqslant_{ff} K = A * B$ implies $B = 1$; we denote by $\mathcal{AE}_{F_n}(H)$ the set of algebraic extensions of H in F_n.

It is known that any finitely generated subgroup H of F_n has finitely many algebraic extensions, i.e., $|\mathcal{AE}_{F_n}(H)| < \infty$. This was proved long time ago by Takahasi, see [39], and reproved independently by Ventura [40], Kapovich–Miasnikov [22] and Margolis–Sapir–Weil [24] with later unification by Miasnikov–Ventura–Weil in [31]. We offer here an sketch of this modern proof.

Given a finitely generated subgroup $H \leqslant_{fg} F_n$ one can depict its Stallings graph $\Gamma(H)$ (which is finite) and start identifying its vertices in all possible ways, each followed by a sequence of Stallings foldings until getting a genuine new Stallings automata. Clearly, each of these finitely many Stallings graph obtained in this way correspond to a new finitely generated subgroup H' such that $H \leqslant_{fg} H' \leqslant_{fg} F_n$; we call it an *overgroup* of H. Consider the finite set of overgroups of H, denoted $\mathcal{O}(H)$, and call it the *fringe of* H; observe that $H \in \mathcal{O}(H)$ corresponding to the trivial identification of vertices (no identification at all).

It is not difficult to see that $\mathcal{AE}_{F_n}(H) \subseteq \mathcal{O}(H)$. However, as seen in the following example, $\mathcal{O}(H)$ could contain pairs of different subgroups H' and H'' one being a free factor of the other, $H' \leqslant_{ff} H''$; in this case, H'' is obviously not an algebraic extension of H and can be eliminated from the list. Following this cleaning process until there are no free factors among the members of the reduced list, one gets the set of *algebraic extension* of H. It is worth to mention that $\mathcal{O}(H)$ depends upon the se of generators of the ambient group but $\mathcal{AE}_{F_n}(H)$ does not depend on the set of generators.

Example 3.2.5. Let $H = \langle b^2, ac^{-1}ac^{-1}, bac^{-1}\rangle$, and Fig. 3.3 represents the Stallings graph $\Gamma(H)$ for H as a subgroup of F_3 with respect to the ambient free basis $A = \{a, b, c\}$. Successively identifying pairs of vertices of $\Gamma(H)$ and reducing the resulting A-labeled graph in all possible ways, one concludes that $\Gamma(H)$ has nine congruences, whose corresponding quotient graphs are depicted in Figs. 3.3 and 3.4; this is the so-called fringe of H, $\mathcal{O}(H)$.

Thus the A-fringe of H consists of $\mathcal{O}(H) = \{H_0, H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8\}$, where $H_0 = H$, $H_1 = \langle a, bc, b^2, bac^{-1}\rangle$, $H_2 = \langle b, ac^{-1}\rangle$, $H_3 = \langle c, ba, b^2, acb^{-1}\rangle$, $H_4 = \langle b^2, ba^{-1}, bca^{-1}, ca^{-1}b^{-1}\rangle$, $H_5 = \langle ac^{-1}b^{-1}, bab^{-1}, b^2, bc^{-1}\rangle$, $H_6 = \langle b^2, a, bab^{-1}, bc^{-1}, bc\rangle$, $H_7 = \langle c, b^2, ab^{-1}, ba, bcb^{-1}\rangle$ and $H_8 = \langle a, b, c\rangle = F_3$.

Let us clean now this set $\mathcal{O}(H)$. It is a well know fact that if N is obtained from M by a single identification of two vertices followed by the necessary foldings (i.e., if N is generated by M and a single extra generator) then $M \leqslant_{ff} N$ whenever $r(N) = r(M)+1$,

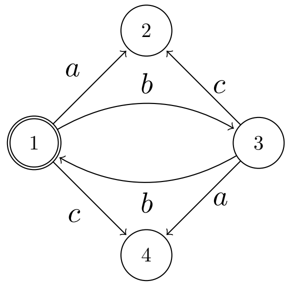

Figure 3.3: Stallings graph of H

and $M \leqslant_{alg} N$ otherwise. With this fact we deduce that, $H_1, H_3, H_4, H_5 \leqslant_{ff} H$, $H_6 \leqslant_{ff} H_1, H_5$, $H_7 \leqslant_{ff} H_3, H_4$ and $H_8 \leqslant_{ff} H_2$ and obtain $\mathcal{AE}_{F_n}(H) = \{H, \langle b, ac^{-1}\rangle\}$.

Theorem 3.2.6 (Takahasi, [39]; see also [22], [31], [24], [41])**.** *Every $H \leqslant_{fg} F_n$ has finitely many algebraic extensions, say $\mathcal{AE}_{F_n}(H) = \{H = H_0, H_1, \ldots, H_r\}$ (r depending on H), each H_i is finitely generated, and free bases for all of them are algorithmically computable from a given set of generators for H. Furthermore, for every extension $H \leqslant K \leqslant F_n$, there exists a unique (and computable) $0 \leqslant i \leqslant r$ such that $H \leqslant_{alg} H_i \leqslant_{ff} K$; this H_i is called the K*-algebraic closure of H.

Sketch of the proof. The original proof by M. Takahasi [39] was combinatorial, playing with words and cancellation in the free group. We sketch the modern proof given in [31] following ideas of Ventura [40], Kapovich–Miasnikov [22] and Margolis–Sapir–Weil [24].

We have the alphabet Z fixed as a free basis for the ambient free group, $F_n = F(Z)$. Now, given generators for $H \leqslant_{fg} F(Z)$, one can compute the Stallings graph $\Gamma(H)$ for H (denote the basepoint by $\odot$). Attaching the necessary infinite hanging trees so that it becomes a complete graph (i.e., with all vertices having an incoming and an outgoing edge labelled a for every $a \in Z$), we obtain the Schreier graph $\chi(F_n, H, Z)$ (which is finite if and only if H is of finite index in F_n). Of course, $\chi(F_n, H, Z)$ is a covering, $\chi(F_n, H, Z) \twoheadrightarrow R(Z)$, of the bouquet $R(Z)$, the graph with a single vertex and one loop labelled a for every $a \in Z$; more precisely, it is the covering of $R(Z)$ corresponding to

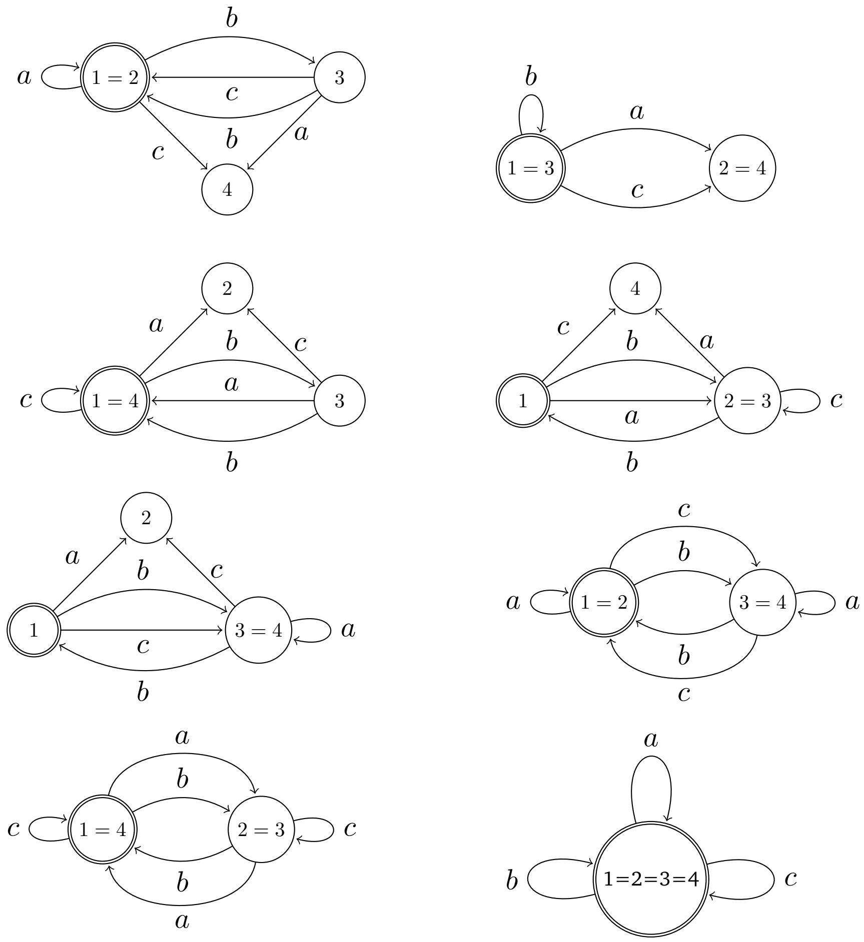

Figure 3.4: The eight non-trivial quotients of $\Gamma_A(H)$

the subgroup $H \leqslant_{fg} \pi(R(Z)) = F_n$. By standard covering theory, $K \leftrightarrow \chi(F_n, K, Z)$ is a bijection between intermediate subgroups $H \leqslant K \leqslant F_n$ and intermediate coverings, $\chi(F_n, H, Z) \twoheadrightarrow \chi(F_n, K, Z) \twoheadrightarrow R(Z)$ (mapping finitely generated subgroups to graphs with finite core, and vice-versa).

Fix $H \leqslant_{fg} K \leqslant_{fg} F_n$, and consider their Stallings graphs $\Gamma(H) = \mathrm{core}(\chi(F_n, H, Z))$ and $\Gamma(K) = \mathrm{core}(\chi(F_n, K, Z))$, both being finite graphs. The above bijection means that $\chi(F_n, K, Z)$ is a quotient of $\chi(F_n, H, Z)$, i.e., the result of $\chi(F_n, H, Z)$ after identifying vertices and edges in a compatible way (i.e., modulo a *congruence*, an equivalence relation satisfying that if $p \sim q$ and e_1 and e_2 are edges with the same label and $\iota e_1 = \iota e_2 = p$, then $e_1 \sim e_2$). There are two cases: if no pair of vertices in $\Gamma(H) \leqslant \chi(F_n, H, Z)$ become identified then $\Gamma(H)$ is a subgraph of $\Gamma(K) = \mathrm{core}(\chi(F_n, K, Z))$ and so, $H \leqslant_{ff} K$; otherwise, we loose H from the picture, but we can still say that some compatible quotient of $\Gamma(H)$ will be visible as a subgraph of $\Gamma(K)$. Since $\Gamma(H)$ is finite, it has finitely many compatible quotients and, therefore, computing all of them and computing free basis for their fundamental groups, we obtain a finite list of subgroups $\mathcal{O}_{F_n}(H) = \{H = H_0, H_1, \ldots, H_s\}$ (s depending on H), called *fringe* of H in [31], all of them containing H and satisfying the following property: *for every $H \leqslant_{fg} K \leqslant_{fg} F_n$ there exists (a non necessarily unique) $i = 0, \ldots, s$ such that $H \leqslant H_i \leqslant_{ff} K$.*

It only remains to clean this list by checking, for each pair of indices i, j, whether $H_i \leqslant_{ff} H_j$ and, in this case, delete H_j from the list. It is not difficult to see that the resulting reduced list is precisely $\mathcal{AE}_{F_n}(H) \subseteq \mathcal{O}_{F_n}(H)$. Uniqueness of the K-algebraic closure follows directly from the definition of algebraic extension. $\square$

Takahasi Theorem ensures us that when we are computing $\mathrm{dc}_{F_n}(H)$, we can restrict ourselves only into algebraic extensions of H. Hence we have the following easy Corollary which leads us to prove one of our results, Theorem 3.2.8.

Corollary 3.2.7. *For any subgroup* $H \leqslant_{fg} F_n$*, we have* $\operatorname{dc}_{F_n}(H) = \sup_{H \leqslant K \leqslant_{fg} F_n} \{\tilde{\mathrm{r}}(H)/\tilde{\mathrm{r}}(K)\} = \max_{K \in \mathcal{AE}_{F_n}(H)} \{\tilde{\mathrm{r}}(H)/\tilde{\mathrm{r}}(K)\}$*; furthermore, we can effectively compute* $\operatorname{dc}_{F_n}(H)$ *and a free basis of a subgroup* K *where the maximum is attained.*

Proof. By Theorem 3.2.6, every $H \leqslant K \leqslant_{fg} F_n$ uniquely determines the K-algebraic closure of H, i.e., an $H' \in \mathcal{AE}_{F_n}(H)$ such that $H \leqslant_{alg} H' \leqslant_{ff} K$. Therefore, since $\tilde{\mathrm{r}}(H') \leqslant \tilde{\mathrm{r}}(K)$, we can restrict the supremum in the definition of $\operatorname{dc}_{F_n}(H)$ to those subgroups in $\mathcal{AE}_{F_n}(H)$. Since $|\mathcal{AE}_{F_n}(H)|$ is finite and computable, this supremum is a maximum and we can effectively compute both $\operatorname{dc}_{F_n}(H)$ and a free basis of a subgroup K where the maximum is attained. □

Theorem 3.2.8. *For any finitely generated free group* $G = F_n$*, the function* dc_{F_n} *is computable; more precisely, there is an algorithm which, on input* $h_1, \ldots, h_r \in F_n$*, computes the value of* $\operatorname{dc}_G(\langle h_1, \ldots, h_r \rangle)$ *and outputs a free basis of a subgroup* $K \leqslant_{fg} F_n$ *where it is attained.*

Proof. The proof is immediate from Corollary 3.2.7. □

3.3 Degree of compression in free-abelian times free groups

For the rest of the Chapter we work in *free-abelian times free* groups $G = \mathbb{Z}^m \times F_n$, investigating here the degrees of compression and inertia of subgroups. More precisely, in this section we study the degree of compression of a given subgroup $H \leqslant_{fg} G$. The following lemma says that it is enough to consider those overgroups K such that $H\pi \leqslant K\pi$ is an algebraic extension.

Lemma 3.3.1. *Let $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$. Then,*

$$\mathrm{dc}_G(H) := \sup_{H \leqslant K \leqslant_{fg} G} \left\{ \frac{\tilde{\mathrm{r}}(H)}{\tilde{\mathrm{r}}(K)} \right\} = \max_{\substack{H \leqslant K \leqslant_{fg} G \\ H\pi \leqslant_{alg} K\pi}} \left\{ \frac{\tilde{\mathrm{r}}(H)}{\tilde{\mathrm{r}}(K)} \right\}.$$

Proof. We already observed above that the supremum defining the degree of compression is always a maximum. The inequality $\geqslant$ is clear.

Fix a basis for H, say $\{t^{a_1}u_1, \ldots, t^{a_r}u_r, t^{b_1}, \ldots, t^{b_s}\}$. To see the other inequality, take a subgroup $H \leqslant K \leqslant_{fg} G$ and we shall construct $H \leqslant K' \leqslant_{fg} G$ such that $H\pi \leqslant_{alg} K'\pi$ and $\tilde{\mathrm{r}}(H)/\,\tilde{\mathrm{r}}(K) \leqslant \tilde{\mathrm{r}}(H)/\,\tilde{\mathrm{r}}(K')$.

We have $L_H = H \cap \mathbb{Z}^m = \langle t^{b_1}, \ldots, t^{b_s} \rangle \leqslant K \cap \mathbb{Z}^m = L_K$ and $H\pi \leqslant K\pi$ so, $\mathrm{r}(L_H) \leqslant \mathrm{r}(L_K)$ and $H\pi \leqslant_{alg} J \leqslant_{ff} K\pi$, for some $J \in \mathcal{AE}_{F_n}(H\pi)$. Take a free basis $\{v_1, \ldots, v_p\}$ for J and extend it to a free basis $\{v_1, \ldots, v_p, v_{p+1}, \ldots, v_q\}$ for $K\pi$, $p \leqslant q$. Now, consider a basis for K of the form $\{t^{c_1}v_1, \ldots, t^{c_p}v_p, t^{c_{p+1}}v_{p+1}, \ldots, t^{c_q}v_q, t^{d_1}, \ldots, t^{d_\ell}\}$, where $c_i \in \mathbb{Z}^m$, $i = 1, \ldots, q$, are certain vectors, and $\{t^{d_1}, \ldots, t^{d_\ell}\}$ is a free-abelian basis for L_K.

Let, $K' = \langle t^{c_1}v_1, \ldots, t^{c_p}v_p, t^{d_1}, \ldots, t^{d_\ell} \rangle \leqslant_{fg} K \leqslant G$ and we claim that $H \leqslant K'$. In fact, we already know that $t^{b_i} \in L_H \leqslant L_K = L_{K'} = \langle t^{d_1}, \ldots, t^{d_\ell} \rangle \leqslant K'$ for $i = 1, \ldots, s$. Now, for $i = 1, \ldots, r$ we see that $t^{a_i}u_i \in K'$: write u_i as a word $u_i = w_i(v_1, \ldots, v_p)$ (unique up to reduction) and compute $w_i(t^{c_1}v_1, \ldots, t^{c_p}v_p) = t^{e_i}w_i(v_1, \ldots, v_p) = t^{e_i}u_i \in K' \leqslant K$, where $e_i = |w_i|_{v_1}c_1 + \cdots + |w_i|_{v_p}c_p$. But $t^{a_i}u_i \in H \leqslant K$ so, $t^{e_i - a_i} \in L_K = L_{K'} \leqslant K'$ and hence, $t^{a_i}u_i = (t^{e_i - a_i})^{-1}(t^{e_i}u_i) \in K'$.

So, for every $H \leqslant K \leqslant_{fg} G$ we have found a finitely generated subgroup in between, $H \leqslant K' \leqslant K$, such that $H\pi \leqslant_{alg} J = K'\pi$ and

$$\tilde{\mathrm{r}}(K') = \tilde{\mathrm{r}}(K'\pi) + \mathrm{r}(L_{K'}) = (p-1) + \mathrm{r}(L_{K'}) \leqslant (q-1) + \mathrm{r}(L_K) = \tilde{\mathrm{r}}(K);$$

therefore, $\tilde{\mathrm{r}}(H)/\tilde{\mathrm{r}}(K') \geqslant \tilde{\mathrm{r}}(H)/\tilde{\mathrm{r}}(K)$ and the proof is completed. $\square$

Now we are in position to explain the term $d(A, B, U)$ involved in Theorem 3.3.3. First we fix $H \leqslant_{fg} G$, a basis for it $\{t^{a_1}u_1, \ldots, t^{a_r}u_r, t^{b_1}, \ldots, t^{b_s}\}$, and consider the matrices

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} \in M_{r\times m}(\mathbb{Z}) \text{ and } B = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix} \in M_{s\times m}(\mathbb{Z}).$$

For every $J \in \mathcal{AE}_{F_n}(H\pi)$ given with a free basis, say $J = \langle v_1, \ldots, v_p\rangle$, we can consider the (unique reduced) word expressing each u_i in terms of $v_1, \ldots, v_p$, say $u_i = w_i(v_1, \ldots, v_p)$, abelianize, and get the vector $(|w_i|_{v_1}, \ldots, |w_i|_{v_p}) \in \mathbb{Z}^p$, $i = 1, \ldots, r$; collecting all of them into the rows of a matrix, we have the following matrix U_J:

$$U_J = \begin{pmatrix} |w_1|_{v_1} & \cdots & |w_1|_{v_p} \\ & \vdots & \\ |w_r|_{v_1} & \cdots & |w_r|_{v_p} \end{pmatrix} \in M_{r\times p}(\mathbb{Z}).$$

According to Lemma 3.3.1, to compute $\mathrm{dc}_G(H)$ it is enough to consider the subgroups of the form $K = \langle t^{c_1}v_1, \ldots, t^{c_p}v_p, L_K\rangle \leqslant_{fg} G$ (where $L_K = K \cap \mathbb{Z}^m$ and assume the given set of generators to be a basis of K) such that $H \leqslant K \leqslant G$, $H\pi = \langle u_1, \ldots, u_r\rangle \leqslant_{alg} K\pi = \langle v_1, \ldots, v_p\rangle$, compute $\tilde{\mathrm{r}}(H)/\tilde{\mathrm{r}}(K)$, and take the maximum of these values. Observe that, although $|\mathcal{AE}_{F_n}(H\pi)| < \infty$, there are, possibly, infinitely many such K's; however, $\tilde{\mathrm{r}}(K) = p - 1 + \mathrm{r}(L_K)$ takes only finitely many values.

So, fix such a K and consider the matrix

$$C_K = \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix} \in M_{p\times m}(\mathbb{Z}).$$

Observe that C_K satisfies $\operatorname{row}(A - U_{K\pi}C_K) \leqslant L_K$: in fact, for every $i = 1, \ldots, r$, we have

$$K \ni w_i(t^{c_1}v_1, \ldots, t^{c_p}v_p) = t^{|w_i|_{v_1}c_1+\cdots+|w_i|_{v_p}c_p} w_i(v_1, \ldots, v_p) = t^{(U_{K\pi})_i C_K} u_i,$$

where $(U_{K\pi})_i$ is the i-th row of $U_{K\pi}$; therefore, $H \leqslant K$ implies that $a_i - (U_{K\pi})_i C_K \in L_K$, for $i = 1, \ldots, r$. This motivates the following definition, which allows us to obtain the main result in this section.

Definition 3.3.2. For given matrices $A \in M_{r\times m}(\mathbb{Z})$, $B \in M_{s\times m}(\mathbb{Z})$, and $U \in M_{r\times p}(\mathbb{Z})$, define $d(A, B, U) = \min_{L\leqslant \mathbb{Z}^m}\{\mathrm{r}(L) \mid \exists\, C \in M_{p\times m}(\mathbb{Z}) \text{ such that } \operatorname{row}(A - UC) \leqslant L, \text{ and } \operatorname{row}(B) \leqslant L\}$.

Theorem 3.3.3. *For any given subgroup $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$, with basis $\{t^{a_1}u_1, \ldots, t^{a_r}u_r, t^{b_1}, \ldots, t^{b_s}\}$, and using the notation above, we have*

$$\operatorname{dc}_G(H) = \tilde{\mathrm{r}}(H) \Big/ \min_{J \in \mathcal{AE}_{F_n}(H\pi)} \{\tilde{\mathrm{r}}(J) + d(A, B, U_J)\}.$$

Moreover, $\operatorname{dc}_G(H)$ is algorithmically computable.

Proof. By Lemma 3.3.1, we know that the supremum in $\mathrm{dc}_G(H)$ is attained in a certain $H \leqslant K \leqslant_{fg} G$ such that $K\pi \in \mathcal{AE}_{F_n}(H\pi)$. And, for every such K, $\tilde{\mathrm{r}}(K) = \tilde{\mathrm{r}}(K\pi) + \mathrm{r}(L_K)$ so,

$$\mathrm{dc}_G(H) = \max_{\substack{H \leqslant K \leqslant_{fg} G \\ H\pi \leqslant_{alg} K\pi}} \left\{ \frac{\tilde{\mathrm{r}}(H)}{\tilde{\mathrm{r}}(K)} \right\} = \max_{J \in \mathcal{AE}_{F_n}(H\pi)} \left\{ \frac{\tilde{\mathrm{r}}(H)}{\tilde{\mathrm{r}}(J) + d(A, B, U_J)} \right\} =$$

$$= \frac{\tilde{\mathrm{r}}(H)}{\min_{J \in \mathcal{AE}_{F_n}(H\pi)} \{\tilde{\mathrm{r}}(J) + d(A, B, U_J)\}} \tag{3.1}$$

since, by the argument above, every K with $K\pi = J \in \mathcal{AE}_{F_n}(H\pi)$ satisfies $\mathrm{r}(L_K) \geqslant d(A, B, U_J)$, one of them with equality.

In order to compute the value of $\mathrm{dc}_G(H)$ we can do the following: first compute $\mathcal{AE}_{F_n}(H\pi)$; for each member $J = \langle v_1, \ldots, v_p \rangle$, write each u_i in the free basis of $H\pi$ in terms of the free basis $\{v_1, \ldots, v_p\}$ of J, and obtain the matrix U_J; then compute $d(A, B, U_J) + \tilde{\mathrm{r}}(J)$ (which is effectively doable by the following Proposition 3.3.4). When this procedure is done for each of the finitely many $J \in \mathcal{AE}_{F_n}(H\pi)$, take the minimum of the values $d(A, B, U_J) + \tilde{\mathrm{r}}(J)$ and, by (3.1), we are done. $\square$

Proposition 3.3.4. *For any given matrices $A \in M_{r \times m}(\mathbb{Z})$, $B \in M_{s \times m}(\mathbb{Z})$, and $U \in M_{r \times p}(\mathbb{Z})$, the value of $d(A, B, U)$ is algorithmically computable, together with a free-abelian basis of an $L \leqslant \mathbb{Z}^m$ attaining the minimum, and the corresponding matrix $C \in M_{p \times m}(\mathbb{Z})$.*

Proof. Recall that $d(A, B, U)$ is the minimum rank of those subgroups $L \leqslant \mathbb{Z}^m$ satisfying $\mathrm{row}(B) \leqslant L$, and $\mathrm{row}(A - UC) \leqslant L$ for some $C \in M_{p \times m}(\mathbb{Z})$. Observe first that, replacing B by B' with $\mathrm{row}(B) \leqslant_{fi} \mathrm{row}(B') \leqslant_{\oplus} \mathbb{Z}^m$, we have $d(A, B', U) = d(A, B, U)$; in fact, $d(A, B', U) \geqslant d(A, B, U)$ is clear from the definition, and for every $L \leqslant \mathbb{Z}^m$ containing $\mathrm{row}(B)$ and $\mathrm{row}(A - UC)$ for some $C \in M_{p \times m(\mathbb{Z})}$, we have the subgroup $L + \mathrm{row}(B') \leqslant \mathbb{Z}^m$ which contains $\mathrm{row}(B')$ and $\mathrm{row}(A - UC)$ for the same matrix C, and has the same rank, $\mathrm{r}(L + \mathrm{row}(B')) = \mathrm{r}(L)$, since $L \leqslant_{fi} L + \mathrm{row}(B')$; this proves the equality.

Let us do a few reductions to the problem. Compute matrices $P \in \mathrm{GL}_r(\mathbb{Z})$, $Q \in \mathrm{GL}_p(\mathbb{Z})$, and positive integers $d_1, \ldots, d_\ell \in \mathbb{N}$, $\ell \leqslant \min\{r, p\}$, satisfying $1 \leqslant d_1 | d_2 | \cdots | d_\ell \neq 0$, such that $PUQ = U'$, where $U' = \mathrm{diag}(d_1, \ldots, d_\ell) \in M_{r \times p}(\mathbb{Z})$ (understanding the last $r - \ell \geqslant 0$ rows and the last $p - \ell \geqslant 0$ columns full of zeros); this is the Smith normal form of U, see [2] for details. Writing $A' = PA$, $B' = B$, and doing the change of variable $C = QC'$, we have $\mathrm{row}(A - UC) = \mathrm{row}(PA - PUQC') = \mathrm{row}(A' - U'C')$. So, $d(A, B, U) = d(A', B', U')$.

To compute $d(A', B', U')$, we have to find a subgroup $L \leqslant \mathbb{Z}^m$ of the minimum possible rank, and vectors $c'_1, \ldots, c'_p \in \mathbb{Z}^m$, such that $\mathrm{row}(B') \leqslant L$,

$$\left.\begin{array}{c} a'_1 - d_1 c'_1 \in L \\ \cdots \\ a'_\ell - d_\ell c'_\ell \in L \end{array}\right\}, \tag{3.2}$$

and

$$\left.\begin{array}{c} a'_{\ell+1} \in L \\ \cdots \\ a'_r \in L \end{array}\right\}. \tag{3.3}$$

Note that the last $p - \ell \geqslant 0$ columns of U' are full of zeroes and so, no condition concerns the vectors $c'_{\ell+1}, \ldots, c'_p$ and we can take them to be arbitrary (say zero, for example). That is, taking $c'_{\ell+1} = \cdots = c'_p = 0$, denoting $A'' = A' \in M_{r \times m}(\mathbb{Z})$, $B'' = B' \in M_{s \times m}(\mathbb{Z})$, $U'' \in M_{r \times \ell}(\mathbb{Z})$ the matrix U' after deleting the last $p - \ell \geqslant 0$ columns (and $C'' \in M_{\ell \times m}(\mathbb{Z})$ the matrix C' after deleting the last $p - \ell \geqslant 0$ rows), we have $d(A', B', U') = d(A'', B'', U'')$.

Now, we can ignore conditions (3.3) by adding the vectors $a''_{\ell+1}, \ldots, a''_r$ as extra rows at the bottom of B: let $A''' \in M_{\ell \times m}(\mathbb{Z})$ be A'' after deleting the last $r - \ell \geqslant 0$ rows, $B''' \in M_{(s+r-\ell) \times m}(\mathbb{Z})$ be B'' enlarged with $r - \ell$ extra rows with the vectors $a''_{\ell+1}, \ldots, a''_r$, (and $C''' = C''$), and we have that $d(A'', B'', U'') = d(A''', B''', U''')$.

Finally, if $d_1 = 1$ we can take $c'_1 = a'_1$ and the first condition in (3.2) becomes trivial; so, deleting the possible ones at the beginning of the list $d_1|d_2|\cdots|d_\ell$ (and their rows and columns from U'''), and deleting also the corresponding first rows of A and C, we can assume $d_1 \neq 1$.

Altogether, and resetting the notation to the original one, we are reduced to compute $d(A,B,U)$ in the special situation where $A \in M_{r\times m}$, $B \in M_{s\times m}$, and $U = \operatorname{diag}(d_1,\ldots,d_r) \in M_{r\times r}$, with $1 \neq d_1|d_2|\cdots|d_r \neq 0$, and further, by the argument in the first paragraph of the present proof, with $\operatorname{row}(B)$ being a direct summand of $\mathbb{Z}^m$. That is, we have to compute a subgroup $L \leqslant \mathbb{Z}^m$ of the minimum possible rank, and vectors $c_1,\ldots,c_p \in \mathbb{Z}^m$ satisfying $\operatorname{row}(B) \leqslant L$ and

$$\left.\begin{array}{c} a_1 - d_1c_1 \in L \\ \cdots \\ a_r - d_rc_r \in L \end{array}\right\}, \tag{3.4}$$

where a_i is the i-th row of A. Let us think the conditions in (3.4) as saying that $a_i \in L$ modulo $d_i\mathbb{Z}^m$, $i = 1,\ldots,r$. To solve this, let us start with $L_0 = \operatorname{row}(B) \leqslant_\oplus \mathbb{Z}^m$ and let us increase it the minimum possible in order to fulfill conditions (3.4).

Since $d_1|d_2|\cdots|d_r$, the natural projections $\pi_i\colon \mathbb{Z}^m \twoheadrightarrow (\mathbb{Z}/d_i\mathbb{Z})^m$ factorize through the chain of morphisms $\mathbb{Z}^m \twoheadrightarrow (\mathbb{Z}/d_r\mathbb{Z})^m \twoheadrightarrow (\mathbb{Z}/d_{r-1}\mathbb{Z})^m \twoheadrightarrow \cdots \twoheadrightarrow (\mathbb{Z}/d_1\mathbb{Z})^m$. Starting with $L \geqslant L_0$ and collecting the last condition in (3.4), we deduce that L must further satisfy $L\pi_r \geqslant L_0\pi_r + \langle v_r^0\pi_r\rangle$, where $v_r^0 = a_r \in \mathbb{Z}^m$. Now the second condition from below in (3.4) adds the requirement $L\pi_{r-1} \ni a_{r-1}\pi_{r-1}$. But $a_{r-1}\pi_{r-1} \in (\mathbb{Z}/d_{r-1}\mathbb{Z})^m$ has finitely many (more precisely, $(d_r/d_{r-1})^m$) pre-images in $(\mathbb{Z}/d_r\mathbb{Z})^m$; compute them all, take pre-images v_{r-1} up in $\mathbb{Z}^m$, and we get that L must further satisfy $L\pi_r \geqslant L_0\pi_r + \langle v_r^0\pi_r,\, v_{r-1}\pi_r\rangle$, where $v_{r-1}\pi_r$ is one of these $(d_r/d_{r-1})^m$ pre-images. Repeat this same argument with all the conditions in (3.4), working from bottom to top: we deduce that L must further satisfy $L\pi_r \geqslant L_0\pi_r + \langle v_r^0\pi_r, v_{r-1}\pi_r, \ldots, v_1\pi_r\rangle$, where $v_i \in \mathbb{Z}^m$ is a vector such that $v_i\pi_r$ is one of the computed $(d_r/d_i)^m$ pre-images of $a_i\pi_i \in (\mathbb{Z}/d_i\mathbb{Z})^m$ up in $(\mathbb{Z}/d_r\mathbb{Z})^m$, $i = r-1, \ldots, 1$, i.e., $v_i \equiv a_i \mod d_i$. This makes a total of $(d_r/d_{r-1})^m \cdots (d_r/d_1)^m$ possible lower bounds for $L\pi_r$: compute them all, find one with minimal possible rank, say $L\pi_r \geqslant L_0\pi_r + \langle v_r^0\pi_r, v_{r-1}^0\pi_r, \ldots, v_1^0\pi_r\rangle$, and we deduce that $d(A, U, B) \geqslant \mathrm{r}(L_1\pi_r)$, where $L_1 = L_0 + \langle v_r^0, v_{r-1}^0, \ldots, v_1^0\rangle \leqslant \mathbb{Z}^m$.

We claim that this lower bound is tight, i.e., $d(A, B, U) = \mathrm{r}(L_1\pi_r)$. To see this, we have to construct a subgroup $L \leqslant \mathbb{Z}^m$ of rank exactly $\mathrm{r}(L_1\pi_r)$, containing L_0 and satisfying (3.4) for some vectors $c_1, \ldots, c_r \in \mathbb{Z}^m$ (which must also be computed). Since L_0 is a direct summand of $\mathbb{Z}^m$, say with free-abelian basis $\{w_1, \ldots, w_k\}$, we deduce that $L_0\pi_r$ is a direct summand of $(\mathbb{Z}/d_r\mathbb{Z})^m$ with abelian basis $\{w_1\pi_r, \ldots, w_k\pi_r\}$. So, $L_0\pi_r$ is also a direct summand of $L_1\pi_r \leqslant (\mathbb{Z}/d_r\mathbb{Z})^m$; compute a complement and get vectors $v_1', \ldots, v_l' \in \mathbb{Z}^m$, $l \leqslant r$, such that $\{w_1\pi_r, \ldots, w_k\pi_r, v_1'\pi_r, \ldots, v_l'\pi_r\}$ is an abelian basis of $L_1\pi_r = L_0\pi_r \oplus V$; in particular, $\mathrm{r}(L_1\pi_r) = k + l$.

Finally, take $L = \langle w_1, \ldots, w_k, v_1', \ldots, v_l'\rangle \leqslant \mathbb{Z}^m$. This subgroup has the desired rank $\mathrm{r}(L) = k + l = \mathrm{r}(L_1\pi_r)$ (since the given generators are linearly independent because their π_r-projections are so), and satisfies the required conditions: on one hand, $L_0 = \langle w_1, \ldots, w_k\rangle \leqslant L$; on the other, for every $i = 1, \ldots, r$, $v_i^0\pi_r \in L_1\pi_r = \langle w_1\pi_r, \ldots, w_k\pi_r\rangle \oplus \langle v_1'\pi_r, \ldots, v_l'\pi_r\rangle$ so,

$$\begin{aligned} v_i^0\pi_r &= \lambda_1(w_1\pi_r) + \cdots + \lambda_k(w_k\pi_r) + \mu_1(v_1'\pi_r) + \cdots + \mu_l(v_l'\pi_r) \\ &= (\lambda_1 w_1 + \cdots + \lambda_k w_k + \mu_1 v_1' + \cdots + \mu_l v_l')\pi_r, \end{aligned}$$

for some integers $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_l \in \mathbb{Z}$; thus, L contains the vector $c_i = \lambda_1 w_1 + \cdots + \lambda_k w_k + \mu_1 v_1' + \cdots + \mu_l v_l'$ which satisfies $c_i \equiv v_i^0 \mod d_r$ and so, $c_i \equiv v_i^0 \mod d_i$ too; since $v_i^0 \equiv a_i \mod d_i$, we deduce $c_i \equiv a_i \mod d_i$ and we are done. □

It is natural to ask whether the minimum $\min_{J \in \mathcal{AE}_{F_n}(H\pi)}\{\tilde{\mathrm{r}}(J) + d(A, B, U_J)\}$ in Theorem 3.3.3 is attained at an algebraic extension $J \in \mathcal{AE}_{F_n}(H\pi)$ of minimal rank. Unfortunately, this is not always the case, as shown in the following example. In order to compute $\mathrm{dc}_G(H)$, this forces us to run over *all* algebraic extensions J of $H\pi$, and compute $d(A, B, U_J)$ following the algorithm given in Proposition 3.3.4, for *each one*. We do not see any shortcut to this procedure, for the general case.

Example 3.3.5. We exhibit an explicit example of a subgroup $H \leqslant_{fg} G$ having two $J, J' \in \mathcal{AE}_{F_n}(H\pi)$ with $\tilde{\mathrm{r}}(J) < \tilde{\mathrm{r}}(J')$ but $\tilde{\mathrm{r}}(J) + d(A, B, U_J) > \tilde{\mathrm{r}}(J') + d(A, B, U_{J'})$.

Let $H = \langle t^{(-1,0)}b^2, t^{(1,0)}ac^{-1}ac^{-1}, t^{(0,1)}bac^{-1}\rangle \leqslant_{fg} G = \mathbb{Z}^2 \times F_3$. Projecting, we have $H\pi = \langle b^2, ac^{-1}ac^{-1}, bac^{-1}\rangle$, and Fig. 3.3 represents the Stallings' graph $\Gamma_A(H\pi)$ for $H\pi$ as a subgroup of F_3 with respect to the ambient free basis $A = \{a, b, c\}$. The fringe of $H\pi$ is depicted in Figs. 3.3 and 3.4. Now from example 3.2.5, we get the set of algebraic extensions for $H\pi$, namely $\mathcal{AE}(H\pi) = \{H\pi, J\}$, where $J = \langle b, ac^{-1}\rangle\}$.

Following the notation above, we have

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \emptyset, \qquad U_{H\pi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U_J = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}.$$

According to Theorem 3.3.3,

$$dc_G(H) = \tilde{\mathrm{r}}(H)/\min\{\tilde{\mathrm{r}}(H\pi) + d(A, B, U_{H\pi}),\, \tilde{\mathrm{r}}(J) + d(A, B, U_J)\}. \tag{3.5}$$

Since $H \leqslant H$, $d(A, B, U_{H\pi}) = \mathrm{r}(L_H) = 0$ and the first term on the minimum in (3.5) is $\tilde{\mathrm{r}}(H\pi) + d(A, B, U_{H\pi}) = (3-1) + 0 = 2$.

Following the algorithm given in Proposition 3.3.4, let us compute now $d(A, B, U_J)$, where $J = \langle b, ac^{-1} \rangle$; we have $r = 3$, $m = 2$, $s = 0$, and $p = 2$. Computing the Smith normal form for U_J, we get

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Z}), \qquad Q = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}), \qquad U' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix},$$

with $d_1 = 1$, $d_2 = 2$, and $\ell = \min\{r, p\} = 2$. Diagonalyzing the problem, we obtain

$$A' = PA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad B' = B = \emptyset, \qquad U' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix},$$

and $d(A, B, U_J) = d(A', B', U')$ (under the change of variable $C = QC'$). Since $p = \ell = 2$ the next reduction is empty and $A'' = A'$, $B'' = B'$, and $U'' = U'$. Applying the following reduction to delete the last $r - \ell = 3 - 2 = 1$ zero rows in U'', we get

$$A''' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B''' = \begin{pmatrix} 0 & -2 \end{pmatrix}, \qquad U''' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Finally, in order to delete $d_1 = 1$, we take $c_1''' = (0, 1)$ and get

$$A'''' = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad B'''' = \begin{pmatrix} 0 & -2 \end{pmatrix}, \qquad U'''' = \begin{pmatrix} 2 \end{pmatrix}.$$

Going up by finite index, we replace the matrix B'''' to $(0, 1)$, and are reduced to compute $d(A'''', (0, 1), U'''')$; this is the smallest rank of a subgroup $L \leqslant \mathbb{Z}^2$ such that $\langle (0, 1) \rangle \leqslant L$ and $(1, 0) - 2c_2 \in L$ for some $c_2 \in \mathbb{Z}^2$. Clearly, $d(A'''', (0, 1), U'''') = 2$, and one (non unique) solution is given by $L = \mathbb{Z}^2$ and $c_2'''' = (1,0)$. Collecting the c_1 computed before, and undoing the change of variable, we get

$$C = QC' = QC'''' = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

We conclude that $d(A, B, U_J) = 2$ and one of the subgroups K with the smallest possible rank satisfying $K\pi = J$ and $H \leqslant K \leqslant \mathbb{Z}^2 \times F_3$ is $K = \langle t^{(-1,1)}b^2,\, t^{(1,0)}ac^{-1},\, t^{(1,0)},\, t^{(0,1)}\rangle$. So, the second term on the minimum in (3.5) is $\tilde{\mathrm{r}}(J) + d(A, B, U_J) = (2-1) + 2 = 3$. Therefore,

$$\begin{aligned} dc_G(H) &= \frac{\tilde{\mathrm{r}}(H)}{\min\{\tilde{\mathrm{r}}(H\pi) + d(A, B, U_{H\pi}),\ \tilde{\mathrm{r}}(J) + d(A, B, U_J)\}} \\ &= \frac{3-1}{\min\{(3-1)+0,\ (2-1)+2)\}} \\ &= \frac{2}{2} = 1. \end{aligned}$$

In particular, H is compressed in G.

As seen in this example, the algebraic extension J looks better than the other one $H\pi$ because it contributes to the free rank in 2 units instead of 3. However, in order to match the free abelian part, J forces us to take two more units of rank, while $H\pi$ requires zero units. Note that in this example, $d(A, B, U_J)$ is as big as it could be since, in general, $d(A, B, U_J) \leqslant m = 2$. The example can easily be extended to an arbitrary m.

3.4 Degree of inertia in free-abelian times free groups

In this section, we study the degree of inertia for subgroups H of $G = \mathbb{Z}^m \times F_n$ and relate it to the corresponding degree of inertia of $H\pi$ in F_n; it turns out that the index of

$H \cap \mathbb{Z}^m$ in $\mathbb{Z}^m$ (whether finite or infinite) is closely related to the degree of inertia of H. Unfortunately, the situation here is more complicated and we can only prove an upper bound for di_G in terms of di_{F_n} and the previously mentioned index; the computability of this function remains open, as in the free case.

Lemma 3.4.1. *For positive real numbers $a, b, c, d > 0$,*

$$\frac{a}{b} \leqslant \frac{c}{d} \;\Rightarrow\; \frac{a}{b} \leqslant \frac{a+c}{b+d} \leqslant \frac{c}{d}. \quad \square$$

Theorem 3.4.2. *Let $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$, and let $L_H = H \cap \mathbb{Z}^m$.*

(i) If $\mathrm{r}(H\pi) \leqslant 1$ then $\mathrm{di}_G(H) = 1$, (i.e., H is inert in G);

(ii) if $\mathrm{r}(H\pi) \geqslant 2$ and $[\mathbb{Z}^m : L_H] = \infty$ then $\mathrm{di}_G(H) = \infty$;

(iii) if $\mathrm{r}(H\pi) \geqslant 2$ and $[\mathbb{Z}^m : L_H] = l < \infty$ then $\mathrm{di}_G(H) \leqslant l\, \mathrm{di}_{F_n}(H\pi)$.

Proof. (i). The hypothesis $\mathrm{r}(H\pi) \leqslant 1$ implies that $H = \langle t^a u, L_H \rangle$, for some $a \in \mathbb{Z}^m$ and $u \in F_n$ (possibly trivial). Then, for every $K \leqslant_{fg} G$, we have $(H \cap K)\pi \leqslant H\pi \cap K\pi \leqslant \langle u \rangle$. So, $(H \cap K)\pi = \langle u^r \rangle$ for some $r \in \mathbb{Z}$. Then, $H \cap K = \langle t^b u^r, L_H \cap L_K \rangle$ for some $b \in \mathbb{Z}^m$ and we get $\mathrm{r}(H \cap K) \leqslant \mathrm{r}(K)$. Therefore, $\tilde{\mathrm{r}}(H \cap K)/\,\tilde{\mathrm{r}}(K) \leqslant 1$, which is valid for every $K \leqslant_{fg} G$. Thus, $\mathrm{di}_G(H) = 1$.

(ii). Consider the subgroup $\tilde{L}_H$ satisfying $L_H \leqslant_{fi} \tilde{L}_H \leqslant_{\oplus} \mathbb{Z}^m$, and take a free-abelian basis $\{b_1, \ldots, b_s\}$ of $\tilde{L}_H$, such that $\{\lambda_1 b_1, \ldots, \lambda_s b_s\}$ is a free-abelian basis of L_H for appropriate choices of $\lambda_1, \ldots, \lambda_s \in \mathbb{Z}$ (there is always a basis like this by standard linear algebra arguments). By hypothesis, $s = \mathrm{r}(L_H) < m$ and, completing to a free-abelian basis $\{b_1, \ldots, b_s, b_{s+1}, \ldots, b_m\}$ of the ambient $\mathbb{Z}^m$, we get at list one extra vector b_{s+1} (which, of course, is primitive in $\mathbb{Z}^m$ and so has relatively prime coordinates).

Now fix a basis for H of the form $\{t^{a_1}u_1, \ldots, t^{a_{n_1}}u_{n_1}, t^{\lambda_1 b_1}, \ldots, t^{\lambda_s b_s}\}$, where $a_1, \ldots, a_{n_1} \in \mathbb{Z}^m$, and $\{u_1, \ldots, u_{n_1}\}$ is a free basis for $H\pi$; in particular, $\mathrm{r}(H\pi) = n_1 \geqslant 2$, $\mathrm{r}(L_H) = s < m$, and $\mathrm{r}(H) = n_1 + s$.

For proving $\mathrm{di}_G(H) = \infty$, we shall construct a family of subgroups $K_N \leqslant_{fg} \mathbb{Z}^m \times F_n$, indexed by $N \in \mathbb{N}$, all of them with constant rank 3 (i.e., $\tilde{\mathrm{r}}(K_N) = 2$), with all the intersections $H \cap K_N$ being finitely generated, but with $\tilde{\mathrm{r}}(H \cap K_N)$ tending to ∞, as $N \to \infty$.

Let $K_N = \langle t^{a'_1}u_1, t^{a'_2}u_2, L_{K_N}\rangle \leqslant \mathbb{Z}^m \times F_n$, where the vectors $a'_1, a'_2 \in \mathbb{Z}^m$ and the subgroup $L_{K_N} \leqslant \mathbb{Z}^m$ are to be determined (note that for all choices $\mathrm{r}(K_N\pi) = 2$, and here we are already using the hypothesis $n_1 \geqslant 2$).

Let us understand the intersection $H \cap K_N$ (see the figure 3.5). We have $n_2 = \mathrm{r}(K_N\pi) = 2$, $H\pi \cap K_N\pi = \langle u_1, u_2\rangle$ and so $n_3 = \mathrm{r}(H\pi \cap K_N\pi) = 2$, and we consider the matrices

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_{n_1} \end{pmatrix} \in M_{n_1 \times m}(\mathbb{Z}), \quad A' = \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} \in M_{2 \times m}(\mathbb{Z}).$$

Let $\rho_1 \colon H\pi \twoheadrightarrow \mathbb{Z}^{n_1}$, $\rho_2 \colon K_N\pi \twoheadrightarrow \mathbb{Z}^2$ and $\rho_3 \colon H\pi \cap K_N\pi \twoheadrightarrow \mathbb{Z}^2$ be the corresponding abelianization maps. Clearly, the inclusion maps $\iota_H \colon H\pi \cap K_N\pi \hookrightarrow H\pi$ and $\iota_K \colon H\pi \cap K_N\pi \hookrightarrow K_N\pi$ abelianize, respectively, to the morphisms $\mathbb{Z}^2 \to \mathbb{Z}^{n_1}$ and $\mathbb{Z}^2 \to \mathbb{Z}^2$ given by the matrices

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in M_{2 \times n_1}(\mathbb{Z}), \quad P' = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_{2 \times 2}(\mathbb{Z}).$$

$$
\begin{tikzcd}
 & (H\cap K_N)\pi \arrow[d, phantom, "\rotatebox{90}{$\leqslant$}"] & \\
H\pi \arrow[d, two heads, "\rho_1"'] \arrow[dr, phantom, "///"] & H\pi\cap K_N\pi \arrow[l, hook', "\iota"'] \arrow[r, hook, "\iota'"] \arrow[d, two heads, "\rho_3"] \arrow[dr, phantom, "///"] & K_N\pi \arrow[d, two heads, "\rho_2"] \\
\mathbb{Z}^{n_1} \arrow[dr, "A"] & \mathbb{Z}^2 \arrow[l, "P"'] \arrow[r, "P'"] \arrow[d, "R"] & \mathbb{Z}^2 \arrow[dl, "A'"'] \\
 & \mathbb{Z}^m &
\end{tikzcd}
$$

Figure 3.5: Intersection diagram of H and K_N

Moreover, let

$$R = PA - P'A' = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} a_1 - a_1' \\ a_2 - a_2' \end{pmatrix} \in M_{2\times m}(\mathbb{Z}),$$

and let us put all these ingredients into the diagram 3.5.

According to the argument in [10, Thm. 4.5], the subgroup $(H \cap K_N)\pi \leqslant H\pi \cap K_N\pi$ is, precisely, the full preimage by R and ρ_3 of $L_H + L_{K_N} \leqslant \mathbb{Z}^m$.

Let us choose now the vectors $a_1' = a_1 - b_{s+1}$ and $a_2' = a_2$, so that the matrix R becomes

$$R = \begin{pmatrix} b_{s+1} \\ 0 \end{pmatrix},$$

and let us choose the subgroup $L_{K_N} = \langle Nb_{s+1}\rangle$. We have $L_H + L_{K_N} = \langle \lambda_1 b_1, \ldots, \lambda_s b_s, Nb_{s+1}\rangle$ and then,

$$\begin{aligned}
(L_H + L_{K_N})R^{-1} &= \{(x,y)\in \mathbb{Z}^2 \mid (x \;\; y)R \in L_H + L_{K_N}\} \\
&= \{(x,y)\in \mathbb{Z}^2 \mid xb_{s+1} \in L_H + L_{K_N}\} \\
&= \{(x,y)\in \mathbb{Z}^2 \mid xb_{s+1} \in \langle Nb_{s+1}\rangle\} \\
&= N\mathbb{Z}\times\mathbb{Z} \quad \leqslant_N \; \mathbb{Z}^2
\end{aligned}$$

(the last equality being true because b_{s+1} has relatively prime coordinates). As ρ_3 is onto, taking ρ_3-preimages preserves the index and we have

$$(H\cap K_N)\pi = (L_H + L_{K_N})R^{-1}{\rho_3}^{-1} \leqslant_N H\pi \cap K_N\pi.$$

Thus, by the Schreier index formula, $\tilde{\mathrm{r}}((H\cap K_N)\pi) = N\,\tilde{\mathrm{r}}(H\pi\cap K_N\pi) = N$ and we deduce that $\tilde{\mathrm{r}}(H\cap K_N) = N + \mathrm{r}(L_H\cap L_{K_N}) = N+0 = N$ tends to ∞, as $N\to\infty$. This completes the proof that $\mathrm{di}_G(H) = \infty$.

(iii). Fix a basis for H, say $\{t^{a_1}u_1, \ldots, t^{a_{n_1}}u_{n_1}, t^{b_1}, \ldots, t^{b_m}\}$, where $a_1, \ldots, a_{n_1} \in \mathbb{Z}^m$, $\{u_1, \ldots, u_{n_1}\}$ is a free basis for $H\pi$, and $\{b_1, \ldots, b_m\}$ is a free-abelian basis for $L_H \leqslant_l \mathbb{Z}^m$; in particular, $\mathrm{r}(H\pi) = n_1 \geqslant 2$, $\mathrm{r}(L_H) = m$, and $\mathrm{r}(H) = n_1 + m$.

In order to prove the inequality $\mathrm{di}_G(H) \leqslant l\,\mathrm{di}_{F_n}(H\pi)$, let us take an arbitrary subgroup $K\leqslant_{fg} G$, assume that $H\cap K$ is finitely generated, and let us prove that $\tilde{\mathrm{r}}(H\cap K)/\,\tilde{\mathrm{r}}(K) \leqslant l\,\mathrm{di}_{F_n}(H\pi)$. Fix a basis for K, say $K = \langle t^{a_1'}v_1, \ldots, t^{a_{n_2}'}v_{n_2}, L_K\rangle$ and we have

$$\frac{\tilde{\mathrm{r}}(H\cap K)}{\tilde{\mathrm{r}}(K)} = \frac{\tilde{\mathrm{r}}((H\cap K)\pi) + \mathrm{r}(L_H\cap L_K)}{\tilde{\mathrm{r}}(K\pi) + \mathrm{r}(L_K)}. \tag{3.6}$$

As in the proof of part (ii), we consider the following diagram to understand $H\cap K$:

$$\begin{tikzcd}
 & (H \cap K)\pi & \\
H\pi \arrow[d, two heads, "\rho_1"'] \arrow[dr, phantom, "///"] & H\pi \cap K\pi \arrow[l, hook', "\iota"'] \arrow[r, hook, "\iota'"] \arrow[d, two heads, "\rho_3"] \arrow[u, phantom, "\rotatebox{-90}{$\leqslant$}"] \arrow[dr, phantom, "///"] & K\pi \arrow[d, two heads, "\rho_2"] \\
\mathbb{Z}^{n_1} \arrow[dr, "A"] & \mathbb{Z}^{n_3} \arrow[l, "P"'] \arrow[r, "P'"] \arrow[d, "\mathbf{R}"] & \mathbb{Z}^{n_2} \arrow[dl, "A'"'] \\
 & \mathbb{Z}^m &
\end{tikzcd}$$

Figure 3.6: Intersection diagram of H and K

where $\rho_1 \colon H\pi \twoheadrightarrow \mathbb{Z}^{n_1}$, $\rho_2 \colon K\pi \twoheadrightarrow \mathbb{Z}^{n_2}$, and $\rho_3 \colon H\pi \cap K\pi \twoheadrightarrow \mathbb{Z}^{n_3}$ are the corresponding abelianization maps (here, $n_3 = \mathrm{r}(H\pi \cap K\pi) < \infty$), where ι and ι' are the natural inclusions and $P \in M_{n_3 \times n_1}(\mathbb{Z})$ and $P' \in M_{n_3 \times n_2}(\mathbb{Z})$ are the matrices of their respective abelianizations (note that ι and ι' being injective do not imply P and P' necessarily being so), where $A \in M_{n_1 \times m}(\mathbb{Z})$ and $A' \in M_{n_2 \times m}(\mathbb{Z})$ are the matrices with rows $\{a_1, \ldots, a_{n_1}\}$ and $\{a'_1, \ldots, a'_{n_2}\}$ respectively, and where $R = PA - P'A' \in M_{n_3 \times m}(\mathbb{Z})$. According to the argument in [10, Thm. 4.5], the crucial property of diagram 3.6 is the fact that $(H \cap K)\pi = (L_H + L_K)R^{-1}\rho_3^{-1}$.

From the hypothesis, $L_H \leqslant_l \mathbb{Z}^m$ and so, $L_H + L_K \leqslant_{l'} \mathbb{Z}^m$, where $1 \leqslant l' \leqslant l$. As in general R is not necessarily onto, $(L_H + L_K)R^{-1} \leqslant_{l''} \mathbb{Z}^{n_3}$ with $1 \leqslant l'' \leqslant l'$. And, since ρ_3 is onto, $(H \cap K)\pi = (L_H + L_K)R^{-1}\rho_3^{-1} \leqslant_{l''} H\pi \cap K\pi$. Therefore, by the Schreier index formula,

$$\tilde{\mathrm{r}}((H \cap K)\pi) = l''\, \tilde{\mathrm{r}}(H\pi \cap K\pi) \quad = \quad l'' \frac{\tilde{\mathrm{r}}(H\pi \cap K\pi)}{\tilde{\mathrm{r}}(K\pi)}\, \tilde{\mathrm{r}}(K\pi) \quad \leqslant \quad l''\, \mathrm{di}_{F_n}(H\pi)\, \tilde{\mathrm{r}}(K\pi). \tag{3.7}$$

Now, using (3.6), we have

$$\frac{\tilde{\mathrm{r}}(H\cap K)}{\tilde{\mathrm{r}}(K)} \leqslant \frac{l''\,\mathrm{di}_{F_n}(H\pi)\,\tilde{\mathrm{r}}(K\pi)+\mathrm{r}(L_H\cap L_K)}{\tilde{\mathrm{r}}(K\pi)+\mathrm{r}(L_K)} \leqslant \frac{l''\,\mathrm{di}_{F_n}(H\pi)\,\tilde{\mathrm{r}}(K\pi)}{\tilde{\mathrm{r}}(K\pi)} = l''\,\mathrm{di}_{F_n}(H\pi), \tag{3.8}$$

where the second inequality is an equality if $L_K = \{0\}$, and follows from applying Lemma 3.4.1 to $\frac{\mathrm{r}(L_H\cap L_K)}{\mathrm{r}(L_K)} \leqslant 1 \leqslant l''\,\mathrm{di}_{F_n}(H\pi)$ otherwise. Therefore,

$$\frac{\tilde{\mathrm{r}}(H\cap K)}{\tilde{\mathrm{r}}(K)} \leqslant l''\,\mathrm{di}_{F_n}(H\pi) \leqslant l'\,\mathrm{di}_{F_n}(H\pi) \leqslant l\,\mathrm{di}_{F_n}(H\pi), \tag{3.9}$$

as we wanted. □

3.5 Restricted degree of inertia for free-abelian times free group

The present section is dedicated to develop several lemmas about intersections of subgroups of F_n, which will be used later. A well-known tool for understanding intersections of finitely generated subgroups of F_n is the pull-back of graphs.

Definition 3.5.1. Let $N, M \leqslant_{fg} F_n$ and consider its Stallings graphs $\Gamma(N), \Gamma(M)$. Consider the direct product $\Gamma(N)\times\Gamma(M)$, which is defined as the new graph having as set of vertices $V\Gamma(N)\times V\Gamma(M)$, set of a-labelled edges $E_a\Gamma(N)\times E_a\Gamma(M)$ (here, $E_a\Gamma$ denotes the set of edges in Γ labelled by the letter a), and with the natural incidence functions $\iota(e,f) = (\iota e, \iota f)$ and $\tau(e,f) = (\tau e, \tau f)$.

It is well known (see, for example, [22] for details) that $\Gamma(N)\times\Gamma(M)$ is folded; but neither connected nor free of degree one vertices, in general. After taking the connected component of the basepoint $(\odot,\odot)$ and trimming (i.e., repeatedly deleting vertices of degree one different from the basepoint), one gets the Stallings graph for $(N\cap M)$,i.e.,

$\Gamma(N \cap M)$ which is also denoted by $\Gamma(N) \wedge \Gamma(M)$. In particular, $N \cap M$ is always finitely generated (proving the Howson property for F_n).

Definition 3.5.2. Let $\Gamma(N)$ be the Stallings graph of $N \leqslant_{fg} F_n$. For every vertex $p \in V\Gamma(N)$ and every element $w \in F_n$, we define pw to be the terminal vertex of the unique reduced path γ in $\Gamma(N)$ starting at p and having label w, in case it exists; otherwise, pw is *undefined*. Note that $w \in N$ if and only if $\odot w$ is defined and equals $\odot$. Also, note that N has finite index in F_n if and only if $\Gamma(N)$ is complete and if and only if $\odot w$ is defined in $\Gamma(N)$, for every $w \in F_n$.

Lemma 3.5.3. *For $N, M \leqslant_{fg} F_n$, $N \cap M$ has infinite index in N if and only if there exists $w \in N$ such that $\odot w$ is undefined in $\Gamma(M)$.*

Proof. The implication to the left is clear: if $N \cap M$ has finite index in N then $\odot w$ would be defined in $\Gamma(N \cap M)$, and hence in $\Gamma(M)$, for every $w \in N$.

For the implication to the right, suppose the conclusion is not true, i.e., for every $w \in N$, $\odot w \in \{p_0 = \odot, p_1, \ldots, p_r\} \subseteq V\Gamma(M)$. Choosing a maximal tree T in $\Gamma(M)$ and defining $w_i = \ell(T[\odot, p_i]) \in F_n$ for $i = 0, \ldots, r$ (note that $w_0 = 1$), we have $N \subseteq M \sqcup Mw_1 \sqcup \cdots \sqcup Mw_r$. Intersecting with N, we get $N \subseteq (N \cap M) \sqcup (N \cap M)v_1 \sqcup \cdots \sqcup (N \cap M)v_s$ for some $v_i \in N$ and $s \leqslant r$ (where we have deleted the possibly empty intersections). Since the other inclusion is immediate, we deduce that $N \cap M$ has finite index in N, a contradiction. □

Proposition 3.5.4. *(p-Expansion). Let $N, M \leqslant_{fg} F_n$, and suppose that $\mathrm{r}(N) \geqslant 2$, the basepoint $\odot$ has degree at least 3 in $\Gamma(N)$, and $N \cap M$ has infinite index in N. Then, for every $1 \leqslant p \leqslant \infty$, there exist p freely independent elements $w_1, \ldots, w_p \in N$ such that $M \leqslant_{ff} M' = M * \langle w_1, \ldots, w_p \rangle$ and $N \cap M \leqslant_{ff} (N \cap M) * \langle w_1, \ldots, w_p \rangle \leqslant_{ff} N \cap M'$. Furthermore, there exists w in N, such that $\odot w$ is undefined in $\Gamma(M')$.*

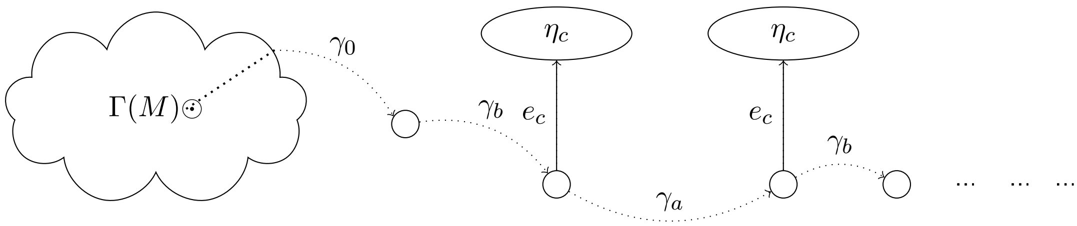

Figure 3.7: Expansion of $\Gamma(M)$.

Proof. Let e_a, e_b, e_c be three different edges going out from $\odot$ in $\Gamma(N)$, $\iota e_a = \iota e_b = \iota e_c = \odot$, with pairwise different labels $a, b, c \in X^{\pm 1}$, respectively. By Lemma 3.5.3, there is $u_0 \in N$ such that $\odot u_0$ is undefined in $\Gamma(M)$. Realize u_0 as a reduced closed path γ_0 at $\odot$ in $\Gamma(N)$ and, without lost of generality, we can assume it finishes with e_a^{-1}. For $\alpha = a, b, c$, take a non-trivial reduced path η_α in the graph $\Gamma(N) \setminus \{e_\alpha\}$ and closed at τe_α (there always exists such a path because $\mathrm{r}(N) \geqslant 2$, even if e_α is a bridge since $\Gamma(N)$ has no vertices of degree 1 except possibly $\odot$); now consider $\gamma_\alpha = e_\alpha \eta_\alpha e_\alpha^{-1}$, a reduced closed path at $\odot$ in $\Gamma(N)$, beginning with e_α and ending with e_α^{-1} (so, its label $u_\alpha = \ell(\gamma_\alpha) \in N$ is a reduced word on $X^{\pm 1}$ beginning with α and ending with α^{-1}). Note that then the paths $\gamma_0, \gamma_1 = \gamma_0\gamma_b, \gamma_2 = \gamma_0\gamma_b\gamma_a, \gamma_3 = \gamma_0\gamma_b\gamma_a\gamma_b, \ldots$, and also the paths $\gamma_i\gamma_c\gamma_i^{-1}, i \geqslant 0$, are reduced as written; furthermore, all of them are closed paths at $\odot$ in $\Gamma(N)$ so, the elements $w_i = \ell(\gamma_i\gamma_c\gamma_i^{-1}) \in F_n$ belong to N, for all $i \geqslant 1$.

Now, let us extend the graph $\Gamma(M)$ by adding the necessary vertices and edges so that we can read all the paths $\gamma_i\gamma_c\gamma_i^{-1}$ from $\odot, i = 1, \ldots, p$: since $\odot u_0$ was undefined in $\Gamma(M)$, possibly an initial segment of γ_0 is readable in $\Gamma(M)$ but not the entire path, forcing us to append at least a new edge sticking out of $\Gamma(M)$; behind it, we add the rest of the construction, see Fig 3.7 (this is infinitely many new vertices and edges, if $p = \infty$). Since the added paths were all reduced, the resulting graph presents no foldings and so it is a (possibly infinite) Stallings graph, having $\Gamma(M)$ as a subgraph. Hence, M is a free factor of its fundamental group, $M \leqslant_{ff} M' = M * \langle w_1, \ldots, w_p \rangle$.

Now we will look at the Stallings graph $\Gamma(N \cap M') = \Gamma(N) \wedge \Gamma(M')$. Since $w_i \in N$ for all $i \geqslant 0$, it is clear that $\Gamma(N) \wedge \Gamma(M')$ contains, as a subgraph, $\Gamma(N) \wedge \Gamma(M)$ with the same additions as in Fig. 3.7 (and possibly more edges which we do not control). Therefore, $N \cap M \leqslant_{ff} (N \cap M) * \langle w_1, \ldots, w_p \rangle \leqslant_{ff} N \cap M'$.

Finally we can choose $w = \ell(\gamma_0 \gamma_c)$. Clearly, $w \in N$ and from the Fig. 3.7, $\odot w$ is not defined in $\Gamma(M')$. $\square$

Observation 3.5.5. *Let $K \trianglelefteq_d F_n$, for any $M \leqslant F_n$, $KM = \langle K, M \rangle$.*

Proof. It is obvious that $KM \leqslant \langle K, M \rangle$. We will just proof the other inclusion. As $K \trianglelefteq_d F_n$, $mkm^{-1} \in K$ for any $m \in M$ and $k \in K$. In other words, $mk = k'm$, for some $k' \in K$. Let $x \in \langle K, M \rangle$, then x is a word of the form $x = k_1 m_1 \cdots k_r m_r$ for $k_i \in K$, $m_i \in M$. But in this word any sub-word of the form mk can be replaced by $k'm$ as $K \trianglelefteq_d F_n$. Repeatedly doing this replacement operation, we can take all the letters from K to left side together and the letters from M to the right side. And in this way we can express x as an element in KM. Thus we have $KM = \langle K, M \rangle$. $\square$

Lemma 3.5.6. *Let G be a group and $N, M \leqslant G$. Then, $[N : N \cap M] \leqslant [G : M]$, with equality if $MN = G$. (If additionally $[N : N \cap M]$ is finite, the equality holds if and only if $MN = G$.)*

Proof. Let $G = \sqcup_{i \in I} M x_i$ be the coset decomposition of G modulo M, where $|I| = [G : M] \leqslant \infty$. Intersecting with N (and removing the possibly empty intersections), we have $N = \sqcup_{i \in I} (N \cap M x_i) = \sqcup_{i' \in I'} (N \cap M) y_i$, for some $I' \subseteq I$. So, $[N : N \cap M] = |I'| \leqslant |I| = [G : M]$.

Furthermore, Mg intersects N non-trivially for $g \in G$ if $g \in MN$. So, $[N : N \cap M] = |I'| = |I| = [G : M]$ if $G = MN$ (with converse also true when $|I'| < \infty$). $\square$

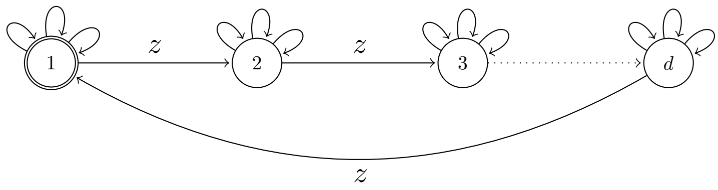

Figure 3.8: Stallings graph for K_d^z

Corollary 3.5.7. *Let $K \trianglelefteq_d F_n$, and $M \leqslant F_n$, then $[M : M \cap K] = d$ if and only if $\langle K, M \rangle = F_n$.*

Proof. The proof follows directly from Observation 3.5.5 and Lemma 3.5.6. $\square$

To proof our Theorem 3.5.14, first we will consider the "easy" normal subgroup $K_d^z = \{w \in F_n \mid |w|_z \in d\mathbb{Z}\}$, where z is one of the letters in the alphabet defining $F_n = F(\mathbb{Z})$. The Stallings graph of K_d^z is depicted in the Fig. 3.8 (the loops at each vertex present the other $n-1$ generators (except z) of F_n). From the construction of K_d^z, it is very clear that $K_d^z \trianglelefteq_d F_n$.

Lemma 3.5.8. *Let $M \leqslant_{fg} F_n$, $d \in \mathbb{N}$. Then the following conditions are equivalent :*

(i) $K_d^z M = \langle K_d^z, M \rangle = F_n$,

(ii) $M \cap K_d^z \leqslant_d M$,

(iii) There exists a word $m \in M$ such that $gcd(|m|_z, d) = 1$,

(iv) The pullback of $\Gamma(M)$ and $\Gamma(K_d^z)$ is connected.

Proof. $(i) \Leftrightarrow (ii)$: True by observation 3.5.5 and corollary 3.5.7.

$(i) \Rightarrow (iii)$: From the hypothesis, z can be written as $z = km$, for some $k \in K_d^z$ and $m \in M$ and from the construction, $|k|_z = \lambda d$ for some $\lambda \in \mathbb{Z}$. Thus we have $|m|_z = 1 - \lambda d$, which in turn implies that $gcd(|m|_z, d) = 1$.

$(iii) \Rightarrow (iv)$: The pull-back of of $\Gamma(M)$ and $\Gamma(K_d^z)$ can be consider as a block-picture, where there are d many blocks, with the $i-th$ block corresponding to the subgraph whose vertices are of the form (p,i) for $p \in V\Gamma(M)$, $i = 1,\ldots,d$ and $\odot$ is the base point of $\Gamma(M)$. Let $m \in M$ such that $gcd(|m|_z, d) = 1$. From Bezout's inequality, there exist $\alpha, \beta \in \mathbb{Z}$ such that $\alpha|m|_z + \beta d = 1$, in other words, $|m^\alpha|_z = 1 - \beta d$. Let γ_{m^α} denotes the path whose label is m^α. Now γ_{m^α} is a closed path at the base point of $\Gamma(M)$. On the other hand, γ_{m^α} is readable in $\Gamma(K_d^z)$, as K_d^z is finite index subgroup of F_n. But it is not readable as a closed path because of $|m^\alpha|_z = 1 - \beta d$. Due to the construction of K_d^z, if $\iota(\gamma_{m^\alpha}) = i$-th vertex, then $\tau(\gamma_{m^\alpha}) = i+1 \pmod d$-th vertex of $\Gamma(K_d^z)$, where $i = 1, 2, \ldots d$ (see Fig. 3.8). In this way m^α will appear in the pull-back connecting each block to the following one (mod d). It remains to prove that each block of the pull back is connected.

Let p be any arbitrary vertex of $\Gamma(M)$. As $\Gamma(M)$ is connected, there will be a path, say γ, from $\iota(\gamma) = \odot$ to $\tau(\gamma) = p$. Let $w_0 = \ell(\gamma) \in F_n$ and consider $w = m^{-s}w_0 \in F_n$ and the path γ_w starting at $\odot$ of $\Gamma(M)$ and reading w (and ending at p of $\Gamma(M)$ because $m \in M$). Since,

$$|m^{-s}w_0|_z = -s|m|_z + |w_0|_z = -s + s \pmod d = 0 \pmod d. \tag{3.10}$$

γ_w is a closed path in $\Gamma(K_d^z)$ because of (3.10). Hence γ_w is present in the pull back connecting $(\odot, i)$ to (p, i) at any block i. Therefore the pull back of $\Gamma(M)$ and $\Gamma(K_d^z)$ is connected.

$(iv) \Rightarrow (ii)$: The number of edges labelled by each letter in $\Gamma(K_d^z)$ is d and also $|V\Gamma(K_d^z)| = d$ (see Fig. 3.8). As the pull-back is connected, we have the following,

$$\begin{aligned} \tilde{\mathrm{r}}(M \cap K_d^z) &= d|E\Gamma(M)| - d|V\Gamma(M)| \\ &= d\,\tilde{\mathrm{r}}(M). \end{aligned}$$

By Schreier Index formula we have, $(M \cap K_d^z) \leqslant_d M$. $\square$

Definition 3.5.9. Let G be a group, $\pi\colon G \to G/Z(G)$ where $Z(G)$ is the center of the group G. Let $H \leqslant_{fg} G$ such that $H\pi$ is not virtually cyclic and $H\pi \nleqslant [G\pi, G\pi]$. The *restricted degree of inertia* of H in G is $\mathrm{di}'_G(H) = sup_K\{\tilde{\mathrm{r}}(H \cap K)/\tilde{\mathrm{r}}(K)\}$, where the supremum is taken over all $K \leqslant_{fg} G$ satisfying $H \cap K \leqslant_{fg} G$, $[H\pi : H\pi \cap K\pi] = \infty$ and $H\pi \cap K\pi \nleqslant [G\pi, G\pi]$ and here $0/0$ is understood to be 1.

Remark 3.5.10. If H is finitely generated, $H\pi$ is not virtually cyclic and $H\pi \nleqslant [G\pi, G\pi]$, then there always exist one such finitely generated K. In fact, let $h \in H\pi$ such that $h \notin [G\pi, G\pi]$ and take $K = \langle h \rangle$, which is finitely generated because it is cyclic. For the same reason $H \cap K$ is also finitely generated. Now, $H\pi \cap K\pi = \langle h \rangle \leqslant_\infty H\pi$ (because $H\pi$ is not virtually cyclic). Also, $H\pi \cap K\pi \nleqslant [G\pi, G\pi]$ as $h \notin [G\pi, G\pi]$.

Observation 3.5.11. *Let $H \leqslant_{fg} G$, be such that $H\pi$ is not virtually cyclic and $H\pi \nleqslant [G\pi, G\pi]$. Then, $\mathrm{di}'_G(H) \geqslant 1$.*

Proof. The previous remark gives a subgroup K with the appropriate conditions and such that $\frac{\tilde{\mathrm{r}}(H\cap K)}{\tilde{\mathrm{r}}(K)} = \frac{0}{0} = 1$. □

Lemma 3.5.12. *Let $\phi\colon G_1 \to G_2$ be an isomorphism of groups and $\pi_i\colon G_i \to G_i/Z(G_i)$ be the natural projection map, where $Z(G_i)$ is the center of the group G_i, for $i = 1, 2$. For every $H \leqslant_{fg} G_1$ such that $H\pi_1$ is not virtually cyclic and $H\pi_1 \nleqslant [G_1\pi_1, G_1\pi_1]$, $\mathrm{di}'_{G_2}(H\phi) = \mathrm{di}'_{G_1}(H)$.*

Proof. Let $K \leqslant_{fg} G_1$ with $H \cap K \leqslant_{fg} G_1$, $H\pi_1 \cap K\pi_1 \leqslant_\infty H\pi_1$ and $H\pi_1 \cap K\pi_1 \nleqslant [G_1\pi_1, G_1\pi_1]$. Since, ϕ is the isomorphism, $K\phi \leqslant_{fg} G_2$ and $H\phi \cap K\phi = (H \cap K)\phi \leqslant_{fg} G_2$. Now since ϕ maps $Z(G_1)$ onto $Z(G_2)$, there exists another isomorphism $\bar{\phi}\colon G_1/Z(G_1) \to G_2/Z(G_2)$ such that $g_1\pi_1\bar{\phi} = g_1\phi\pi_2$ for every $g_1 \in G_1$. As $H\pi_1 \cap K\pi_1 \leqslant_\infty H\pi_1$, we also deduce that, $H\phi\pi_2 \cap K\phi\pi_2 = H\pi_1\bar{\phi} \cap K\pi_1\bar{\phi} = (H\pi_1 \cap K\pi_1)\bar{\phi} \leqslant_\infty H\pi_1\bar{\phi} = H\phi\pi_2$. And also $H\phi\pi_2 \cap K\phi\pi_2 = H\pi_1\bar{\phi} \cap K\pi_1\bar{\phi} = (H\pi_1 \cap K\pi_1)\bar{\phi} \nleqslant [G_1\pi_1, G_1\pi_1]\bar{\phi} = [G_1\phi\pi_2, G_1\phi\pi_2] = [G_2\pi_2, G_2\pi_2]$.

Hence we have, $\tilde{\mathrm{r}}(H \cap K) = \tilde{\mathrm{r}}((H \cap K)\phi) = \tilde{\mathrm{r}}(H\phi \cap K\phi) \leqslant \mathrm{di}'_{G_2}(H\phi) \cdot \tilde{\mathrm{r}}(K\phi) = \mathrm{di}'_{G_2}(H\phi) \cdot \tilde{\mathrm{r}}(K)$. Therefore, $\mathrm{di}'_{G_1}(H) \leqslant \mathrm{di}'_{G_2}(H\phi)$. By symmetry, we deduce the other inequality, $\mathrm{di}'_{G_2}(H) \leqslant \mathrm{di}'_{G_1}(H\phi)$. $\square$

Corollary 3.5.13. *Let G be a group. For every $H \leqslant_{fg} G$ such that $H\pi$ is not virtually cyclic and $H\pi \nleqslant [G\pi, G\pi]$, and for every $g \in G$, $\mathrm{di}'_G(H^g) = \mathrm{di}'_G(H)$.* $\square$

In the case of our interest, where $G = \mathbb{Z}^m \times F_n$, observe that $Z(G) = \mathbb{Z}^m$ and so $\pi\colon G \to G/Z(G)$ is the standard projection to the free part $\pi\colon G \mapsto F_n$. Furthermore, $[G\pi, G\pi] = [F_n, F_n]$ and the hypothesis of $H\pi$ not being virtually cyclic is equivalent to say that $H\pi$ not cyclic, i.e., $\mathrm{r}(H\pi) \geqslant 2$.

With this definition of *restricted degree of inertia*, we can reprove Theorem 3.4.2 improving the last statement into an equality, as desired. The proof goes along the same line but it is much more technical and tricky: the extra technical conditions allow us to do the arguments but, at the same time, we have to worry about their presentation through each one of the manipulations done along the proof.

Theorem 3.5.14. *Let $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$, such that $H\pi$ is not cyclic and $H\pi \nleqslant [F_n, F_n]$ and let $L_H = H \cap \mathbb{Z}^m$;*

(i) if $[\mathbb{Z}^m : L_H] = \infty$ then $\mathrm{di}'_G(H) = \infty$;

(ii) if $[\mathbb{Z}^m : L_H] = l$ then $\mathrm{di}'_G(H) = l\, \mathrm{di}'_{F_n}(H\pi)$.

Proof. *(i)*. Consider the subgroup $\tilde{L}_H$ satisfying $L_H \leqslant_{fi} \tilde{L}_H \leqslant_{\oplus} \mathbb{Z}^m$, and take a free-abelian basis $\{b_1, \ldots, b_s\}$ of $\tilde{L}_H$, such that $\{\lambda_1 b_1, \ldots, \lambda_s b_s\}$ is a free-abelian basis of L_H for appropriate choices of $\lambda_1, \ldots, \lambda_s \in \mathbb{Z}$ (there is always a basis like this by standard linear algebra arguments). By hypothesis, $s = \mathrm{r}(L_H) < m$ and, completing to a free-abelian basis $\{b_1, \ldots, b_s, b_{s+1}, \ldots, b_m\}$ of the ambient $\mathbb{Z}^m$, we get at list one extra vector b_{s+1} (which, of course, is primitive in $\mathbb{Z}^m$ and so has relatively prime coordinates).

Now fix a basis for H of the form $\{t^{a_1}u_1, \ldots, t^{a_{n_1}}u_{n_1}, t^{\lambda_1 b_1}, \ldots, t^{\lambda_s b_s}\}$, where $a_1, \ldots, a_{n_1} \in \mathbb{Z}^m$, and $\{u_1, \ldots, u_{n_1}\}$ is a free basis for $H\pi$; in particular, $\mathrm{r}(H\pi) = n_1 \geqslant 2$, $\mathrm{r}(L_H) = s < m$, and $\mathrm{r}(H) = n_1 + s$. Further, $H\pi \nleqslant [F_n, F_n]$, so without loss of generality we can assume that $u_1 \notin [F_n, F_n]$.

For proving $\mathrm{di}'_G(H) = \infty$, we shall construct a family of subgroups $K_N \leqslant_{fg} \mathbb{Z}^m \times F_n$, indexed by $N \in \mathbb{N}$, all of them having rank 3 (i.e., $\tilde{\mathrm{r}}(K_N) = 2$), with all the intersections $H \cap K_N$ being finitely generated, all of them satisfying that $[H\pi : H\pi \cap K_N\pi] = \infty$, $(H\pi \cap K_N\pi) \nleqslant [F_n, F_n]$ with $\tilde{\mathrm{r}}(H \cap K_N)$ tending to ∞, as $N \to \infty$. The construction of these K_N's will be similar to that in Theorem 3.4.2(ii), but with slight technical modifications in order to get the extra conditions.

Let $K_N = \langle t^{a'_1}u_1^2, t^{a'_2}u_2^2, L_{K_N}\rangle \leqslant \mathbb{Z}^m \times F_n$, where the vectors $a'_1, a'_2 \in \mathbb{Z}^m$ and the subgroup $L_{K_N} \leqslant \mathbb{Z}^m$ are to be determined (note that for all choices $\mathrm{r}(K_N\pi) = 2$, and here we are already using the hypothesis $n_1 \geqslant 2$).

Hence we have $H\pi \cap K_N\pi = \langle u_1^2, u_2^2\rangle \leqslant_\infty H\pi$, and also $H\pi \cap K_N\pi \nleqslant [F_n, F_n]$ as $u_1^2 \notin [F_n, F_n]$ (since $u_1 \notin [F_n, F_n]$ and $F_n/[F_n, F_n] = \mathbb{Z}^n$ is torsion-free).

Let us understand the intersection $H \cap K_N$ (see the figure 3.9). We have $n_2 = \mathrm{r}(K_N\pi) = 2$, $H\pi \cap K_N\pi = \langle u_1^2, u_2^2\rangle$ and so $n_3 = \mathrm{r}(H\pi \cap K_N\pi) = 2$, and we consider the matrices

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_{n_1} \end{pmatrix} \in M_{n_1 \times m}(\mathbb{Z}), \quad A' = \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} \in M_{2 \times m}(\mathbb{Z}).$$

Let $\rho_1 \colon H\pi \twoheadrightarrow \mathbb{Z}^{n_1}$, $\rho_2 \colon K_N\pi \twoheadrightarrow \mathbb{Z}^2$ and $\rho_3 \colon H\pi \cap K_N\pi \twoheadrightarrow \mathbb{Z}^2$ be the corresponding abelianization maps. From 3.9, clearly the inclusion maps $\iota_H \colon H\pi \cap K_N\pi \hookrightarrow H\pi$ and

$$\begin{tikzcd}
 & (H \cap K_N)\pi \arrow[d, phantom, sloped, "\leqslant"] & \\
H\pi \arrow[d, two heads, "\rho_1"'] \arrow[dr, phantom, "///"] & H\pi \cap K_N\pi \arrow[l, hook', "\iota"'] \arrow[r, hook, "\iota'"] \arrow[d, two heads, "\rho_3"] \arrow[dr, phantom, "///"] & K_N\pi \arrow[d, two heads, "\rho_2"] \\
\mathbb{Z}^{n_1} \arrow[dr, "A"] & \mathbb{Z}^2 \arrow[l, "P"'] \arrow[r, "P'"] \arrow[d, "\mathbf{R}"] & \mathbb{Z}^2 \arrow[dl, "A'"'] \\
 & \mathbb{Z}^m &
\end{tikzcd}$$

Figure 3.9: Diagram for $(H \cap K_N)\pi$

$\iota_K \colon H\pi \cap K_N\pi \hookrightarrow K_N\pi$ abelianize, respectively, to the morphisms $\mathbb{Z}^2 \to \mathbb{Z}^{n_1}$ and $\mathbb{Z}^2 \to \mathbb{Z}^2$ given by the matrices

$$P = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \end{pmatrix} \in M_{2\times n_1}(\mathbb{Z}), \quad P' = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}).$$

Moreover, let

$$R = PA - P'A' = \begin{pmatrix} 2a_1 \\ 2a_2 \end{pmatrix} - \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} 2a_1 - a_1' \\ 2a_2 - a_2' \end{pmatrix} \in M_{2\times m}(\mathbb{Z}),$$

and let us put all these ingredients into the diagram 3.9. According to the argument in [10, Thm. 4.5], the subgroup $(H \cap K_N)\pi \leqslant H\pi \cap K_N\pi$ is, precisely, the full preimage by R and ρ_3 of $L_H + L_{K_N} \leqslant \mathbb{Z}^m$.

Let us choose now the vectors $a_1' = 2a_1 - b_{s+1}$ and $a_2' = 2a_2$, so that the matrix R becomes

$$R = \begin{pmatrix} b_{s+1} \\ 0 \end{pmatrix},$$

and let us choose the subgroup $L_{K_N} = \langle Nb_{s+1}\rangle \leqslant \mathbb{Z}^m$. Therefore, we have $L_H + L_{K_N} = \langle \lambda_1 b_1, \ldots, \lambda_s b_s, Nb_{s+1}\rangle$ and then,

$$\begin{aligned}(L_H + L_{K_N})R^{-1} &= \{(x,y)\in\mathbb{Z}^2 \mid (x\ \ y)R \in L_H + L_{K_N}\}\\ &= \{(x,y)\in\mathbb{Z}^2 \mid xb_{s+1} \in L_H + L_{K_N}\}\\ &= \{(x,y)\in\mathbb{Z}^2 \mid xb_{s+1} \in \langle Nb_{s+1}\rangle\}\\ &= N\mathbb{Z}\times\mathbb{Z} \quad \leqslant_N \ \mathbb{Z}^2\end{aligned}$$

(the last equality being true because b_{s+1} has relatively prime coordinates). As ρ_3 is onto, taking ρ_3-preimages preserves the index and we have

$$(H\cap K_N)\pi = (L_H + L_{K_N})R^{-1}{\rho_3}^{-1} \leqslant_N H\pi \cap K_N\pi.$$

Thus, by the Schreier index formula, $\tilde{\mathrm{r}}((H\cap K_N)\pi) = N\,\tilde{\mathrm{r}}(H\pi\cap K_N\pi) = N$ and we deduce that $\tilde{\mathrm{r}}(H\cap K_N) = N + \mathrm{r}(L_H \cap L_{K_N}) = N$ tends to ∞, as $N\to\infty$. This completes the proof that $\mathrm{di}'_G(H) = \infty$.

Proof of (ii). The base point $\odot$ of $\Gamma(H\pi)$ may be of degree 1 or 2 but, without loss of generality, we can assume that it has degree at least 3 because of the following:

From the hypothesis $H\pi$ is not cyclic, i.e., $\mathrm{r}(H\pi) \geqslant 2$. Thus, there exists at least one vertex p of degree at least 3 in $\Gamma(H\pi)$ and we can conjugate H and $H\pi$ appropriately so that the vertex p becomes the base point. With this consideration, we will compute $\mathrm{di}'_{F_n}(H\pi^w)$ and $\mathrm{di}'_G(H^w)$ instead of $\mathrm{di}'_{F_n}(H\pi)$ and $\mathrm{di}'_G(H)$. But from Corollary 3.5.13, we have $\mathrm{di}'_{F_n}(H\pi^w) = \mathrm{di}'_{F_n}(H\pi)$ and $\mathrm{di}'_G(H^w) = \mathrm{di}'_G(H)$.

So, assume the base point of $\Gamma(H\pi)$ has degree at least 3. Fix a basis for H, say $\{t^{a_1}u_1, \ldots, t^{a_{n_1}}u_{n_1}, t^{b_1}, \ldots, t^{b_m}\}$, where $a_1,\ldots,a_{n_1}\in\mathbb{Z}^m$, $\{u_1,\ldots,u_{n_1}\}$ is a free basis for $H\pi$, and $\{b_1,\ldots,b_m\}$ is a free-abelian basis for $L_H \leqslant_l \mathbb{Z}^m$; in particular, $\mathrm{r}(H\pi) = n_1 \geqslant 2$, $\mathrm{r}(L_H) = m$, and $\mathrm{r}(H) = n_1 + m$.

In order to prove the inequality $\operatorname{di}'_G(H) \leqslant l \operatorname{di}'_{F_n}(H\pi)$, let us take an arbitrary subgroup $K \leqslant_{fg} G$, assume that $H \cap K$ is finitely generated, $[H\pi : H\pi \cap K\pi] = \infty$ and $(H\pi \cap K\pi) \nleqslant [F_n : F_n]$, and let us prove that $\tilde{\mathrm{r}}(H \cap K)/\tilde{\mathrm{r}}(K) \leqslant l \operatorname{di}'_{F_n}(H\pi)$. Fix a basis for K, say $K = \langle t^{a'_1}v_1, \ldots, t^{a'_{n_2}}v_{n_2}, L_K \rangle$ and we have

$$\frac{\tilde{\mathrm{r}}(H \cap K)}{\tilde{\mathrm{r}}(K)} = \frac{\tilde{\mathrm{r}}((H \cap K)\pi) + \mathrm{r}(L_H \cap L_K)}{\tilde{\mathrm{r}}(K\pi) + \mathrm{r}(L_K)}. \tag{3.11}$$

We consider the same diagram (see Fig. 3.6) to understand $H \cap K$ as in the proof of part (iii) of Theoem 3.4.2. According to the argument in [10, Thm. 4.5], the crucial property of diagram (3.6) is the fact that $(H \cap K)\pi = (L_H + L_K)R^{-1}\rho_3^{-1}$.

From the hypothesis, $L_H \leqslant_l \mathbb{Z}^m$ and so, $L_H + L_K \leqslant_{l'} \mathbb{Z}^m$, where $1 \leqslant l' \leqslant l$. As in general R is not necessarily onto, $(L_H + L_K)R^{-1} \leqslant_{l''} \mathbb{Z}^{n_3}$ with $1 \leqslant l'' \leqslant l'$. And, since ρ_3 is onto, $(H \cap K)\pi = (L_H + L_K)R^{-1}\rho_3^{-1} \leqslant_{l''} H\pi \cap K\pi$. Therefore, by the Schreier index formula,

$$\tilde{\mathrm{r}}((H \cap K)\pi) = l''\,\tilde{\mathrm{r}}(H\pi \cap K\pi) \quad = \quad l''\tfrac{\tilde{\mathrm{r}}(H\pi \cap K\pi)}{\tilde{\mathrm{r}}(K\pi)}\,\tilde{\mathrm{r}}(K\pi) \quad \leqslant \quad l'' \operatorname{di}'_{F_n}(H\pi)\,\tilde{\mathrm{r}}(K\pi). \tag{3.12}$$

The last inequality of equation (3.12) holds because by construction $K\pi \leqslant_{fg} F_n$ further satisfying $H\pi \cap K\pi \leqslant_\infty H\pi$ and $H\pi \cap K\pi \nleqslant [F_n, F_n]$. Now, using (3.11), we have

$$\frac{\tilde{\mathrm{r}}(H \cap K)}{\tilde{\mathrm{r}}(K)} \leqslant \frac{l'' \operatorname{di}'_{F_n}(H\pi)\,\tilde{\mathrm{r}}(K\pi) + \mathrm{r}(L_H \cap L_K)}{\tilde{\mathrm{r}}(K\pi) + \mathrm{r}(L_K)} \leqslant \frac{l'' \operatorname{di}'_{F_n}(H\pi)\,\tilde{\mathrm{r}}(K\pi)}{\tilde{\mathrm{r}}(K\pi)} = l'' \operatorname{di}'_{F_n}(H\pi), \tag{3.13}$$

where the second inequality is an equality if $L_K = \{0\}$, and follows from applying Lemma 3.4.1 to $\frac{\mathrm{r}(L_H \cap L_K)}{\mathrm{r}(L_K)} \leqslant 1 \leqslant l'' \operatorname{di}'_{F_n}(H\pi)$ otherwise. Therefore,

$$\frac{\tilde{\mathrm{r}}(H \cap K)}{\tilde{\mathrm{r}}(K)} \leqslant l'' \operatorname{di}'_{F_n}(H\pi) \leqslant l' \operatorname{di}'_{F_n}(H\pi) \leqslant l \operatorname{di}'_{F_n}(H\pi). \tag{3.14}$$

To prove the other inequality, $\operatorname{di}'_G(H) \geqslant l \operatorname{di}'_{F_n}(H\pi)$, we fix $\epsilon > 0$ and will construct a subgroup $K_\epsilon \leqslant_{fg} G$ such that $H \cap K_\epsilon$ is again finitely generated, $[H\pi : H\pi \cap K_\epsilon\pi] = \infty$, $H\pi \cap K_\epsilon\pi \not\leqslant [G\pi, G\pi]$ and $\tilde{\mathrm{r}}(H \cap K_\epsilon)/\,\tilde{\mathrm{r}}(K_\epsilon) > l \operatorname{di}'_{F_n}(H\pi) - \epsilon$. For any candidate K, equations (3.12) and (3.14) above contain all the possible reasons why the quotient $\tilde{\mathrm{r}}(H \cap K)/\,\tilde{\mathrm{r}}(K)$ may be less than $l \operatorname{di}'_{F_n}(H\pi)$, namely:

(1) $\tilde{\mathrm{r}}(H\pi \cap K\pi)/\,\tilde{\mathrm{r}}(K\pi) \leqslant \operatorname{di}'_{F_n}(H\pi)$;

(2) $\frac{\tilde{\mathrm{r}}(H \cap K)}{\tilde{\mathrm{r}}(K)} \leqslant l'' \operatorname{di}'_{F_n}(H\pi)$;

(3) $l'' \leqslant l'$;

(4) $l' \leqslant l$.

To control these four possible losses and construct a subgroup $K_\epsilon \leqslant_{fg} G$ making them arbitrary small, we follow this strategy:

(1) choose $M' \leqslant_{fg} F_n$ such that $[H\pi : H\pi \cap M'] = \infty$, $H\pi \cap M' \not\leqslant [F_n, F_n]$ and $\tilde{\mathrm{r}}(H\pi \cap M')/\,\tilde{\mathrm{r}}(M') > \operatorname{di}'_{F_n}(H\pi) - \epsilon/l$ and construct $K_\epsilon \leqslant_{fg} G$ with $K_\epsilon\pi = M'$ (and with $H \cap K_\epsilon$ finitely generated);

(2,4) take $L_{K_\epsilon} = 0$;

(3) choose the matrix A' (i.e., the vectors $\{a'_1, \ldots, a'_{n_2}\}$) so that $R = PA - P'A'$ is onto (note that, in particular, this requires to choose M' in such a way that $n_3 = \mathrm{r}(H\pi \cap M') \geqslant m$).

Therefore if, for every $\epsilon > 0$, we succeed constructing such a $K_\epsilon \leqslant G$ we will be done:

$$\frac{\tilde{\mathrm{r}}(H \cap K_\epsilon)}{\tilde{\mathrm{r}}(K_\epsilon)} = \frac{\tilde{\mathrm{r}}((H \cap K_\epsilon)\pi) + \mathrm{r}(L_H \cap L_{K_\epsilon})}{\tilde{\mathrm{r}}(K_\epsilon\pi) + \mathrm{r}(L_{K_\epsilon})} = \frac{\tilde{\mathrm{r}}((H \cap K_\epsilon)\pi)}{\tilde{\mathrm{r}}(K_\epsilon\pi)} = \frac{l''\,\tilde{\mathrm{r}}(H\pi \cap K_\epsilon\pi)}{\tilde{\mathrm{r}}(K_\epsilon\pi)} =$$

$$= \frac{l'\,\tilde{\mathrm{r}}(H\pi \cap K_\epsilon\pi)}{\tilde{\mathrm{r}}(K_\epsilon\pi)} = \frac{l\,\tilde{\mathrm{r}}(H\pi \cap K_\epsilon\pi)}{\tilde{\mathrm{r}}(K_\epsilon\pi)} > l(\operatorname{di}'_{F_n}(H\pi) - \epsilon/l) = l \operatorname{di}'_{F_n}(H\pi) - \epsilon$$

Now we will consider two cases; namely $\mathrm{di}'_{F_n}(H\pi) > 1$ and $\mathrm{di}'_{F_n}(H\pi) = 1$.

Case-1 **:** $\mathrm{di}'_{F_n}(H\pi) > 1$.

For any small enough $\epsilon > 0$, there always exists a subgroup $M \leqslant_{fg} F_n$ such that $[H\pi : H\pi \cap M] = \infty$, $(N \cap M) \nleqslant [F_n, F_n]$ and

$$\frac{\tilde{\mathrm{r}}(H\pi \cap M)}{\tilde{\mathrm{r}}(M)} > \mathrm{di}'_{F_n}(H\pi) - \frac{\epsilon}{2l} > 1. \tag{3.15}$$

Hence, both the reduced ranks are greater than 1 (from the definition, $0/0$ is understood to be 1),i.e., $\tilde{\mathrm{r}}(H\pi \cap M) > 1$ and $\tilde{\mathrm{r}}(M) > 0$. As $(H\pi \cap M) \nleqslant [F_n, F_n]$, this guarantees us the existence of $v \in H\pi \cap M$ such that $v^{ab} \neq (0, \ldots, 0)$, where $v^{ab} = v\rho$ and ρ is the abelianization map of F_n. Thus we can assume that $|v|_z = \lambda \neq 0$ for some letter z in the alphabet generating F_n. Let us write $\lambda = p_1^{\alpha_1} \ldots p_n^{\alpha_n}$, where each p_i is a prime divisor of λ. Now we choose a prime $d \gg 0$, such that $gcd\ (\lambda, d) = 1$ and $d > 2lm\,\mathrm{di}'_{F_n}(H\pi)/\epsilon$. The following computations (using repeatedly the fact $\tilde{\mathrm{r}}(M) > 0$) show that $(d\,\tilde{\mathrm{r}}(H\pi \cap M) + m)/(d\,\tilde{\mathrm{r}}(M) + m) > \tilde{\mathrm{r}}(H\pi \cap M)/\,\tilde{\mathrm{r}}(M) - \epsilon/2l$.

$$\epsilon\,\tilde{\mathrm{r}}(M)(d\,\tilde{\mathrm{r}}(M) + m) \geqslant \epsilon d\,\tilde{\mathrm{r}}(M) > 2lm\,\mathrm{di}'_{F_n}(H\pi)\,\tilde{\mathrm{r}}(M) \geqslant 2lm\,\tilde{\mathrm{r}}(H\pi \cap M). \tag{3.16}$$

In the above inequality, the second inequality holds because of the choice $d > 2lm\,\mathrm{di}'_{F_n}(H\pi)/\epsilon$ and the third inequality holds from the definition of $\mathrm{di}'_{F_n}(H\pi)$. So,

$$\begin{aligned}
2l(d\,\tilde{\mathrm{r}}(H\pi \cap M) + m)\,\tilde{\mathrm{r}}(M) &\geqslant 2ld\,\tilde{\mathrm{r}}(H\pi \cap M)\,\tilde{\mathrm{r}}(M) \\
&\geqslant 2ld\,\tilde{\mathrm{r}}(H\pi \cap M)\,\tilde{\mathrm{r}}(M) + 2lm\,\tilde{\mathrm{r}}(H\pi \cap M) - \epsilon\,\tilde{\mathrm{r}}(M)(d\,\tilde{\mathrm{r}}(M) + m) \\
&= 2l\,\tilde{\mathrm{r}}(H\pi \cap M)(d\,\tilde{\mathrm{r}}(M) + m) - \epsilon\,\tilde{\mathrm{r}}(M)(d\,\tilde{\mathrm{r}}(M) + m)
\end{aligned} \tag{3.17}$$

In (3.17) the second inequality holds as from (3.16) we have, $2lm\,\tilde{\mathrm{r}}(H\pi \cap M) - \epsilon\,\tilde{\mathrm{r}}(M)(d\,\tilde{\mathrm{r}}(M) + m) < 0$. Dividing both sides by $2l(d\,\tilde{\mathrm{r}}(M) + m)\,\tilde{\mathrm{r}}(M) \neq 0$ we get

$$\frac{(d\,\tilde{\mathrm{r}}(H\pi \cap M) + m)}{(d\,\tilde{\mathrm{r}}(M) + m)} > \frac{\tilde{\mathrm{r}}(H\pi \cap M)}{\tilde{\mathrm{r}}(M)} - \frac{\epsilon}{2l}. \tag{3.18}$$

Now we consider K_d^z as in Lemma 3.5.8. As $gcd\,(\lambda, d) = 1$, applying Lemma 3.5.8 we have,

$$M \cap K_d^z \leqslant_d M. \tag{3.19}$$

As that particular v also belongs to $H\pi \cap M$, again from the Lemma 3.5.8 we have,

$$(H\pi \cap M) \cap K_d^z \leqslant_d (H\pi \cap M) \tag{3.20}$$

Let us consider $M_0 = M \cap K_d^z$. By (3.19) and (3.20), we have $M_0 \leqslant_d M$ and $H\pi \cap M_0 \leqslant_d H\pi \cap\ M$. From the hypothesis we have, $H\pi \cap M \leqslant_\infty H\pi$, which implies that $H\pi \cap M_0 \leqslant_\infty H\pi$. Note that, the pull-back $\Gamma(H\pi \cap M) \times \Gamma(K_d^z)$ is connected and it is $\Gamma(H\pi \cap M \cap K_d^z) = \Gamma(H\pi \cap M_0)$. The pull-back $\Gamma(H\pi) \times \Gamma(M_0)$ is not necessarily connected, but we are interested in the connected component containing its base point $(\odot, \odot)$, which is $\Gamma(H\pi \cap M_0)$. Note that, $\mathrm{r}(H\pi) \geqslant 2$, and the degree of the base point $\odot$ of $\Gamma(H\pi)$ is at least 3. Hence we can apply Proposition 3.5.4 and do an $m{-}expansion$ to M_0, getting new freely independent elements $w_1, \ldots, w_m \in F_n$ such that $M_0 \leqslant_{ff} M' = M_0 * \langle w_1, \ldots, w_m \rangle$ and $H\pi \cap M_0 \leqslant_{ff} (H\pi \cap M_0) * \langle w_1, \ldots, w_m \rangle \leqslant_{ff} (H\pi \cap M')$. By construction, we have $\tilde{\mathrm{r}}(M') = \tilde{\mathrm{r}}(M_0) + m$ and $\tilde{\mathrm{r}}(H\pi \cap M') \geqslant \tilde{\mathrm{r}}(H\pi \cap M_0) + m$.

Now we have to vectorize this subgroup $M' \leqslant F_n$ and construct the desired K_ϵ. Take a free basis $\{v_1, \ldots, v_k\}$ of M_0 and the free basis $\{v_1, \ldots, v_k, w_1, \ldots, w_m\}$ for M'; we have $n_1 = \mathrm{r}(H\pi)$, $k = \mathrm{r}(M_0)$ and $n_2 = \mathrm{r}(M') = k + m$. Similarly, as a free basis for $H\pi \cap M'$ let us take a free basis for $H\pi \cap M_0$ followed by possibly some more, say p, free generators, and finally by $\{w_1, \ldots, w_m\}$; we have $n_3 := \mathrm{r}(H\pi \cap M') = \mathrm{r}(H\pi \cap M_0) + p + m \geqslant m$. Then, let Q be the lower $m \times m$ block in the matrix PA, define

$$A' = \left(\begin{array}{c} 0 \\ \hline -I_m + Q \end{array}\right) \in M_{n_2 \times m}(\mathbb{Z}),$$

and consider the intersection diagram

$$\begin{tikzcd}
 & (H \cap K_\epsilon)\pi \arrow[d, phantom, "\rotatebox{-90}{$\leqslant$}"] & \\
H\pi \arrow[d, two heads, "\rho_1"'] \arrow[dr, phantom, "///"] & H\pi \cap M' \arrow[l, hook', "\iota"'] \arrow[r, hook, "\iota'"] \arrow[d, two heads, "\rho_3"] \arrow[dr, phantom, "///"] & M' \arrow[d, two heads, "\rho_2"] \\
\mathbb{Z}^{n_1} \arrow[dr, "A"] & \mathbb{Z}^{n_3} \arrow[l, "P"'] \arrow[r, "P'"] \arrow[d, red, "R"] & \mathbb{Z}^{n_2} \arrow[dl, "A'"'] \\
 & \mathbb{Z}^m &
\end{tikzcd}$$

Separating the n_3 rows of PA, P', and $P'A'$ into the first $\mathrm{r}(H\pi \cap M_0)$, then the following p, and finally the last m, and separating the n_2 columns of P' into the first k and then the last m, we deduce that

$$R = PA - P'A' = \left(\begin{array}{c} * \\ \hline * \\ \hline Q \end{array}\right) - \left(\begin{array}{c|c} * & 0 \\ \hline * & * \\ \hline 0 & I_m \end{array}\right) \left(\begin{array}{c} 0 \\ \hline -I_m + Q \end{array}\right) =$$

$$= \left(\begin{array}{c} * \\ \hline * \\ \hline Q \end{array}\right) - \left(\begin{array}{c} 0 \\ \hline * \\ \hline -I_m + Q \end{array}\right) = \left(\begin{array}{c} * \\ \hline * \\ \hline I_m \end{array}\right)$$

is a surjective map from $\mathbb{Z}^{n_3}$ onto $\mathbb{Z}^m$. Hence, taking $L_{K_\epsilon} = \{0\}$, we construct our desired subgroup $K_\epsilon = \langle t^{a'_1}v_1, \dots, t^{a'_k}v_k, t^{a'_{k+1}}w_1, \dots, t^{a'_{n_2}}w_m \rangle \leqslant_{fg} G$ where $a'_1, \dots, a'_{n_2} \in \mathbb{Z}^m$ are the rows of A' and $K_\epsilon\pi = M'$. From the construction of M' and from Proposition 3.5.4, there exists $w \in H\pi$ such that $\odot w$ is not defined in $\Gamma(M')$, hence by Lemma 3.5.3, we have $H\pi \cap M' \leqslant_\infty H\pi$, i.e., $H\pi \cap K_\epsilon\pi \leqslant_\infty H\pi$. Also, $H\pi \cap M \nleqslant [F_n, F_n]$ and $H\pi \cap M_0 \leqslant_d H\pi \cap M$, hence $H\pi \cap M_0 \nleqslant [F_n, F_n]$. This implies that $H\pi \cap M' \nleqslant [F_n, F_n]$ as $H\pi \cap M_0 \leqslant H\pi \cap M'$. In other words, $H\pi \cap K_\epsilon\pi \nleqslant [F_n, F_n]$.

Finally the subgroup K_ϵ makes the job because of the following:

$$\frac{\tilde{\mathrm{r}}(H \cap K_\epsilon)}{\tilde{\mathrm{r}}(K_\epsilon)} = \frac{\tilde{\mathrm{r}}((H \cap K_\epsilon)\pi) + \mathrm{r}(L_H \cap L_{K_\epsilon})}{\tilde{\mathrm{r}}(K_\epsilon\pi) + \mathrm{r}(L_{K_\epsilon})} = \frac{\tilde{\mathrm{r}}((H \cap K_\epsilon)\pi)}{\tilde{\mathrm{r}}(K_\epsilon\pi)} =$$

$$= \frac{l\ \tilde{\mathrm{r}}(H\pi \cap K_\epsilon\pi)}{\tilde{\mathrm{r}}(M_0) + m} = l\frac{\tilde{\mathrm{r}}(H\pi \cap M_0) + p + m}{\tilde{\mathrm{r}}(M_0) + m} = l\frac{d\ \tilde{\mathrm{r}}(H\pi \cap M) + p + m}{d\ \tilde{\mathrm{r}}(M) + m}.$$

For the chosen $d > 0$ and using (3.18), we have

$$l\frac{d\ \tilde{\mathrm{r}}(H\pi \cap M) + p + m}{d\ \tilde{\mathrm{r}}(M) + m} \geqslant l\frac{d\ \tilde{\mathrm{r}}(H\pi \cap M) + m}{d\ \tilde{\mathrm{r}}(M) + m} \geqslant l\left(\frac{\tilde{\mathrm{r}}(H\pi \cap M)}{\tilde{\mathrm{r}}(M)} - \frac{\epsilon}{2l}\right)$$

Finally using (3.15), we have

$$\frac{\tilde{\mathrm{r}}(H \cap K_\epsilon)}{\tilde{\mathrm{r}}(K_\epsilon)} \geqslant l\left(\frac{\tilde{\mathrm{r}}(H\pi \cap M)}{\tilde{\mathrm{r}}(M)} - \frac{\epsilon}{2l}\right) > l(\mathrm{di}'_{F_n}(H\pi) - \frac{\epsilon}{2l} - \frac{\epsilon}{2l}) = l\,\mathrm{di}'_{F_n}(H\pi) - \epsilon,$$

and as this is true for any $\epsilon > 0$, we can conclude that $\frac{\tilde{\mathrm{r}}(H \cap K_\epsilon)}{\tilde{\mathrm{r}}(K_\epsilon)} \geqslant l\,\mathrm{di}'_{F_n}(H\pi)$.

Case-2 **:** $\mathrm{di}'_{F_n}(H\pi) = 1$.

We already fixed a basis for H as $\{t^{a_1}u_1, \dots, t^{a_{n_1}}u_{n_1}, t^{b_1}, \dots, t^{b_m}\}$. As $H\pi \nleqslant [F_n, F_n]$, again without loss of generality, we can assume that $u_1 \notin [F_n, F_n]$. Let M be any finitely generated subgroup of $H\pi$ such that $u_1 \in M$, $M \leqslant_\infty H\pi$ and $\mathrm{r}(M) = m$, e.g, $M = \langle u_1, {u_2}^{-1}u_1u_2, \dots, {u_2}^{-(m-1)}u_1{u_2}^{m-1} \rangle$. Let $K =$

$\langle t^{a'_1}u_1, t^{a'_2}u_2{}^{-1}u_1u_2, \dots, t^{a'_m}u_2{}^{-(m-1)}u_1u_2{}^{m-1}\rangle$, where $a'_1, \dots, a'_m \in \mathbb{Z}^m$ will be determined later. Clearly, K is finitely generated, $H\pi \cap K\pi = H\pi \cap M = M = \langle u_1, u_2{}^{-1}u_1u_2, \dots, u_2{}^{-(m-1)}u_1u_2{}^{m-1}\rangle \leqslant_\infty H\pi$ and also $H\pi \cap K\pi \nleqslant [F_n, F_n]$ as $u_1 \notin [F_n, F_n]$.

Now if we consider Fig. 3.6, $n_3 = m$ and

$$P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \in M_{m\times n_1}(\mathbb{Z}), \quad P' = I_m \in M_{m\times m}(\mathbb{Z}).$$

Moreover,

$$R = PA - P'A' = \begin{pmatrix} a_1 \\ \vdots \\ a_1 \end{pmatrix} - \begin{pmatrix} a'_1 \\ \vdots \\ a'_m \end{pmatrix} \in M_{m\times m}(\mathbb{Z}),$$

Taking, $a'_i = a_1 - (0, \dots, 0, 1, 0, \dots, 0)$ (1 is at the $i-th$ position), we get $R = I_m \in M_{m\times m}(\mathbb{Z})$. As, $L_K = \{0\}$, $L_H + L_K = L_H \leqslant_l \mathbb{Z}^m$. So, $(H \cap K)\pi = (L_H + L_K)R^{-1}\rho_3{}^{-1} \leqslant_l H\pi \cap K\pi = K\pi = M$. Hence, $H \cap K$ is finitely generated and

$$\frac{\tilde{\mathrm{r}}(H\cap K)}{\tilde{\mathrm{r}}(K)} = \frac{\tilde{\mathrm{r}}((H\cap K)\pi)}{\tilde{\mathrm{r}}(K\pi)} = \frac{l\,\tilde{\mathrm{r}}(M)}{\tilde{\mathrm{r}}(M)} = l.$$

In fact, for this particular case we prove that $\mathrm{di}'_G(H)$ is maximum and its value exactly equal to $l\,\mathrm{di}'_{F_n}(H\pi) = l \cdot 1 = l$. $\square$

In Theorem 3.4.2 we proved the inequality $\mathrm{di}_G(H) \leqslant l\,\mathrm{di}_{F_n}(H\pi)$, where $\mathrm{r}(H\pi) \geqslant 2$ and $L_H \leqslant_l \mathbb{Z}^m$. And did not succeed in proving that this is in fact an equality. Our failed attempts to do so worked well under the two extra technical assumptions about the subgroup K satisfying $H\pi \cap K\pi \leqslant_\infty H\pi$ and $H\pi \cap K\pi \nleqslant [F_n : F_n]$. We crystallized this partial proof into Theorem 3.5.14, where we get the desired equality but with the definition of degree of inertia conveniently modified (hence the notion of restricted degree of inertia) so that we have at hand the two extra technical conditions needed. However, we still hope it should be possible to arrange our arguments and prove the equality with the genuine definition of degree of inertia. Intuitively, here are two possible ideas to skip using these two extra assumptions:

(1) In the proof of Theorem 3.5.14(i) is essentially the same as that of Theorem 3.4.2(ii) but playing with $\langle u_1^2, u_2^2 \rangle$ instead of $\langle u_1, u_2 \rangle$; this allows us to get the extra condition $H\pi \cap K_N\pi \leqslant_\infty H\pi$ for free. This trick does not work like this in the general situation but maybe a more clever application of this idea could allow us to remove the extra hypothesis $H\pi \cap K\pi \leqslant_\infty H\pi$ in the definition of restricted degree of inertia.

(2) In the proof of Theorem 3.5.14(ii) we manipulated the original subgroup M into another one M' in such a way that the rank of the intersection $n_3 = \mathrm{r}(H\pi \cap M')$ became bigger than or equal to m: this was a mandatory step to do because we needed the linear map $R\colon \mathbb{Z}^{n_3} \to \mathbb{Z}^m$ to be onto in order not to loose index (from l' to l''). However, this change from M to M' must at the same time be done in such a way that the quotients of ranks $\tilde{\mathrm{r}}(H\pi \cap M)/\,\tilde{\mathrm{r}}(M)$ and $\tilde{\mathrm{r}}(H\pi \cap M')/\,\tilde{\mathrm{r}}(M')$ remain close to each other. This goal was achieved by using an m-expansion (see Proposition 3.5.4) which gives

$$\frac{\tilde{\mathrm{r}}(H\pi \cap M')}{\tilde{\mathrm{r}}(M')} = \frac{\tilde{\mathrm{r}}(H\pi \cap M) + p + m}{\tilde{\mathrm{r}}(M) + m}.$$

But to get $\frac{\tilde{\mathrm{r}}(H\pi \cap M')}{\tilde{\mathrm{r}}(M')}$ arbitrarily close to $\frac{\tilde{\mathrm{r}}(H\pi \cap M)}{\tilde{\mathrm{r}}(M)}$, since m is a fixed integer, it is mandatory to assume both the numerator and denominator of the fraction $\frac{\tilde{\mathrm{r}}(H\pi \cap M)}{\tilde{\mathrm{r}}(M)}$ big enough. We achieved this by previously replacing M (and $H\pi \cap M$) by its intersection with the normal finite index subgroup $K_d^z \trianglelefteq_d F_n$, where d is a appropriate big enough integer. Appropriate here means satisfying the equivalent conditions of Lemma 3.5.8 and, to this end, it is mandatory to have the condition $H\pi \cap M \nleqslant [F_n : F_n]$ (otherwise, there is no such d available).

The idea to try to avoid this hypothesis is to look at the descending sequence of iterated commutators in the free group, $F_n' = [F_n, F_n]$, $F_n'' = [F_n' : F_n]$, $F_n''' = [F_n'' : F_n]$, $\ldots$. It is well known that $F_n \geqslant F_n' \geqslant F_n'' \geqslant \cdots$ and the intersection is 1. Hence, our non-trivial subgroup $H\pi$ will contain an element $h \in H\pi$ outside $F_n^{(r)}$ for some r (without assuming necessarily $r = 1$). Now we need to locate a normal subgroup of arbitrarily big index in F_n which could play a similar role to K_d^z but now with respect to $F_n^{(r)}$ instead of F_n'; of course this would require an analogue to Lemma 3.5.8 understanding when the index gets preserved under arbitrary intersections in terms of an explicit condition potentially satisfied by the element h. It seems that a project like will require playing with finite nilpotent groups (much in the same way that travelling around the graph $\Gamma(K_d^z)$ is analogous to playing in the finite abelian group $F_n/K_d^z \simeq \mathbb{Z}/d\mathbb{Z}$).

Conjecture 3.5.15. *For any finitely generated subgroup H of $G = \mathbb{Z}^m \times F_n$,* $\mathrm{di}_G(H) = \mathrm{di}'_G(H)$.

4 Fixed subgroups and computation of auto-fixed closures in free-abelian times free groups

"Groups, as men, will be known by their actions."

— **Guillermo Moreno**

The classical result by Dyer–Scott about fixed subgroups of finite order automorphisms of F_n being free factors of F_n is no longer true in $\mathbb{Z}^m \times F_n$. Within this more general context, we prove a relaxed version in the spirit of Bestvina–Handel Theorem: the rank of fixed subgroups of finite order automorphisms is uniformly bounded in terms of m, n. We also study periodic points of endomorphisms of $\mathbb{Z}^m \times F_n$, and give an algorithm to compute auto-fixed closures of finitely generated subgroups of $\mathbb{Z}^m \times F_n$. On the way, we prove the analog of Day's Theorem for real elements in $\mathbb{Z}^m \times F_n$, contributing a modest step into the project of doing so for any right angled Artin group (as McCool did with respect to Whitehead's Theorem in the free context).

The goal of this chapter is to investigate the structure of the fixed subgroups by endomorphisms (and automorphisms) of G. At a first glance, it may seem that the problems concerning automorphisms will easily reduce to the corresponding problems for $\mathbb{Z}^m$ and F_n; and, in fact, this is the case when the problem considered is easy or rigid enough. When one considers automorphisms; $\operatorname{Aut}(\mathbb{Z}^m \times F_n)$ naturally contains $\mathrm{GL}_m(\mathbb{Z}) \times \operatorname{Aut}(F_n)$, but there are many more automorphisms other than those preserving the factors $\mathbb{Z}^m$ and F_n. This causes potential complications when studying problems involving automorphisms: apart from understanding the problem in both the free-abelian and the free parts, one has to be able to control how is it affected by the interaction between the two parts.

4.1 Preliminaries on $\mathrm{GL}_m(\mathbb{Z})$

In this section we collect well known and folklore results about the general linear group over the integers, $\mathrm{GL}_m(\mathbb{Z})$. This group is very well studied in the literature, but we are interested in highlighting several subtleties coming from the fact that $\mathbb{Z}$ is not a field, but just an integral domain.

Lemma 4.1.1. *Let $Q \in \mathrm{GL}_m(\mathbb{Z})$ be a matrix such that $Q^k = I_m$. Then, we have the decomposition $\mathbb{Z}^m = \ker(Q - I_m) \oplus \ker(Q^{k-1} + \cdots + Q + I_m)$.*

Proof. Since $\gcd(x^{k-1} + \cdots + x + 1,\ x - 1) = 1$, Bezout's equality gives us two polynomials $\alpha(x), \beta(x) \in \mathbb{Z}[x]$ such that $1 = \alpha(x)(x^{k-1} + \cdots + x + 1) + \beta(x)(x - 1)$. Plugging Q, we obtain the matrix equality $I_m = \alpha(Q)(Q^{k-1} + \cdots + Q + I_m) + \beta(Q)(Q - I_m)$. Now, for every vector $v \in \mathbb{Z}^m$, we have $v = v\alpha(Q)(Q^{k-1} + \cdots + Q + I_m) + v\beta(Q)(Q - I_m)$. And, since $(Q - I_m)(Q^{k-1} + \cdots + Q + I_m) = (Q^{k-1} + \cdots + Q + I_m)(Q - I_m) = Q^k - I_m = 0$, the first summand is in $\ker(Q - I_m)$ and the second one in $\ker(Q^{k-1} + \cdots + Q + I_m)$; hence, $\mathbb{Z}^m = \ker(Q - I_m) + \ker(Q^{k-1} + \cdots + Q + I_m)$.

Now let $v \in \ker(Q - I_m) \cap \ker(Q^{k-1} + \cdots + Q + I_m)$. This means that $v(Q - I_m) = 0$ and $v(Q^{k-1} + \cdots + Q + I_m) = 0$, which imply $v = v(Q^{k-1} + \cdots + Q + I_m)\alpha(Q) + v(Q - I_m)\beta(Q) = 0$. Thus, $\mathbb{Z}^m = \ker(Q - I_m) \oplus \ker(Q^{k-1} + \cdots + Q + I_m)$. □

To proof our Propositions 4.1.3 and 4.1.4 we will use one result about Euler φ-function from the literature.

Lemma 4.1.2. $\lim_{n\to\infty} \varphi(n) = \infty$*, where φ is the Euler φ-function.*

Proof. If p is a prime and $k \geqslant 1$, $\varphi(p^k) = p^{k-1}(p-1) = p^k(1-\frac{1}{p})$. The fundamental theorem of arithmetic states that if $n \in \mathbb{N}$ and $n > 1$ there is a unique expression for n,

$$n = {p_1}^{k_1} \cdots {p_r}^{k_r},$$

where $p_1 < p_2 < \cdots < p_r$ are prime numbers and each $k_i \geqslant 1$. (The case n = 1 corresponds to the empty product.) Repeatedly using the multiplicative property of φ and the formula for $\varphi(p^k)$ gives

$$\begin{aligned}
\varphi(n) &= \varphi({p_1}^{k_1}) \cdots \varphi({p_r}^{k_r}) \\
&= {p_1}^{k_1}(1-\tfrac{1}{p_1}){p_2}^{k_2}(1-\tfrac{1}{p_2}) \cdots {p_r}^{k_r}(1-\tfrac{1}{p_r}) \\
&= {p_1}^{k_1}{p_2}^{k_2} \cdots {p_r}^{k_r}(1-\tfrac{1}{p_1})(1-\tfrac{1}{p_2}) \cdots (1-\tfrac{1}{p_r}) \\
&= n(1-\tfrac{1}{p_1})(1-\tfrac{1}{p_2}) \cdots (1-\tfrac{1}{p_r}).
\end{aligned}$$

As $p_1 \geqslant 2$ and $p_1 < p_2 < \cdots < p_r$, $n \geqslant 2^r$. Applying logarithm function, we have $\frac{\log n}{\log 2} \geqslant r$. Now we have the following inequality

$$\begin{aligned}
\varphi(n) &= n(1-\tfrac{1}{p_1})(1-\tfrac{1}{p_2}) \cdots (1-\tfrac{1}{p_r}) \\
&\geqslant n(1-\tfrac{1}{2})(1-\tfrac{1}{3})^{r-1} \\
&\geqslant n(1-\tfrac{1}{2})(1-\tfrac{1}{3})^{\frac{\log n}{\log 2}-1} \\
&= n \times \tfrac{1}{2} \times (\tfrac{2}{3})^{\frac{\log n}{\log 2}-1} \\
&= \tfrac{n}{2} \times \tfrac{3}{2} \times (\tfrac{2}{3})^{\frac{\log n}{\log 2}} \\
&= \tfrac{3}{4}n^{1+1-\frac{\log 3}{\log 2}} \\
&= \tfrac{3}{4}n^{2-\frac{\log 3}{\log 2}}.
\end{aligned} \tag{4.1}$$

Hence, $\varphi(n) \geqslant \frac{3}{4}n^c$ where $c > 0$ and so $\lim_{n\to\infty} \varphi(n) = \infty$. $\square$

With the help of the Lemma 4.1.1, we are able to bound the order of any arbitrary matrix $Q \in \mathrm{GL}_m(\mathbb{Z})$.

Proposition 4.1.3. *Consider the integral linear group* $\mathrm{GL}_m(\mathbb{Z})$*,* $m \geqslant 1$*.*

(i) There exists a computable constant $L_1 = L_1(m)$ *such that, for every matrix* $Q \in \mathrm{GL}_m(\mathbb{Z})$ *of finite order,* $\mathrm{ord}(Q) \leqslant L_1$*.*

(ii) There exists a computable constant $L_2 = L_2(m)$ *such that, for every matrix* $Q \in \mathrm{GL}_m(\mathbb{Z})$ *of finite order, say* $k = \mathrm{ord}(Q) \leqslant L_1$*, we have that* $M = \mathrm{Im}(Q - I_m)$ *is a finite index subgroup of* $\ker(Q^{k-1} + \cdots + Q + I_m)$ *with* $[\ker(Q^{k-1} + \cdots + Q + I_m) : M] \leqslant L_2$*.*

Proof. (i) is a well known fact about integral matrices; we offer here a self-contained proof mixed with that of (ii).

Let $Q \in \mathrm{GL}_m(\mathbb{Z})$ be a matrix of order $k < \infty$ (i.e., $Q^k = I_m$ but $Q^i \neq I_m$ for $i = 1, \ldots, k-1$).

Since $(Q - I_m)(Q^{k-1} + \cdots + Q + I_m) = Q^k - I_m = 0$, we have $M = \mathrm{Im}(Q - I_m) \leqslant \ker(Q^{k-1} + \cdots + Q + I_m)$. But, by Lemma 4.1.1 and the Rank-Nullity Theorem, $\mathrm{r}(M) = \mathrm{r}(\mathrm{Im}(Q - I_m)) = m - \mathrm{r}(\ker(Q - I_m)) = \mathrm{r}(\ker(Q^{k-1} + \cdots + Q + I_m))$ and so, $M \leqslant_{fi} \ker(Q^{k-1} + \cdots + Q + I_m)$. This is the index we have to bound globally in terms of m.

Let $m_Q(x)$ be the minimal polynomial of Q. Since $Q^k = I_m$, we have $m_Q(x) \,|\, x^k - 1$ and so, $m_Q(x) = (x - \alpha_1) \cdots (x - \alpha_r)$, where $\alpha_1 \ldots, \alpha_r$ are $r \leqslant m$ pairwise different k-th roots of unity (in particular, all roots of $m_Q(x)$ are simple and so Q diagonalizes over the complex field $\mathbb{C}$). Write $d_i = \mathrm{ord}(\alpha_i)$. Since cyclotomic polynomials $\Phi_{d_i}(x)$ are irreducible over $\mathbb{Z}$, we deduce $\Phi_{d_i}(x) \,|\, m_Q(x)$ and so, $\varphi(d_i) = \deg(\Phi_{d_i}(x)) \leqslant \deg(m_Q(x)) \leqslant m$, where φ is the Euler φ-function. From Lemma 4.1.2, $\lim_{n \to \infty} \varphi(n) = \infty$ from where we can compute a big enough constant $C = C(m)$ such that $d_1, \ldots, d_r \leqslant C$. Finally,

$k = \mathrm{ord}(Q) = \mathrm{lcm}(\mathrm{ord}(\alpha_1) \ldots, \mathrm{ord}(\alpha_r)) = \mathrm{lcm}(d_1, \ldots, d_r) \leqslant d_1 \cdots d_r \leqslant C^r \leqslant C^m$; this is the constant we are looking for in (i), $L_1 = C(m)^m$.

On the other hand, diagonalizing Q, we get an invertible complex matrix $P \in \mathrm{GL}_m(\mathbb{C})$ such that $P^{-1}QP = D = \mathrm{diag}(\alpha_1, \overset{s_1}{\ldots}, \alpha_1, \ldots, \alpha_r, \overset{s_r}{\ldots}, \alpha_r)$, where $s_1, \ldots, s_r$ are the multiplicities in the characteristic polynomial, $\chi_Q(x) = (x - \alpha_1)^{s_1} \cdots (x - \alpha_r)^{s_r}$. Since α_i is a primitive d_i-th root of unity, it can take $\varphi(d_i) \leqslant m$ many values and, since $s_1 + \cdots + s_r = m$, the diagonal matrix D can take only finitely many values; we can make a list of all of them (up to reordering of the α_i's) and, for each one, compute the index $[\ker(D^{k-1} + \cdots + D + I_m) : \mathrm{Im}(D - I_m)]$. The maximum of these indices is the constant $L_2 = L_2(m)$ we are looking for in (ii), because

$$\begin{aligned}
[\ker(Q^{k-1} + \cdots + Q + I_m) : M] &= [(\ker(Q^{k-1} + \cdots + Q + I_m))P : (\mathrm{Im}(Q - I_m))P] \\
&= [\ker(P^{-1}(Q^{k-1} + \cdots + Q + I_m)P) : \mathrm{Im}(P^{-1}(Q - I_m)P)] \\
&= [\ker(D^{k-1} + \cdots + D + I_m) : \mathrm{Im}(D - I_m)] \\
&\leqslant L_2(m).
\end{aligned}$$

This completes the proof. $\square$

Now we study the *periodic* subgroup of a matrix $Q \in \mathrm{M}_{m\times m}(\mathbb{Z})$, namely $\mathrm{Per}\, Q = \{v \in \mathbb{Z}^m \mid vQ^p = v, \text{ for some } p \geqslant 1\}$. The next Proposition states that a uniform single exponent depending only on m, $L_3 = L_3(m)$, is enough to capture *all* the periodicity of *all* $m \times m$ matrices Q.

Proposition 4.1.4. *There exists a computable constant $L_3 = L_3(m)$ such that* $\mathrm{Per}\, Q = \mathrm{Fix}\, Q^{L_3}$*, for every $Q \in \mathrm{M}_{m\times m}(\mathbb{Z})$.*

Proof. As we argued in the proof of Proposition 4.1.3(i), there is a computable constant $C = C(m)$ such that $\varphi(d) > m$ for every $d > C(m)$; in fact from (4.1) we can choose $C(m) = 8m^3$ and then $\varphi(d) > \frac{3}{4}(8m^3)^{(2-\frac{\log 3}{\log 2})} = 1.78m^{1.26} > m$. Let us prove that the statement is true with the constant $L_3 = C(m)! = (8m^3)!$

Fix a matrix $Q \in \mathrm{M}_{m\times m}(\mathbb{Z})$, and consider its characteristic polynomial factorized over the complex field $\mathbb{C}$, $\chi_Q(x) = (x-\alpha_1)^{s_1}\cdots(x-\alpha_r)^{s_r}$, where $\alpha_i \neq \alpha_j$, $i \neq j$. Standard linear algebra tells us that $\mathbb{C}^m = K_{\alpha_1} \oplus \cdots \oplus K_{\alpha_r}$, where $K_{\alpha_i} = \ker(Q-\alpha_i I_m)^{s_i} \leqslant \mathbb{C}^m$ is the generalized eigenspace of Q with respect to α_i, a Q-invariant $\mathbb{C}$-subspace of $\mathbb{C}^m$. Distinguish now between those α_i's which are roots of unity, say $\alpha_1, \ldots, \alpha_{r'}$, and those which are not, say $\alpha_{r'+1}, \ldots, \alpha_r$, $0 \leqslant r' \leqslant r$. Write $d_i = \mathrm{ord}(\alpha_i)$, for $i = 1, \ldots, r'$, and observe that $d_1, \ldots, d_{r'} \leqslant C$ (since the cyclotomic polynomials $\Phi_{d_i}(x)$ are $\mathbb{Q}$-irreducible and so must divide $\chi_Q(x) \in \mathbb{Z}[X]$, which has degree m); in particular, $\alpha_i^{L_3} = 1$, $i = 1, \ldots, r'$.

Now, let $v \in \mathrm{Per}\, Q$, i.e., $vQ^p = v$ for some $p \geqslant 1$. Applying the above decomposition, $v = v_1 + \cdots + v_r$, where $v_i \in K_{\alpha_i}$, and the Q-invariance of K_{α_i}, we get the alternative decomposition $v = vQ^p = v_1Q^p + \cdots + v_rQ^p$. So, $v_iQ^p = v_i$, i.e., $v_i(Q^p - I_m) = 0$, for $i = 1, \ldots, r$. For a fixed i, distinguish the following two cases:

(i) if $\alpha_i^p \neq 1$, then α_i is not a root of $x^p - 1$ and so, $1 = \gcd\left((x-\alpha_i)^{s_i}, x^p - 1\right)$. By Bezout's equality, there are polynomials $a(x), b(x) \in \mathbb{C}[x]$ such that $1 = (x-\alpha_i)^{s_i}a(x) + (x^p-1)b(x)$. Plugging the matrix Q and multiplying by the vector v_i on the left, we obtain $v_i = v_i(Q-\alpha_i I_m)^{s_i}a(Q) + v_i(Q^p - I_m)b(Q) = 0$.

(ii) if $\alpha_i^p = 1$, then $x - \alpha_i = \gcd\left((x-\alpha_i)^{s_i}, x^p - 1\right)$. By Bezout's equality, there are polynomials $a(x), b(x) \in \mathbb{C}[x]$ such that $x - \alpha_i = (x-\alpha_i)^{s_i}a(x) + (x^p-1)b(x)$. Now, plugging the matrix Q and multiplying by the vector v_i on the left, we have $v_i(Q - \alpha_i I_m) = v_i(Q-\alpha_i I_m)^{s_i}a(Q) + v_i(Q^p - I_m)b(Q) = 0$. That is, $v_iQ = \alpha_i v_i$ and so, $v_iQ^{L_3} = \alpha_i^{L_3}v_i = v_i$.

Altogether we have, $v = v_1 + \cdots + v_r = \sum_{i \,|\, \alpha_i^p = 1} v_i$ and $vQ^{L_3} = \left(\sum_{i \,|\, \alpha_i^p=1} v_i\right)Q^{L_3} = \sum_{i \,|\, \alpha_i^p=1} v_iQ^{L_3} = \sum_{i \,|\, \alpha_i^p=1} v_i = v$, and $v \in \mathrm{Fix}\, Q^{L_3}$. This completes the proof that $\mathrm{Per}\, Q = \mathrm{Fix}\, Q^{L_3}$. $\square$

4.2 Concept of "factor" and Takahashi theorem for $\mathbb{Z}^{\mathbf{m}} \times \mathbf{F_n}$

In this section I introduce the notion of *factor* in $\mathbb{Z}^m \times F_n$, which can be considered as parallel notion of the concepts of *direct summand* in $\mathbb{Z}^m$ and *free factor* in F_n.

Lemma 4.2.1. *Let $G = \mathbb{Z}^m \times F_n$. For given finitely generated subgroups $H \leqslant_{fg} K \leqslant_{fg} G$, the following are equivalent:*

(a) every basis of H extends to a basis of K;

(b) some basis of H extends to a basis of K;

(c) $H\pi \leqslant_{ff} K\pi$ and $L_H \leqslant_{\oplus} L_K$.

In this case, we say that H is a *factor* of K, denoted $H \leqslant_f K$; this is the notion in G corresponding to *free factor* in F_n (denoted $\leqslant_{ff}$), and *direct summand* in $\mathbb{Z}^m$ (denoted $\leqslant_{\oplus}$).

Proof. $(a) \Rightarrow (b)$ is obvious.

Assuming (b), we have $H = \langle t^{a_1}u_1, \ldots, t^{a_r}u_r, t^{b_1}, \ldots, t^{b_s}\rangle$ and $K = \langle t^{a_1}u_1, \ldots, t^{a_r}u_r, t^{a_{r+1}}u_{r+1}, \ldots, t^{a_{r+p}}u_{r+p}, t^{b_1}, \ldots, t^{b_s}, t^{b_{s+1}}, \ldots, t^{b_{s+q}}\rangle$, where $\{u_1, \ldots, u_r\}$ is a free-basis of $H\pi$, $\{b_1, \ldots, b_s\}$ is an abelian-basis of L_H, $\{u_1, \ldots, u_{r+p}\}$ is a free-basis of $K\pi$, and $\{b_1, \ldots, b_{s+q}\}$ is an abelian-basis of L_K. Therefore, $H\pi \leqslant_{ff} K\pi$ and $L_H \leqslant_{\oplus} L_K$. This proves $(b) \Rightarrow (c)$.

Finally, assume (c). Given any basis $\{t^{a_1}u_1, \ldots, t^{a_r}u_r, t^{b_1}, \ldots, t^{b_s}\}$ for H, $\{u_1, \ldots, u_r\}$ is a free-basis of $H\pi$, which can be extended to a free-basis $\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_{r+p}\}$ of $K\pi$ since $H\pi \leqslant_{ff} K\pi$; and $\{b_1, \ldots, b_s\}$ is an abelian-basis of L_H, which can be extended to an abelian-basis $\{b_1, \ldots, b_s, b_{s+1}, \ldots, b_{s+q}\}$

of L_K since $L_H \leqslant_\oplus L_K$. Then, choose vectors $a_{r+1}, \ldots, a_{r+p} \in \mathbb{Z}^m$ such that $t^{a_{r+1}}u_{r+1}, \ldots, t^{a_{r+p}}u_{r+p} \in K$ (this is always possible because $u_{r+1}, \ldots, u_{r+p} \in K\pi$), and $\{t^{a_1}u_1, \ldots, t^{a_r}u_r, t^{a_{r+1}}u_{r+1}, \ldots, t^{a_{r+p}}u_{r+p}, t^{b_1}, \ldots, t^{b_s}, t^{b_{s+1}}, \ldots, t^{b_{s+q}}\}$ is a basis of K (in fact, they generate K, and have the appropriate form). This proves $(c) \Rightarrow (a)$. $\square$

Proposition 4.2.2. *Let $M, A, B \leqslant \mathbb{Z}^m \times F_n$ and $M \leqslant_f A$, $M \leqslant_f B$. Then we have, $M \leqslant_f A \cap B$.*

Proof. From the definition, $M\pi \leqslant_{ff} A\pi$ and $M\pi \leqslant_{ff} B\pi$. Intersection of free factors is again a free factor in free group. As $M\pi, A\pi, B\pi$ all are subgroups of F_n, we have $M\pi \leqslant_{ff} A\pi \cap B\pi$. Now, $M\pi \leqslant (A \cap B)\pi \leqslant A\pi \cap B\pi$, so we have $M\pi \leqslant_{ff} (A \cap B)\pi$. On the other hand, $L_M \leqslant_\oplus L_A$ and also $L_M \leqslant_\oplus L_B$. Hence, $L_M \leqslant_\oplus (L_A \cap L_B) = L_{A\cap B}$. Applying Lemma 4.2.1 we have, $M \leqslant_f A \cap B$. $\square$

Now we will extend the notion of *fringe* from free groups to free-abelian times free groups. For a given $H = \langle t^{a_1}u_1, \ldots, t^{a_r}u_r, L_H\rangle$, $H\pi = \langle u_1, \ldots, u_r\rangle$ and $\{u_1, \ldots, u_r\}$ is the free basis of $H\pi \leqslant F_n$. So we can compute $\mathcal{AE}(H\pi)$.

Let $\mathcal{AE}(H\pi) = \{M_0, M_1, \ldots, M_s\}$, where $M_0 = H\pi$. Let for any $0 \leqslant i \leqslant s$, $M_i = \langle v_1, \ldots, v_p\rangle$, and $\{v_1, \ldots, v_p\}$ is a free basis of M_i. And let $\widetilde{M_i}(c_1, \ldots, c_p) = \langle t^{c_1}v_1, \ldots, t^{c_p}v_p, \tilde{L}\rangle$. Now we construct $\tilde{L}$ in such a way that $\widetilde{M_i}(c_1, \ldots, c_p)$ contains H. Since, $H\pi \leqslant_{alg} M_i$, each u_i can be written as a unique word in $\{v_1, \ldots, v_p\}$. Let $\mathcal{U}_{M_i}$

be the abelianization matrix of $H\pi$ with respect to the ambient basis $\{v_1, \ldots, v_p\}$ and let $L(c_1, \ldots, c_p) = \langle L_H, \text{row}(A - \mathcal{U}_{M_i} C)\rangle$, where

$$\mathcal{U}_{M_i} = \begin{pmatrix} |u_1|_{v_1} & |u_1|_{v_2} & \ldots & |u_1|_{v_p} \\ \vdots & \vdots & & \vdots \\ |u_i|_{v_1} & |u_i|_{v_2} & \ldots & |u_i|_{v_p} \\ \vdots & \vdots & & \vdots \\ |u_r|_{v_1} & |u_r|_{v_2} & \ldots & |u_r|_{v_p} \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix} \text{ and } A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix}.$$

Now $L_H \leqslant L(c_1, \ldots, c_p) \leqslant \mathbb{Z}^m$. Therefore there exists $\tilde{L}$ such that $L(c_1, \ldots, c_p) \leqslant_{fi} \tilde{L} \leqslant_{\oplus} \mathbb{Z}^m$. Choosing $\tilde{L}$ in this way we confirm that $H \leqslant \widetilde{M}_i(c_1, \ldots, c_p)$. Since, H is given, we can not choose L_H and A. So, this $L(c_1, \ldots, c_p)$ depends on the choice of C; in other words L depends on the choice of the vectors $c_1, \ldots, c_p$.

Let $\mathcal{S} = \{S \mid L(c_1, \ldots, c_p) \leqslant_{fi} S \leqslant_{fi} \tilde{L}\}$. And now we are in position to define "*fringe*" of H, $\mathcal{O}(H)$, for any subgroup H in $\mathbb{Z}^m \times F_n$.

Definition 4.2.3. For a given subgroup H of $\mathbb{Z}^m \times F_n$, the *fringe* of H, denoted $\mathcal{O}(H)$, is defined as $\mathcal{O}(H) = \{\langle t^{c_1}v_1, \ldots, t^{c_p}v_p, S\rangle \mid \langle v_1, \ldots, v_p\rangle \in \mathcal{AE}(H\pi)$ and $L(c_1, \ldots, c_p) \leqslant_{fi} S \leqslant_{fi} \tilde{L}\}$.

Remark 4.2.4. This is the natural way to translate the definition of fringe from free groups to free-abelian times free groups. And the following result is the analog of Takahasi's theorem within this more general family of groups: for any given extension of finitely generated subgroups $H \leqslant_{fg} K \leqslant_{fg} \mathbb{Z}^m \times F_n$, maybe H is not a factor of K but at least one of the members in the fringe of H, say $M \in \mathcal{O}(H)$, will be: $H \leqslant M \leqslant_f K$. Takahasi's theorem is quite useful in free groups because, for a given H, there are in general infinitely many bigger subgroups K but only finitely many of them belong to the fringe $\mathcal{O}(H)$. This fact allows arguments like this: in any situation where one has a subgroup H and infinitely many extensions $H \leqslant K_i$, $i \geqslant 1$, infinitely many of them must be free multiples of *a commond* $M \in \mathcal{O}(H)$, namely $H \leqslant M \leqslant_{ff} K_i$ for infinitely many indices i. Unfortunately, it seems we loose this potential in the free-abelian times free version because fringes are, in general, infinitely big because the vector parameters $c_1, \ldots, c_p$ can take arbitrary values in $\mathbb{Z}^m$ (a fact, on the other hand, unavoidable if we want to preserve Takahasi's theorem). We observe that $|\mathcal{O}(H)| = \infty$ but organized in *finitely many* patterns each one with *finitely many* vector parameters running freely over $\mathbb{Z}^m$; however, this seems not to be strong enough to make those kind of arguments work in this more general context. Example 4.2.7 below illustrates this phenomena.

As an obvious consequence of the concept of fringe, we would like to extend Takahashi theorem for our group, $\mathbb{Z}^m \times F_n$.

Theorem 4.2.5. *For every $H \leqslant_{fg} K_{fg} \leqslant \mathbb{Z}^m \times F_n$, there exists $\widetilde{M} \in \mathcal{O}(H)$ such that $H \leqslant \widetilde{M} \leqslant_f K$.*

Proof. Let $\langle t^{a_1}u_1, \ldots, t^{a_r}u_r, L_H\rangle$ be a basis for H; then $\{u_1, \ldots, u_r\}$ is the free-basis of $H\pi$. As $H\pi \leqslant K\pi \leqslant F_n$, there exists $M \in \mathcal{AE}(H\pi)$ such that $H\pi \leqslant_{alg} M \leqslant_{ff} K\pi$. Let $\{v_1, \ldots, v_p\}$ is a free basis of M, and this basis of M can be extended to a basis of $K\pi$. Let $\{v_1, \ldots, v_p, v_{p+1}, \ldots, v_q\}$ be a free-basis of $K\pi$ and let $K = \langle t^{c_1}v_1, \ldots, t^{c_p}v_p, t^{c_{p+1}}v_{p+1}, \ldots, t^{c_q}v_q, L_K\rangle$. Since, $H\pi \leqslant_{alg} M$, each u_i can be written as a unique word in $\{v_1, \ldots, v_p\}$, i.e., $u_i = w_i(v_1, \ldots, v_p)$. Therefore $w_i(t^{c_1}v_1, \ldots, t^{c_p}v_p) = t^{\tilde{c_i}}w_i(v_1, \ldots, v_p) = t^{\tilde{c_i}}u_i$, where $\tilde{c_i} = c_1|u_i|_{v_1} + c_2|u_i|_{v_2} + \ldots + c_p|u_i|_{v_p}$.

Let $L = \langle L_H, \mathrm{row}(A - \mathcal{U}_M C)\rangle$, where

$$\mathcal{U}_M = \begin{pmatrix} |u_1|_{v_1} & |u_1|_{v_2} & \dots & |u_1|_{v_p} \\ \vdots & \vdots & & \vdots \\ |u_i|_{v_1} & |u_i|_{v_2} & \dots & |u_i|_{v_p} \\ \vdots & \vdots & & \vdots \\ |u_r|_{v_1} & |u_r|_{v_2} & \dots & |u_r|_{v_p} \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix} \text{ and } A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix}.$$

Since $H \leqslant K$, $L_H \leqslant L \leqslant L_K$. Therefore there exists $\tilde{L}$ such that $L \leqslant_{fi} \tilde{L} \leqslant_{\oplus} L_K$.

Now take $\widetilde{M} = \langle t^{c_1}v_1, \dots, t^{c_p}v_p, \tilde{L}\rangle$. From the construction $\widetilde{M} \in \mathcal{O}(H)$ and $t^{\tilde{c}_i}u_i \in \widetilde{M}$ and $t^{a_i - \tilde{c}_i} \in \mathrm{row}(A - \mathcal{U}_M C) \leqslant \tilde{L}$, hence we have $t^{a_i}u_i \in \widetilde{M}$, $\forall i = 1, 2, \dots, r$ and it is clear that $L_H \leqslant \tilde{L}$. Then altogether we have $H \leqslant \widetilde{M}$. The only remaining thing is to show that $\widetilde{M} \leqslant_f K$. But, $\widetilde{M} \cap \mathbb{Z}^m = \tilde{L} \leqslant_{\oplus} L_K$ and $\widetilde{M}\pi = M \leqslant_{ff} K$. Hence Lemma 4.2.1 completes the proof. □

Remark 4.2.6. Among many applications of Takahashi theorem for free groups, one application which I want to mention in this context is, A. Martino and E. Ventura [26] proved that, for two given automorphisms (or endomorphisms) f, g, there is a word on them $h = w(f, g)$ such that $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) \leqslant_{ff} \mathrm{Fix}(h)$. This is not true in $\mathbb{Z}^m \times F_n$ as seen in the following example.

Example 4.2.7. This is the example of two automorphisms, f, g of $\mathbb{Z}^m \times F_n$ such that $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is not a free factor of any word $w(f, g)$. Let us define f, g in the following way :

$$\begin{array}{rcl} f\colon \mathbb{Z} \times F_2 & \longrightarrow & \mathbb{Z} \times F_2 \\ a & \longmapsto & ta \\ b & \longmapsto & b \\ t & \longmapsto & t \end{array} \qquad \begin{array}{rcl} g\colon \mathbb{Z} \times F_2 & \longrightarrow & \mathbb{Z} \times F_2 \\ a & \longmapsto & a \\ b & \longmapsto & tb \\ t & \longmapsto & t \end{array}$$

Therefore $\mathrm{Fix}(f) = \langle t\rangle \times \langle\langle b\rangle\rangle$ and $\mathrm{Fix}(g) = \langle t\rangle \times \langle\langle a\rangle\rangle$. So $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) = \langle t\rangle \times [F_2\colon F_2]$. From an easy computation we have, $fg = gf$. Let $h = w(f, g)$ be any word in f and g and we can take $h = f^{\alpha}g^{\beta}$ (say), as f and g commute, we can take all f's in one side. Now $\mathrm{Fix}(f^{\alpha}g^{\beta}) = \langle t\rangle \times \langle w(a,b) \mid (|w|_a \ \ |w|_b)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0\rangle$. Therefore if $(\alpha,\beta) = (1,0)$, $\mathrm{Fix}(f^{\alpha}g^{\beta}) = \langle t\rangle \times \langle\langle b\rangle\rangle$ and if $(\alpha,\beta) = (0,1)$, $\mathrm{Fix}(f^{\alpha}g^{\beta}) = \langle t\rangle \times \langle\langle a\rangle\rangle$. Let, $K_{\alpha\beta} = \langle w(a,b) \mid (|w|_a \ \ |w|_b)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0\rangle$. Now we consider two cases.

Case-1: $(\alpha,\beta) = (0,0)$

$\mathrm{Fix}(f^{\alpha}g^{\beta}) = \mathrm{Fix}\ \ \mathrm{id} = \mathbb{Z} \times F_2$. And $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) = \langle t\rangle \times [F_2\colon F_2]$ is not a *factor* of $\mathbb{Z} \times F_2$.

Case-2: $(\alpha,\beta) \neq (0,0)$

Let gcd $(\alpha,\beta) = \lambda$, then there exists α', β' such that gcd $(\alpha',\beta') = 1$ and $\alpha = \lambda\alpha', \beta = \lambda\beta'$. Therefore, $K_{\alpha\beta} = K_{\alpha'\beta'}$. Hence without loss of generality we can assume that gcd $(\alpha,\beta) = 1$. Since gcd $(\alpha,\beta) = 1$, there exists $x, y \in \mathbb{Z}$ such that $\alpha x + \beta y = 1$. Let $M = \begin{pmatrix} y & \alpha \\ -x & \beta\end{pmatrix} \in GL_2(\mathbb{Z})$, viewed as an automorphism on $\mathbb{Z}^2$ defined as:

$$\begin{aligned} M\colon \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2 \\ (1,0) &\longmapsto (y,\alpha) \\ (0,1) &\longmapsto (-x,\beta) \end{aligned}$$

Since, $\Psi\colon\ \mathrm{Aut}(F_2) \to GL_2(\mathbb{Z})$ is onto, there exists $\phi \in \mathrm{Aut}(F_2)$ such that $\phi\Psi = M$, say

$$\begin{aligned} \phi\colon F_2 &\longrightarrow F_2 \\ a &\longmapsto U \\ b &\longmapsto V \end{aligned}$$

such that $U^{ab} = (y, \alpha)$ and $V^{ab} = (-x, \beta)$. Now we will proof that $K_{\alpha\beta}\phi = K_{01}$. Let $w(a,b) \in K_{\alpha\beta}$ then $|w|_a\alpha + |w|_b\beta = 0$ and

$$\begin{aligned}\begin{pmatrix} |w\phi|_a & |w\phi|_b \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= |w\phi|_b \\ &= |w(U,V)|_b \\ &= |w(U,V)|_U \times |U|_b + |w(U,V)|_V \times |V|_b \\ &= |w|_a\alpha + |w|_b\beta \\ &= 0.\end{aligned}$$

Therefore,

$$K_{\alpha\beta}\phi \leqslant K_{01}. \tag{4.2}$$

Let $a\phi^{-1} = U'$ and $b\phi^{-1} = V'$. Now $\phi^{-1}\Psi = M^{-1} = \begin{pmatrix} \beta & -\alpha \\ x & y \end{pmatrix}$. Therefore, $U'^{ab} = (\beta, -\alpha)$ and $V'^{ab} = (x, y)$. Now $K_{01} = \langle\langle a \rangle\rangle$ and $K_{\alpha\beta} \trianglelefteq F_2 \Rightarrow K_{\alpha\beta}\phi \trianglelefteq F_2$. Therefore to show that $K_{\alpha\beta}\phi \geqslant K_{01}$, we just have to show that $a \in K_{\alpha\beta}\phi$, i.e., $a\phi^{-1} \in K_{\alpha\beta}$. But,

$$\begin{aligned}&|a\phi^{-1}|_a\alpha + |a\phi^{-1}|_b\beta \\ &= |U'|_a\alpha + |U'|_b\beta \\ &= \beta\alpha - \alpha\beta \\ &= 0,\end{aligned}$$

hence, $a\phi^{-1} \in K_{\alpha\beta}$ i.e.,

$$K_{01} \leqslant K_{\alpha\beta}\phi. \tag{4.3}$$

Therefore by (4.2) and (4.3) we have $K_{01} = K_{\alpha\beta}\phi$. And now we show that $[F_2 : F_2]$ is not a free-factor of $K_{01} = \langle\langle a \rangle\rangle = \langle\ \ldots, b^{-2}ab^2, b^{-1}ab; a; bab^{-1}, b^2ab^{-2}, \ldots\ \rangle$. Let $\rho_{\langle\langle a \rangle\rangle}$ be the abelianization map from $\langle\langle a \rangle\rangle$ to $\mathbb{Z}^\infty$. Note that it is not the restriction of the abelianization map ρ of F_2. And take free-basis of $[F_2 : F_2]$ as the one given by the elementary squares in the 2-dimensional integral grid $\mathbb{Z}^2$; two of them namely $bab^{-1}a^{-1} = bab^{-1} \cdot (a)^{-1}$ and $a^{-1}bab^{-1} = (a)^{-1} \cdot bab^{-1}$ abelianize to the same vector $(\ldots, 0, 0; -1; 1, 0, 0, \ldots) \in \mathbb{Z}^\infty$ and so $[F_2 : F_2]$ is not a free-factor of K_{01}. On the other hand, if $\operatorname{Fix}(f) \cap \operatorname{Fix}(g) \leqslant_{ff} \operatorname{Fix}(f^\alpha g^\beta)$, we have the following consequence,

$$\begin{aligned}
\operatorname{Fix}(f) \cap \operatorname{Fix}(g) &\leqslant_{ff} \operatorname{Fix}(f^\alpha g^\beta) \\
\Rightarrow [F_2 : F_2] &\leqslant_{ff} K_{\alpha\beta} \\
\Rightarrow [F_2 : F_2]\phi &\leqslant_{ff} K_{\alpha\beta}\phi \\
\Rightarrow [F_2 : F_2] &\leqslant_{ff} K_{01}
\end{aligned}$$

Hence a contradiction!!!

4.3 Finite order automorphisms of $\mathbb{Z}^{\mathbf{m}} \times \mathbf{F_n}$

A well-known (and deep) result by Bestvina–Handel [3] establishes a uniform bound (in fact, the best possible) for the rank of the fixed subgroup of any automorphism of F_n: for every $\phi \in \operatorname{Aut}(F_n)$, $\operatorname{r}(\operatorname{Fix}\phi) \leqslant n$. This result followed an interesting previously known particular case due to Dyer–Scott [16]: if $\phi \in \operatorname{Aut}(F_n)$ is of finite order then $\operatorname{Fix}\phi$ is a free factor of F_n.

When we move to a free-abelian times free group, $G = \mathbb{Z}^m \times F_n$, the situation degenerates, but still preserving some structure. In Delgado–Ventura [10], the authors gave an example of an automorphism $\Psi \in \operatorname{Aut}(G)$ with $\operatorname{Fix}\Psi$ *not* being finitely generated; so, there is no possible version of Bestvina–Handel result in G. Following the parallelism, we show below an example of an automorphism $\Psi \in \operatorname{Aut}(G)$ of finite order (in fact, of order 2) such that $\operatorname{Fix}\Psi$ is *not* a factor of G; see Example 4.3.2. However, as a positive result, in Theorem 4.3.1(ii) below we prove that finite order automorphisms of G do have finitely generated fixed subgroups, in fact with a computable uniform upper bound for its rank, in terms of m and n.

Restricting ourselves to the case of finite order automorphisms, through Theorem 4.3.1, we simultaneously bound the order of the automorphism and the rank of its fixed point subgroup. The constant $L_2(m)$ of Proposition 4.1.3 seems very technical, but this constant helps us to catch the uniform constant $C_2(m,n)$ concerning the bound of the rank for fixed subgroup of finite ordered automorphisms.

Theorem 4.3.1. *Let* $G = \mathbb{Z}^m \times F_n$*,* $m, n \geqslant 0$*.*

(i) There exists a computable constant $C_1 = C_1(m,n)$ *such that, for every* $\Psi \in \mathrm{Aut}(G)$ *of finite order,* $\mathrm{ord}(\Psi) \leqslant C_1$*.*

(ii) There exists a computable constant $C_2 = C_2(m,n)$ *such that, for every* $\Psi \in \mathrm{Aut}(G)$ *of finite order,* $\mathrm{r}(\mathrm{Fix}\,\Psi) \leqslant C_2$*.*

Proof. (i). By Proposition 4.1.3(i), the set $\{\mathrm{ord}(Q) \mid Q \in \mathrm{GL}_m(\mathbb{Z}) \text{ of finite order}\}$ is bounded above by a computable constant $L_1(m)$. And by Lyndon–Schupp [23, Cor. I.4.15], $\{\mathrm{ord}(\phi) \mid \phi \in \mathrm{Aut}(F_n) \text{ of finite order}\} \subseteq \{\mathrm{ord}(Q) \mid Q \in \mathrm{GL}_n(\mathbb{Z}) \text{ of finite order}\}$, which is bounded above by $L_1(n)$.

If $n \leqslant 1$ then $G = \mathbb{Z}^{m+n}$ is free-abelian and the constant $C_1 = L_1(m+n)$ makes the job; if $m = 0$ then $G = F_n$ is free and the constant $C_1 = L_1(n)$ makes the job.

So, suppose $m \geqslant 1$, $n \geqslant 2$, and take an automorphism $\Psi = \Psi_{\phi,Q,P} \in \mathrm{Aut}(G)$. By Delgado–Ventura [10, Lemma 5.4(ii)], $\Psi^k_{\phi,Q,P} = \Psi_{\phi^k,Q^k,P_k}$, where $P_k = \sum_{i=0}^{k-1} A^i P Q^{k-1-i}$ and $A \in \mathrm{GL}_n(\mathbb{Z})$ is the abelianization of ϕ. In particular, if Ψ is of finite order then ϕ and Q are so too; furthermore, $\mathrm{ord}(\Psi) = \lambda r_3$, where $r_3 = \mathrm{lcm}(r_1, r_2)$, $r_1 = \mathrm{ord}(\phi)$, and $r_2 = \mathrm{ord}(Q)$. But $\Psi^{r_3} = \Psi_{id,id,P_{r_3}}$ and $\Psi^{\lambda r_3} = (\Psi_{id,id,P_{r_3}})^{\lambda} = \Psi_{id,id,\lambda P_{r_3}}$. Hence, Ψ is either of order r_3 or of infinite order. In other words, $\{\mathrm{ord}(\Psi) \mid \Psi \in \mathrm{Aut}(G) \text{ of finite order}\} \subseteq \{\mathrm{lcm}(\mathrm{ord}(\phi), \mathrm{ord}(Q)) \mid \phi \in \mathrm{Aut}(F_n), Q \in \mathrm{GL}_m(\mathbb{Z}), \text{ both of finite order}\}$, which is bounded above by the constant $C_1(m,n) = L_1(n)L_1(m)$.

(ii). If $n \leqslant 1$ then $C_2 = m+n$ makes the job, if $m=0$ then $C_2 = n$ makes the job.

So, suppose $m \geqslant 1$, $n \geqslant 2$. Delgado–Ventura [10, §6] discusses the form of the fixed subgroup of a general automorphism $\Psi_{\phi,Q,P} \in \mathrm{Aut}(G)$, namely, $L_{\mathrm{Fix}\,\Psi} = \mathrm{Fix}(Q) = E_1(Q)$ (the eigenspace of eigenvalue 1 for Q), and $(\mathrm{Fix}\,\Psi)\pi = NP'^{-1}\rho'^{-1}$, where $\rho\colon F_n \twoheadrightarrow \mathbb{Z}^n$ is the abelianization map, ρ' is its restriction to $\mathrm{Fix}\,\phi$, P' is the restriction of P to $\mathrm{Im}\,\rho'$, $M = \mathrm{Im}(Q - I_m)$, $N = M \cap \mathrm{Im}\,P'$, and $(\mathrm{Fix}\,\Psi)\pi = NP'^{-1}\rho'^{-1} \trianglelefteq \mathrm{Fix}\,\phi \leqslant F_n$, see the following diagram,

$$
\begin{tikzcd}
F_n \arrow[r, two heads, "\rho"] \arrow[d, phantom, "\rotatebox{90}{$\leqslant$}"] & \mathbb{Z}^n \arrow[r, "P"] \arrow[d, phantom, "\rotatebox{90}{$\trianglelefteq$}"] & \mathbb{Z}^m \arrow[loop above, "Q-I_m"] \geqslant M = \mathrm{Im}(Q-I_m) \arrow[d, phantom, "\rotatebox{90}{$\trianglelefteq$}"] \\
\mathrm{Fix}\,\phi \arrow[r, two heads, "\rho'"] \arrow[d, phantom, "\rotatebox{90}{$\trianglelefteq$}"] & \mathrm{Im}\,\rho' \arrow[r, two heads, "P'"] \arrow[d, phantom, "\rotatebox{90}{$\trianglelefteq$}"] & \mathrm{Im}\,P' \arrow[d, phantom, "\rotatebox{90}{$\trianglelefteq$}"] \\
(\mathrm{Fix}\,\Psi)\pi = NP'^{-1}\rho'^{-1} & NP'^{-1} \arrow[l, mapsto] & N = M \cap \mathrm{Im}\,P'. \arrow[l, mapsto]
\end{tikzcd}
\tag{4.4}
$$

If $\mathrm{Fix}\,\phi$ is trivial or cyclic, then $\mathrm{r}(\mathrm{Fix}\,\Psi) = \mathrm{r}((\mathrm{Fix}\,\Psi)\pi) + \mathrm{r}(E_1(Q)) \leqslant 1 + m$. So, taking $C_2(m,n) \geqslant 1+m$, we are reduced to the case $\mathrm{r}(\mathrm{Fix}\,\phi) \geqslant 2$.

With this assumption, $(\mathrm{Fix}\,\Psi)\pi \neq 1$ (it always contains the commutator of $\mathrm{Fix}\,\phi$) and so, $\mathrm{Fix}\,\Psi \leqslant G$ is finitely generated if and only if $(\mathrm{Fix}\,\Psi)\pi \leqslant F_n$ is so, which is if and only if the index $\ell := [\mathrm{Fix}\,\phi : (\mathrm{Fix}\,\Psi)\pi] = [\mathrm{Fix}\,\phi : NP'^{-1}\rho'^{-1}] = [\mathrm{Im}\,\rho' : NP'^{-1}] = [\mathrm{Im}\,P' : N]$ is finite. In this case, by the Schreier index formula, $\tilde{\mathrm{r}}(\mathrm{Fix}\,\Psi) = \tilde{\mathrm{r}}((\mathrm{Fix}\,\Psi)\pi) + \mathrm{r}(E_1(Q)) \leqslant \ell\,\tilde{\mathrm{r}}(\mathrm{Fix}\,\phi) + m \leqslant \ell(n-1) + m$. Therefore, we are reduced to bound the index ℓ in terms of n and m.

First, let us prove that Ψ being of finite order implies $\ell = [\mathrm{Im}\,P' : N] < \infty$.

Put $k = \mathrm{ord}(\Psi_{\phi,Q,P})$ so, $\phi^k = \mathrm{id}$, $Q^k = I_m$, and $P_k = \sum_{i=0}^{k-1} A^i P Q^{k-1-i} = 0$, where $A \in \mathrm{GL}_n(\mathbb{Z})$ is the abelianization of ϕ. By Proposition 4.1.3(ii), the subgroup $M = \mathrm{Im}(Q - I_m)$

is a finite index subgroup of $\ker(Q^{k-1}+\cdots+Q+I_m)$, with the index bounded above by a computable constant depending only on m, $[\ker(Q^{k-1}+\cdots+Q+I_m):M]\leqslant L_2(m)$.

We claim that $\operatorname{Im}P'\leqslant\ker(Q^{k-1}+\cdots+Q+I_m)$. In fact, take $u\in\operatorname{Fix}\phi$, note that $u\phi=u$ and so $(u\rho')A=u\phi\rho'=u\rho'$, and split $(u\rho')P'=v_1+v_2$, with $v_1\in\ker(Q-I_m)$ and $v_2\in\ker(Q^{k-1}+\cdots+Q+I_m)$; see Lemma 4.1.1. Multiplying by $Q^{k-1}+\cdots+Q+I_m$ on the right,

$$v_1(Q^{k-1}+\cdots+Q+I_m)=(v_1+v_2)(Q^{k-1}+\cdots+Q+I_m)=(u\rho')P'(Q^{k-1}+\cdots+Q+I_m)=$$

$$=\sum_{i=0}^{k-1}(u\rho')PQ^{k-1-i}=\sum_{i=0}^{k-1}(u\rho')A^iPQ^{k-1-i}=(u\rho')\sum_{i=0}^{k-1}A^iPQ^{k-1-i}=(u\rho')P_k=0,$$

from which we deduce $v_1\in\ker(Q-I_m)\cap\ker(Q^{k-1}+\cdots+Q+I_m)=\{0\}$ so, $(u\rho')P'=v_2\in\ker(Q^{k-1}+\cdots+Q+I_m)$. Therefore, $\operatorname{Im}P'\leqslant\ker(Q^{k-1}+\cdots+Q+I_m)$.

Finally, intersecting the inclusion $M\leqslant_{fi}\ker(Q^{k-1}+\cdots+Q+I_m)$ with $\operatorname{Im}P'$, we get $N=M\cap\operatorname{Im}P'\leqslant_{fi}\operatorname{Im}P'$, and $\ell=[\operatorname{Im}P':N]\leqslant[\ker(Q^{k-1}+\cdots+Q+I_m):M]\leqslant L_2(m)$. Hence, taking $C_2(m,n)\geqslant L_2(m)(n-1)+m$ will suffice for the present case.

Therefore, $C_2(m,n)=L_2(m)(n-1)+m+1$ serves as the upper bound claimed in (ii). □

Example 4.3.2. Here is an example of an order 2 automorphism of $G=\mathbb{Z}^2\times F_3$ whose fixed subgroup *is not* a factor of G. Consider the automorphism $\Psi_{\phi,Q,P}$ determined by

$\phi\colon F_3 \to F_3$, $z_1 \mapsto z_1^{-1}$, $z_2 \mapsto z_2$, $z_3 \mapsto z_3$, $Q = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$, and $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{smallmatrix}\right) \in M_{3\times 2}(\mathbb{Z})$, i.e.,

$$\begin{array}{rcl} \Psi\colon \mathbb{Z}^2 \times F_3 & \longrightarrow & \mathbb{Z}^2 \times F_3 \\ z_1 & \longmapsto & t^{(1,0)} z_1^{-1} \\ z_2 & \longmapsto & t^{(0,1)} z_2 \\ z_3 & \longmapsto & t^{(0,2)} z_3 \\ t^{(1,0)} & \longmapsto & t^{(1,0)} \\ t^{(0,1)} & \longmapsto & t^{(0,-1)}. \end{array}$$

An easy computation shows that $\Psi^2 = \mathrm{id}$, i.e., Ψ has order 2. To compute $\operatorname{Fix}\Psi$, let us follow diagram (4.4): first note that $\operatorname{Fix}\phi = \langle z_2, z_3\rangle$; so, $\operatorname{Im}\rho' = \langle (0,1,0), (0,0,1)\rangle$ and $\operatorname{Im} P' = \langle (0,1), (0,2)\rangle = \langle (0,1)\rangle$. On the other hand, $M = \langle (0,2)\rangle$, $N = \langle (0,2)\rangle$, and $NP'^{-1} = \langle (0,2,0), (0,0,1)\rangle$. Therefore, $(\operatorname{Fix}\Psi)\pi = NP'^{-1}\rho'^{-1} = \{w(z_2,z_3) \mid |w|_{z_2} \text{ is even}\} = \langle z_2^2, z_3, z_2^{-1}z_3z_2\rangle$. So, solving the systems of equations to compute the vectors associated with each element of the free part, we obtain that $t^{(0,1)}z_2^2$, $t^{(0,1)}z_3$, $t^{(0,1)}z_2^{-1}z_3z_2 \in \operatorname{Fix}\Psi$. Finally, since $(\operatorname{Fix}\Psi)\cap\mathbb{Z}^2 = E_1(Q) = \langle (1,0)\rangle$, we deduce that $\operatorname{Fix}\Psi = \langle t^{(0,1)}z_2^2,\ t^{(0,1)}z_3,\ t^{(0,1)}z_2^{-1}z_3z_2,\ t^{(1,0)}\rangle$.

Since $\langle z_2^2, z_3, z_2^{-1}z_3z_2\rangle$ is not a free factor of F_3, $\operatorname{Fix}\Psi$ is not a *factor* of $\mathbb{Z}^2 \times F_3$; see Lemma 4.2.1.

Theorem 4.3.1 has the following easy corollary:

Corollary 4.3.3. *Let $\Psi \in \mathrm{End}(\mathbb{Z}^m \times F_n)$. If $\operatorname{Fix}\Psi^p$ is finitely generated then $\operatorname{Fix}\Psi$ is also finitely generated; the converse is not true.*

Proof. Clearly, Ψ restricts to an automorphism $\Psi_| \in \mathrm{Aut}(\operatorname{Fix}\Psi^p)$ such that $\operatorname{Fix}\Psi_| = \operatorname{Fix}\Psi$ and $(\Psi_|)^p = \mathrm{id}$. Since $\operatorname{Fix}\Psi^p$ is finitely generated, we have $\operatorname{Fix}\Psi^p \simeq \mathbb{Z}^{m'} \times F_{n'}$ for some $m' \leqslant m$ and $n' < \infty$ and, applying Theorem 4.3.1(ii), we get $\mathrm{r}(\operatorname{Fix}\Psi) = \mathrm{r}(\operatorname{Fix}\Psi_|) < \infty$ (in fact, bounded above by $C_2(m', n')$).

The converse is not true as the following example shows. Consider $\Psi\colon \mathbb{Z}\times F_2 \to \mathbb{Z}\times F_2$, $z_1 \mapsto tz_1^{-1}$, $z_2 \mapsto z_2^{-1}$, $t \mapsto t^{-1}$. It is straightforward to see that $\operatorname{Fix}\Psi = 1$. But $\Psi^2\colon \mathbb{Z}\times F_2 \to \mathbb{Z}\times F_2$, $z_1 \mapsto t^{-2}z_1$, $z_2 \mapsto z_2$, $t \mapsto t$ and so, $\operatorname{Fix}\Psi^2 = \langle t\rangle \times \{w(z_1,z_2) \in F_2 \mid |w|_{z_1} = 0\} = \langle t\rangle \times \langle\langle z_2\rangle\rangle$ is not finitely generated. □

Corollary 4.3.3 states that, for $\Psi \in \operatorname{Aut}(G)$, the lattice of fixed subgroups of powers of Ψ could simultaneously contain finitely and non-finitely generated subgroups but, as soon as one of them is finitely generated, the smaller ones must be so.

4.4 Periodic points of endomorphisms of $\mathbb{Z}^{\mathbf{m}} \times \mathbf{F_n}$

In the abelian case $G = \mathbb{Z}^m$, this lattice of fixed subgroups is always finite, and coming from a set of exponents uniformly bounded by m; this is precisely the contents of Proposition 4.1.4. In the free case, combining results from Bestvina–Handel, Culler, Imrich–Turner, and Stallings, the exact analogous statement is true:

Proposition 4.4.1 (Bestvina–Handel–Culler–Imrich–Turner–Stallings [3, 8, 19, 38]; see also [4, Prop. 3.1])**.** *For every* $\phi \in \operatorname{End}(F_n)$*, we have* $\operatorname{Per}\phi = \operatorname{Fix}\phi^{(6n-6)!}$.

Proof. Culler [8] proved that every finite order element in $\operatorname{Out}(F_n)$ has order dividing $(6n-6)!$; and the same is true in $\operatorname{Aut}(F_n)$ since the natural map $\operatorname{Aut}(F_n) \twoheadrightarrow \operatorname{Out}(F_n)$ has torsion-free kernel. On the other hand Stallings [38] proved that, for every $\phi \in \operatorname{Aut}(F_n)$, there exists $s \geqslant 0$ such that $\operatorname{Per}\phi = \operatorname{Fix}\phi^s$. Also, Imrich–Turner [19] proved that the so-called stable image of an endomorphism $\phi \in \operatorname{End}(F_n)$, namely $F_n\phi^\infty = \cap_{p=1}^\infty F_n\phi^p$, has rank at most n, it is ϕ-invariant, it contains $\operatorname{Per}\phi$, and the restriction $\phi_|\colon F_n\phi^\infty \to F_n\phi^\infty$ is bijective. Finally, Bestvina–Handel Theorem (see [3]) states that $\operatorname{r}(\operatorname{Fix}\phi) \leqslant n$, for any $\phi \in \operatorname{Aut}(F_n)$.

Combining these four results we can easily deduce the statement: given an endomorphism $\phi\colon F_n \to F_n$, consider its restrictions $\phi_1\colon F_n\phi^\infty \to F_n\phi^\infty$ and $\phi_2\colon \operatorname{Per}\phi_1 \to \operatorname{Per}\phi_1$, both bijective; furthermore, $\operatorname{Per}\phi_2 = \operatorname{Per}\phi_1 = \operatorname{Fix}\phi_1^s$ (assume $s \geqslant 0$ minimal possible), $\operatorname{r}(\operatorname{Per}\phi_1) \leqslant \operatorname{r}(F\phi^\infty) \leqslant n$, and ϕ_2 has order s. Therefore, s divides $(6\operatorname{r}(\operatorname{Per}\phi_1) - 6)!$ and so $(6n-6)!$ as well. We conclude that $\operatorname{Per}\phi = \operatorname{Per}\phi_1 = \operatorname{Fix}\phi_1^s = \operatorname{Fix}\phi^s \leqslant \operatorname{Fix}\phi^{(6n-6)!} \leqslant \operatorname{Per}\phi$ and so, $\operatorname{Per}\phi = \operatorname{Fix}\phi^{(6n-6)!}$. $\square$

Remark 4.4.2. Modulo missing details, this fact was implicitly contained in an older result by M. Takahasi, who proved that an ascending chain of subgroups of a free group, with rank uniformly bounded above by a fixed constant (like the $\operatorname{Fix}\psi^p$'s), must stabilize; see [23, p. 114].

We close the present section by extending this result to the context of free-abelian times free groups.

Theorem 4.4.3. *There exists a computable constant $C_3 = C_3(m,n)$ such that $\operatorname{Per}\Psi = \operatorname{Fix}\Psi^{C_3}$, for every $\Psi \in \operatorname{End}(\mathbb{Z}^m \times F_n)$.*

Proof. Delgado–Ventura [10, Prop. 5.1] gave a classification of all endomorphisms of $G = \mathbb{Z}^m \times F_n$ in two types. For those of the second type, say $\Psi_{z,l,h,Q,P}$ (see [10] for the notation), it is clear that the subgroup $\langle z, \mathbb{Z}^m\rangle \leqslant \mathbb{Z}^m \times F_n$ is invariant under Ψ (denote $\Psi_|\colon \langle z,\mathbb{Z}^m\rangle \to \langle z,\mathbb{Z}^m\rangle$ its restriction), and it contains $\operatorname{Im}\Psi$. Therefore, by Proposition 4.1.4, $\operatorname{Per}\Psi = \operatorname{Per}\Psi_| = \operatorname{Fix}(\Psi_|)^{L_3(m+1)} = \operatorname{Fix}\Psi^{L_3(m+1)}$, since $\langle z,\mathbb{Z}^m\rangle \simeq \mathbb{Z}^{m+1}$ is abelian. Thus, the computable constant $C_3(n,m) = L_3(m+1)$ satisfies the desired result for all endomorphisms of this second type.

Suppose now that Ψ is of the first type, i.e., $\Psi = \Psi_{\phi,Q,P}$, where $\phi \in \operatorname{End}(F_n)$, $Q \in M_{m\times m}(\mathbb{Z})$, and $P \in M_{n\times m}(\mathbb{Z})$. By Propositions 4.1.4 and 4.4.1, we know that $\operatorname{Per} Q = \operatorname{Fix} Q^{L_3}$ and $\operatorname{Per}\phi = \operatorname{Fix}\phi^{(6n-6)!}$ for some computable constant $L_3 = L_3(m)$. Take $C_3(m,n) = \operatorname{lcm}\left(L_3(m), (6n-6)!\right)$ and let us prove that $\operatorname{Per}\Psi = \operatorname{Fix}\Psi^{C_3}$.

By construction, we have both $\operatorname{Per} Q = \operatorname{Fix} Q^{C_3}$ and $\operatorname{Per}\phi = \operatorname{Fix}\phi^{C_3}$. It remains to see that the matrix P does not affect negatively into the calculations. To prove $\operatorname{Per}\Psi = \operatorname{Fix}\Psi^{C_3}$, it is enough to see that $\operatorname{Fix}\Psi^{k} \leqslant \operatorname{Fix}\Psi^{C_3}$ for all $k \geqslant 1$, which reduces to see that $\operatorname{Fix}\Psi^{\lambda C_3} \leqslant \operatorname{Fix}\Psi^{C_3}$ for every $\lambda \in \mathbb{N}$ (in fact, if this is true then $\operatorname{Fix}\Psi^{k} \leqslant \operatorname{Fix}\Psi^{kC_3} \leqslant \operatorname{Fix}\Psi^{C_3}$, for an arbitrary $k \geqslant 1$).

By Delgado–Ventura [10, Lemma 5.4(ii)], powers work like this: $(\Psi_{\phi,Q,P})^{k} = \Psi_{\phi^k,Q^k,P_k}$, where $P_k = \sum_{i=0}^{k-1} A^i P Q^{(k-1)-i}$ and $A \in M_{n\times n}(\mathbb{Z})$ is the abelianization matrix corresponding to $\phi \in \operatorname{End}(F_n)$. In our situation, $(\Psi_{\phi,Q,P})^{C_3} = \Psi_{\phi^{C_3},Q^{C_3},P_{C_3}}$, and $(\Psi_{\phi,Q,P})^{\lambda C_3} = \Psi_{\phi^{\lambda C_3},Q^{\lambda C_3},P_{\lambda C_3}}$, where

$$\begin{aligned} P_{\lambda C_3} &= \textstyle\sum_{i=0}^{\lambda C_3-1} A^i P Q^{(\lambda C_3-1)-i} \\ &= \textstyle\sum_{j=0}^{\lambda-1}\sum_{i=0}^{C_3-1} A^{jC_3+i} P Q^{(\lambda C_3-1)-(jC_3+i)} \\ &= \textstyle\sum_{j=0}^{\lambda-1}\sum_{i=0}^{C_3-1} A^{jC_3+i} P Q^{(\lambda-j)C_3-1-i} \\ &= \textstyle\sum_{j=0}^{\lambda-1} A^{jC_3}\big(\sum_{i=0}^{C_3-1} A^i P Q^{(C_3-1)-i}\big) Q^{(\lambda-j-1)C_3} \\ &= \textstyle\sum_{j=0}^{\lambda-1} (A^{C_3})^j P_{C_3} (Q^{C_3})^{(\lambda-1)-j}. \end{aligned} \tag{4.5}$$

Take any element $t^a u \in \operatorname{Fix}\Psi^{\lambda C_3}$ and let us prove that $t^a u \in \operatorname{Fix}\Psi^{C_3}$. Our assumption means that $t^{aQ^{\lambda C_3}+u^{\mathrm{ab}}P_{\lambda C_3}}(u\phi^{\lambda C_3}) = t^a u$ and so,

(1) $a(I_m - Q^{\lambda C_3}) = u^{\mathrm{ab}} P_{\lambda C_3}$, and

(2) $u \in \operatorname{Fix}\phi^{\lambda C_3} \leqslant \operatorname{Per}\phi = \operatorname{Fix}\phi^{C_3}$; in particular, $u^{\mathrm{ab}} A^{C_3} = u^{\mathrm{ab}}$.

Now from (4.5) and condition (1) we have,

$$\begin{aligned} a(I_m - Q^{C_3})(I + Q^{C_3} + \cdots + Q^{(\lambda-1)C_3}) &= u^{\mathrm{ab}} \textstyle\sum_{j=0}^{\lambda-1} (A^{C_3})^j P_{C_3} (Q^{C_3})^{(\lambda-1)-j} \\ &= u^{\mathrm{ab}} \textstyle\sum_{j=0}^{\lambda-1} P_{C_3} (Q^{C_3})^{(\lambda-1)-j} \\ &= u^{\mathrm{ab}} P_{C_3} \textstyle\sum_{j=0}^{\lambda-1} (Q^{C_3})^{(\lambda-1)-j} \\ &= u^{\mathrm{ab}} P_{C_3} (I + Q^{C_3} + \cdots + Q^{(\lambda-1)C_3}), \end{aligned}$$

which means that $a(I_m - Q^{C_3}) - u^{\mathrm{ab}} P_{C_3} \in \ker\left(I_m + Q^{C_3} + \cdots + Q^{(\lambda-1)C_3}\right)$. But

$$\ker\left(I_m + Q^{C_3} + \cdots + Q^{(\lambda-1)C_3}\right) \leqslant \ker(I_m - Q^{\lambda C_3}) =$$

$$= \operatorname{Fix} Q^{\lambda C_3} \leqslant \operatorname{Per} Q = \operatorname{Fix} Q^{C_3} = \ker(I_m - Q^{C_3})$$

hence, we also have $a(I_m - Q^{C_3}) - u^{\mathrm{ab}} P_{C_3} \in \ker(I_m - Q^{C_3})$. However, the two polynomials $1 + x^{C_3} + \cdots + x^{(\lambda-1)C_3}$ and $1 - x^{C_3}$ are relatively prime so, from Bezout's equality we deduce that $\ker\left(I_m + Q^{C_3} + \cdots + Q^{(\lambda-1)C_3}\right) \cap \ker(I_m - Q^{C_3}) = \{0\}$. Therefore, $a(I_m - Q^{C_3}) - u^{\mathrm{ab}} P_{C_3} = 0$ and so,

$$(t^a u)\Psi^{C_3} = t^{aQ^{C_3} + u^{\mathrm{ab}} P_{C_3}} (u\phi^{C_3}) = t^a u.$$

This shows that $\operatorname{Fix} \Psi^{\lambda C_3} = \operatorname{Fix} \Psi^{C_3}$ for every $\lambda \in \mathbb{N}$, from which we immediately deduce $\operatorname{Per} \Psi = \operatorname{Fix} \Psi^{C_3}$. This means that the constant $C_3(n,m) = \operatorname{lcm}\left(L_3(m), (6n-6)!\right)$ satisfies the desired result for all endomorphisms of the first type.

Hence, the computable constant $C_3(n,m) = \operatorname{lcm}\left(L_3(m), L_3(m+1), (6n-6)!\right)$ makes the job. □

Corollary 4.4.4. *Let $\Psi \in \operatorname{End}(\mathbb{Z}^m \times F_n)$. Then $\operatorname{Per} \Psi$ is finitely generated if and only if $\operatorname{Fix} \Psi^p$ is finitely generated for all $p \geqslant 1$.*

Proof. This follows immediately from Theorem 4.4.3 and Corollary 4.3.3. □

4.5 The auto-fixed closure of a subgroup of $\mathbb{Z}^{\mathbf{m}} \times \mathbf{F_n}$

Given an endomorphism, it is natural to ask for the computability of (a basis of) its fixed subgroup (or its periodic subgroup). In the abelian case, this can easily be done by just solving a system of linear equations, because the fixed point subgroup of an endomorphism of $\mathbb{Z}^m$ is nothing else but the eigenspace of eigenvalue 1 of the corresponding matrix, $\operatorname{Fix} Q = E_1(Q)$.

In the free case, this is a hard problem solved for automorphisms by making strong use of the train track techniques, see Bogopolski–Maslakova [5] (amending the previous wrong version Maslakova [29]) and, alternatively, Feingh–Handel [17, Prop. 7.7].

Theorem 4.5.1 (Bogopolski–Maslakova, [5]; Feingh–Handel, [17])**.** *Let $\phi\colon F_n \to F_n$ be an automorphism. Then, a free-basis for* $\operatorname{Fix}\phi$ *is computable.*

Finally, the free-abelian times free case was studied by Delgado–Ventura who solved the problem (including the decision on whether the fixed subgroup is finitely generated or not), modulo a solution for the free case. More precisely,

Theorem 4.5.2 (Delgado–Ventura, [10])**.** *Let $G = \mathbb{Z}^m \times F_n$. There is an algorithm which, on input an automorphism $\Psi\colon G \to G$, decides whether* $\operatorname{Fix}\Psi$ *is finitely generated or not and, if so, computes a basis for it.*

We note that Theorems 4.5.1 and 4.5.2 work for automorphisms; as far as we know, the computability of the fixed subgroup of an endomorphism, both in the free and in the free-abelian times free cases, remains open.

In the present section, we are interested in the dual problem: given a subgroup, decide whether it can be realized as the fixed subgroup of an endomorphism (resp., an automorphism, a family of endomorphisms, a family of automorphisms) and in the affirmative case, compute such an endomorphism (resp., automorphism, family of endomorphisms, family of automorphisms).

Generalizing the terminology introduced in Martino–Ventura [26] to an arbitrary group G, a subgroup $H \leqslant G$ is called *endo-fixed* (resp., *auto-fixed*) if $H = \operatorname{Fix} S$ for some set of endomorphisms $S \subseteq \operatorname{End}(G)$ (resp., automorphisms $S \subseteq \operatorname{Aut}(G)$). Simillarly, a subgroup $H \leqslant G$ is said to be *1-endo-fixed* (resp., *1-auto-fixed*) if $H = \operatorname{Fix} \phi$, for some $\phi \in \operatorname{End}(G)$ (resp., some $\phi \in \operatorname{Aut}(G)$). Notice that an auto-fixed (resp., endo-fixed) subgroup of G is an intersection of 1-auto-fixed (resp., 1-endo-fixed) subgroups of G, and vice-versa.

Of course, it is straightforward to see that all these notions do coincide in the abelian case: a subgroup $H \leqslant \mathbb{Z}^m$ is endo-fixed if and only if it is auto-fixed, if and only if it is 1-endo-fixed, if and only if it is 1-auto-fixed, and if and only if it is a direct summand, $H \leqslant_{\oplus} \mathbb{Z}^m$.

In the free case (and so, in the free-abelian times free as well) the situation is much more delicate: in Martino–Ventura [26], the authors conjectured that the families of auto-fixed and 1-auto-fixed subgroups of F_n do coincide; in other words, the family of 1-auto-fixed subgroups of F_n is closed under arbitrary intersections. (A similar conjecture can be stated for endomorphisms.) As far as we know, this still remains an open problem, with no progress made since the paper [26] itself, where the authors showed that, for any submonoid $S \leqslant \operatorname{End}(F_n)$, there exists $\phi \in S$ such that $\operatorname{Fix}(S)$ is a free factor of $\operatorname{Fix} \phi$; however, they also gave an explicit example of a 1-auto-fixed subgroup of F_n admitting a free factor which *is not* even endo-fixed. In this context it is worth mentioning the result Martino–Ventura [28, Cor. 4.2] showing that we can always restrict ourselves to consider finite intersections.

In this context, I want to mention that the conjecture is true if we consider *maximum-rank* auto-fixed subgroups of F_n. Because of the fact that a 1-auto-fixed subgroup H

of F_n has rank at most n, by the Bestvina–Handel [3] theorem and on the other hand, Martini–Ventura [26] proved that, for every autos (or endos) f, g there is another one h such that $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is a free factor of $\mathrm{Fix}(h)$.

Definition 4.5.3. Let H be a 1-auto-fixed subgroup of F_n and H has rank exactly n, then H is said to be a *maximum-rank* 1-auto-fixed subgroup of F_n and similar definition for auto-fixed subgroups.

The typical example of a maximum-rank 1-auto-fixed subgroup is given by the following automorphism of $F_2 = \langle a, b\rangle \colon a \mapsto a, b \mapsto a^r b$, where $r \neq 0$ is an integer. Its fixed subgroup is $H = \langle a, [a, b]\rangle = \langle a, b^{-1}ab\rangle$.

Theorem 4.5.4 (Collins–Turner, [7].)**.** *Every automorphism of F_n with fixed subgroup of rank n fixes a primitive element of F_n.*

In fact in the maximum rank case all kinds of fixed point families coincide.

Theorem 4.5.5 (Martino–Ventura, [26].)**.** *Let $H \leqslant F_n$ be a subgroup of F_n with $\mathrm{r}(H) = n$. The following are equivalent:*

(a) H is a 1-auto-fixed subgroup of F_n,

(b) H is a 1-mono-fixed subgroup of F_n,

(c) H is a 1-endo-fixed subgroup of F_n,

(d) H is an auto-fixed subgroup of F_n,

(e) H is a mono-fixed subgroup of F_n,

(f) H is an endo-fixed subgroup of F_n.

Let $H \leqslant G$. We denote by $\mathrm{Aut}_H(G)$ the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms of G which fix H pointwise, $\mathrm{Aut}_H(G) = \{\phi \in \mathrm{Aut}(G) \mid H \leqslant \mathrm{Fix}\,\phi\}$, usually called the *(pointwise) stabilizer* of H. Analogously, we denote by $\mathrm{End}_H(G)$ the submonoid of $\mathrm{End}(G)$ consisting of all endomorphisms of G which fix every element of H. Clearly, $\mathrm{Aut}_H(G) \leqslant \mathrm{End}_H(G)$.

Theorem 4.5.6 (Martino–Ventura, [26].). *Let H be a maximum-rank auto-fixed subgroup of F_n, and let m denote the rank of the (free abelian) image of H in $F_n^{ab} \simeq \mathbb{Z}^n$. Then $\mathrm{End}_H(F_n) = \mathrm{Aut}_H(F_n)$ is a free abelian subgroup of $\mathrm{Aut}(F_n)$ of rank $n - m$.*

In this realm, the following is a well-known result about stabilizers in the free group case, which will be used later:

Theorem 4.5.7 (McCool, [30]; see also [23, Prop. I.5.7]). *Let $H \leqslant_{fg} F_n$, given by a finite set of generators. Then the stabilizer, $\mathrm{Aut}_H(F_n)$, of H is also finitely generated (in fact, finitely presented), and a finite set of generators (and relations) is algorithmically computable.*

Following with the terminology from [26], the *auto-fixed closure* of H in G, denoted $\text{a-Cl}_G\,(H)$, is the subgroup

$$\text{a-Cl}_G\,(H) = \mathrm{Fix}(\mathrm{Aut}_H(G)) = \bigcap_{\substack{\phi\,\in\,\mathrm{Aut}(G)\\ H\,\leqslant\,\mathrm{Fix}\,\phi}} \mathrm{Fix}\,\phi,$$

i.e., the smallest auto-fixed subgroup of G containing H. Similarly, the *endo-fixed closure* of H in G, is $\text{e-Cl}_G\,(H) = \mathrm{Fix}(\mathrm{End}_H(G))$. Since $\mathrm{Aut}_H(G) \leqslant \mathrm{End}_H(G)$, it is obvious that $\text{e-Cl}_G(H) \leqslant \text{a-Cl}_G(H)$. However, the equality does not hold in general (for example, the free group $F_n, n \geqslant 3$ admits 1-endo-fixed subgroups which are not auto-fixed; see Martino–Ventura [27]).

In Ventura [42], fixed closures in free groups are studied from the algorithmic point of view. More precisely, the following results were proven:

Theorem 4.5.8 (Ventura, [42]). *Let $H \leqslant_{fg} F_n$, given by a finite set of generators. Then, a free-basis for the auto-fixed closure $\mathrm{a\text{-}Cl}_{F_n}(H)$ (resp., the endo-fixed closure $\mathrm{e\text{-}Cl}_{F_n}(H)$) of H is algorithmically computable, together with a set of $k \leqslant 2n$ automorphisms $\phi_1, \ldots, \phi_k \in \mathrm{Aut}(F_n)$ (resp., endomorphisms $\phi_1, \ldots, \phi_k \in \mathrm{End}(F_n)$), such that $\mathrm{a\text{-}Cl}_{F_n}(H) = \mathrm{Fix}\, \phi_1 \cap \cdots \cap \mathrm{Fix}\, \phi_k$ (resp., $\mathrm{e\text{-}Cl}_{F_n}(H) = \mathrm{Fix}\, \phi_1 \cap \cdots \cap \mathrm{Fix}\, \phi_k$).*

Corollary 4.5.9 (Ventura, [42]). *It is algorithmically decidable whether a given $H \leqslant_{fg} F_n$ is auto-fixed (resp., endo-fixed) or not.*

For example it is well known that, for every $w \in F_n$ and $r \in \mathbb{Z}$, the equation $x^r = w^r$ has a *unique* solution in F_n, which is the obvious one $x = w$; this means that any endomorphism $\phi\colon F_n \to F_n$ fixing w^r *must* also fix w. Therefore, the auto-fixed and endo-fixed closures of a cyclic subgroup of F_n are equal to the maximal cyclic subgroup where it is contained; in other words, a cyclic subgroup of F_n is auto-fixed, if and only if it is endo-fixed, and if and only if it is maximal cyclic.

In the present section, we prove the analog of Theorem 4.5.8 for free-abelian time free groups, and only in the automorphism case. Our main results in the present section are:

Theorem. *(4.5.18) Let $G = \mathbb{Z}^m \times F_n$. There is an algorithm which, given a finite set of generators for a subgroup $H \leqslant_{fg} G$, outputs a finite set of automorphisms $\Psi_1, \ldots, \Psi_k \in \mathrm{Aut}(G)$ such that $\mathrm{a\text{-}Cl}_G(H) = \mathrm{Fix}\, \Psi_1 \cap \cdots \cap \mathrm{Fix}\, \Psi_k$, decides whether this is finitely generated or not and, in case it is, computes a basis for it.*

Corollary. *(4.5.19) One can algorithmically decide whether a given $H \leqslant_{fg} G$ is auto-fixed or not, and in case it is, compute a finite set of automorphisms $\Psi_1, \ldots, \Psi_k \in \mathrm{Aut}(G)$ such that $H = \mathrm{Fix}\, \Psi_1 \cap \cdots \cap \mathrm{Fix}\, \Psi_k$.*

We want to emphasize that we did not succeed in the task of constructing an example of a finitely generated subgroup $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$ such that $\mathrm{a\text{-}Cl}_G(H)$ is *not* finitely generated; it could be that such examples do not exist so the following is an interesting open question:

Question 4.5.10. *Is it true that, for every $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$, the auto-fixed closure* $\text{a-Cl}_G(H)$ *is again finitely generated ? What about the endo-fixed closure* $\text{e-Cl}_G(H)$ *?*

To prove Theorem 4.5.18 and Corollary 4.5.19, we plan to follow the same strategy as in the free case, which is conceptually quite easy: given $H \leqslant_{fg} F_n$, use Theorem 4.5.7 to compute a set of generators for the stabilizer, say $\operatorname{Aut}_H(F_n) = \langle \phi_1, \ldots, \phi_k \rangle$, then use Theorem 4.5.1 to compute $\operatorname{Fix} \phi_i$ for each $i = 1, \ldots, k$, and finally intersect them all in order to get the auto-fixed closure, $\text{a-Cl}_{F_n}(H) = \operatorname{Fix} \phi_1 \cap \cdots \cap \operatorname{Fix} \phi_k$ (the bound $k \leqslant 2n$ comes from free group arguments and will be lost in the more general free-abelian times free context).

To make this strategy work in the free-abelian times free case, we have to overcome two extra difficulties not present at the free case:

(1) We need an analog to McCool's result for the group $\mathbb{Z}^m \times F_n$; stabilizers are going to be still finitely presented and computable, but more complicated than in the free case. The natural approach to this problem, trying to analyze directly how does an automorphism in $\operatorname{Aut}_H(G)$ look like, ends up with a tricky matrix equation with which we were unable to solve the problem; instead, our approach will be indirect, making use of another two more powerful results from the literature.

(2) When trying to compute $\operatorname{Fix} \Psi_1 \cap \cdots \cap \operatorname{Fix} \Psi_k$, it may very well happen that some of the individual $\operatorname{Fix} \Psi_i$'s are not finitely generated; in this case, Theorem 4.5.2 recognizes this fact and stops, giving us nothing else, while we still have to decide whether the full intersection $\operatorname{Fix} \Psi_1 \cap \cdots \cap \operatorname{Fix} \Psi_k$ is finitely generated or not (and compute a basis for it in case it is so).

We succeed overcoming these two difficulties in Theorem 4.5.14 and Proposition 4.5.16, respectively.

The versions of Theorem 4.5.18 and Corollary 4.5.19 for endomorphisms seem to be much more tricky and remain open (their versions for the free group, contained in Theorem 4.5.8 and Corollary 4.5.9, are already much more complicated because the monoid $\mathrm{End}_{F_n}(H)$ is not necessarily finitely generated, even with H being so, and also computability of fixed subgroups is not known for endomorphisms).

Question 4.5.11. *Let $G = \mathbb{Z}^m \times F_n$. Is there an algorithm which, given a finite set of generators for a subgroup $H \leqslant_{fg} G$, decides whether*

(i) the monoid $\mathrm{End}_H(G)$ is finitely generated or not and, in case it is, computes a set of endomorphisms $\Psi_1, \ldots, \Psi_k \in \mathrm{End}(G)$ such that $\mathrm{End}_H(G) = \langle \Psi_1, \ldots, \Psi_k \rangle$?

(ii) $\text{e-Cl}_G(H)$ is finitely generated or not and, in case it is, computes a basis for it ?

(iii) H is endo-fixed or not ?

Let us begin by understanding stabilizers in $G = \mathbb{Z}^m \times F_n$. For this, we need to remind a couple of other results from the literature.

Given a tuple of conjugacy classes $W = ([g_1], \ldots, [g_k])$ from a group G, the stabilizer of W, denoted $\mathrm{Aut}_W(G)$, is the group of automorphisms fixing all the $[g_i]$'s, i.e., sending the elements g_i to conjugates of themselves (with possibly different conjugators); more precisely,

$$\mathrm{Aut}_W(G) = \{\phi \in \mathrm{Aut}(G) \mid g_1\phi \sim g_1, \ldots, g_k\phi \sim g_k\},$$

where $\sim$ stands for conjugation in G ($g \sim h$ if and only if $g = x^{-1}hx = h^x$ for some $x \in G$). Of course, if $H = \langle h_1, \ldots, h_k \rangle \leqslant_{fg} G$, and $W = ([h_1], \ldots, [h_k])$, then $\mathrm{Aut}_H(G) \leqslant \mathrm{Aut}_W(G)$, without equality, in general.

McCool's Theorem 4.5.7 was a variation and an extension of a much earlier result: back in the 1930's, Whitehead already solved the orbit problem for conjugacy classes in the free group: given two tuples of conjugacy classes $V = ([v_1], \ldots, [v_k])$ and $W = ([w_1], \ldots, [w_k])$ in F_n, one can algorithmically decide whether there is an automorphism $\phi \in \mathrm{Aut}(F_n)$ such that $v_i\phi \sim w_i$, for every $i = 1, \ldots, k$; see [23, Prop. 4.21] or [43]; this was based on the so-called Whitehead automorphisms and the peak reduction technique. McCool's work 40 years later consisted of (1) deducing as a corollary that $\mathrm{Aut}_W(F_n)$ if finitely presented and a finite presentation is computable from the given W; and (2) extending everything to real elements instead of conjugacy classes and so, getting a solution to the orbit problem for tuples of elements, and the finite presentability (and computability) for stabilizers of subgroups, stated in Theorem 4.5.7.

Much more recently, a new version of these peak reduction techniques has been developed by M. Day [9] for right-angled Artin groups, extending McCool result (1) above to this bigger class of groups; we are interested in the stabilizer part:

Theorem 4.5.12 (Day, [9, Thm. 1.2])**.** *There is an algorithm that takes in a tuple W of conjugacy classes from a right-angled Artin group $A(\Gamma)$ and produces a finite presentation for its stabilizer $\mathrm{Aut}_W(A(\Gamma))$.*

Of course, we can make good use of Day's result in our case, because free-abelian times free groups are (a very special kind of) right-angled Artin groups; namely, $\mathbb{Z}^m \times F_n = A(\Gamma_{m,n})$ where $\Gamma_{m,n}$ is the complete graph on m vertices and the null graph on n vertices, together with mn edges joining each pair of vertices one in each side. The problem in doing this is that Day's result works only for conjugacy classes and the corresponding result for real elements is not known in general for right-angled Artin groups; while we need the finite generation (and computability) of stabilizers of *subgroups* in $\mathbb{Z}^m \times F_n$. We overcome this difficulty by using a result from Bogopolski–Ventura [6] relating stabilizers of subgroups and of tuples of conjugacy classes, in torsion-free hyperbolic groups:

Theorem 4.5.13 (Bogopolski–Ventura [6, Thm. 1.2]). *Let G be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Let $g_1, \ldots, g_r$ and $g'_1, \ldots, g'_r$ be elements of G such that $g_i \sim g'_i$ for every $i = 1, \ldots, r$. Then, there is a uniform conjugator for them if and only if $w(g_1, \ldots, g_r) \sim w(g'_1, \ldots, g'_r)$ for every word w in r variables and length up to a computable constant $C = C(\delta, |S|, \sum_{i=1}^{r} |g_i|)$, depending only on δ, $|S|$, and $\sum_{i=1}^{r} |g_i|$.*

Using these results we can effectively compute generators for the stabilizer of a given subgroup $H \leqslant_{fg} \mathbb{Z}^m \times F_n$. For our purposes, we do not need at all any set of relations; however, for completeness with respect to Day's result, we further prove that these stabilizers are also finitely presented and compute a full set of relations.

Theorem 4.5.14. *Let $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$, given by a finite set of generators. Then the stabilizer, $\mathrm{Aut}_H(G)$, of H is finitely presented, and a finite set of generators and relations is algorithmically computable.*

Proof. From the given set of generators, compute a basis for H, say $\{t^{a_1}u_1, \ldots, t^{a_r}u_r, t^{b_1}, \ldots, t^{b_s}\}$; in particular, we have a free-basis $\{u_1, \ldots, u_r\}$ for $H\pi$, and an abelian-basis $\{t^{b_1}, \ldots, t^{b_s}\}$ for $L_H = H \cap \mathbb{Z}^m$.

If $r = 0$ then $H = L_H$ and, clearly, $\Psi_{\phi,Q,P} \in \mathrm{Aut}_H(G)$ if and only if $Q \in \mathrm{Aut}_{L_H}(\mathbb{Z}^m)$. So, $\mathrm{Aut}_H(G)$ is generated by the following finite set of automorphisms of G: (1) $\Psi_{\phi,I_m,0}$, with ϕ running over the Nielsen automorphisms of F_n; (2) $\Psi_{id,Q,0}$, with Q running over the generators of $\mathrm{Aut}_{L_H}(\mathbb{Z}^m)$ computed by Theorem 4.5.12 (note that, since $\mathbb{Z}^m$ is abelian, $\mathrm{Aut}_{L_H}(\mathbb{Z}^m) = \mathrm{Aut}_{([b_1],\ldots,[b_s])}(\mathbb{Z}^m)$); and (3) $\Psi_{id,I_m,1_{i,j}}$, with $1_{i,j}$ being the zero $n \times m$ matrix with a single 1 at position (i,j), $i = 1, \ldots, n$, $j = 1, \ldots, m$. Therefore, from [10, Thm. 5.5], we deduce that $\mathrm{Aut}_H(G) \simeq \mathrm{M}_{n\times m}(\mathbb{Z}) \rtimes \big(\mathrm{Aut}_{L_H}(\mathbb{Z}^m) \times \mathrm{Aut}(F_n)\big)$ with the natural action. Hence, we can easily compute an explicit finite presentation for this group by using the presentation for $\mathrm{Aut}_{L_H}(\mathbb{Z}^m)$ we got from Day's Theorem 4.5.12, any known presentation for $\mathrm{Aut}(F_n)$ (see, for example, [1]), and the standard presentation for $\mathrm{M}_{n\times m}(\mathbb{Z}) \simeq \mathbb{Z}^{nm}$.

Assume that $r = \mathrm{r}(H\pi) \geqslant 1$. Apply Theorem 4.5.13 to the free group F_n and words $u_1, \ldots, u_r$, and compute the constant $C = C(0, n, \sum_{i=1}^{r} |u_i|)$. Consider the tuple of elements from G given by $W = (w_1(t^{a_1}u_1, \ldots, t^{a_r}u_r), \ldots, w_M(t^{a_1}u_1, \ldots, t^{a_r}u_r), t^{b_1}, \ldots, t^{b_s})$, where $w_1, \ldots, w_M$ is the sequence (in any order) of all reduced words on r variables and of length up to C. We claim that

$$\mathrm{Aut}_W(G) = \mathrm{Aut}_H(G) \cdot \mathrm{Inn}(G). \tag{4.6}$$

In fact, the inclusion $\geqslant$ is obvious. To see $\leqslant$, take $\Psi = \Psi_{\phi,Q,P} \in \mathrm{Aut}_W(G)$, that is, an automorphism Ψ satisfying $w_i(t^{a_1}u_1, \ldots, t^{a_r}u_r)\Psi \sim w_i(t^{a_1}u_1, \ldots, t^{a_r}u_r)$ for $i = 1, \ldots, M$, and $t^{b_j}\Psi \sim t^{b_j}$ for $j = 1, \ldots, s$. We have $t^{b_j}\Psi = t^{b_j}$ (since these are central elements from G), and $w_i(u_1, \ldots, u_r)\phi \sim w_i(u_1, \ldots, u_r)$ so, by Theorem 4.5.13, $w_i(u_1, \ldots, u_r)\phi = x^{-1}w_i(u_1, \ldots, u_r)x$ for a common conjugator $x \in F_n$; in particular, $u_i\phi = x^{-1}u_ix$ for $i = 1, \ldots, r$ and so, $\phi = (\phi\gamma_{x^{-1}})\gamma_x$, with $\phi\gamma_{x^{-1}} \in \mathrm{Aut}_{H\pi}(F_n)$. Therefore, $\Psi = (\Psi\Gamma_{x^{-1}})\Gamma_x$, with $\Psi\Gamma_{x^{-1}} \in \mathrm{Aut}_H(G)$.

Now, by Theorem 4.5.12, this stabilizer is finitely presented and a finite presentation

$$\mathrm{Aut}_W(G) = \langle \Psi_1, \ldots, \Psi_\ell \mid R_1, \ldots, R_d \rangle \tag{4.7}$$

can be computed, where the Ψ_i's are explicit automorphisms of G, and the R_j's are words on them satisfying $R_j(\Psi_1, \ldots, \Psi_\ell) = \mathrm{id}_G$, $j = 1, \ldots, d$. From the previous paragraph, we can algorithmically rewrite $\Psi_i = \Psi_i'\Gamma_{x_i}$ for some $\Psi_i' \in \mathrm{Aut}_H(G)$ and some $x_i \in F_n$, $i = 1, \ldots, \ell$ (note that some Ψ_i' could be the identity, corresponding to Ψ_i being possibly a genuine conjugation of G). Finally, let us distinguish two cases.

Suppose $r = \mathrm{r}(H\pi) \geqslant 2$. We claim that $\mathrm{Aut}_H(G) = \langle \Psi'_1, \ldots, \Psi'_\ell \rangle$: the inclusion $\geqslant$ is trivial; for the other, take $\Psi \in \mathrm{Aut}_H(G) \leqslant \mathrm{Aut}_W(G)$ and, since $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$, we have $\Psi = w(\Psi_1, \ldots, \Psi_\ell) = w(\Psi'_1 \Gamma_{x_1}, \ldots, \Psi'_\ell \Gamma_{x_\ell}) = w(\Psi'_1, \ldots, \Psi'_\ell)\Gamma_x$ for some $x \in F_n$. But both Ψ and $w(\Psi'_1, \ldots, \Psi'_\ell)$ fix $t^{a_1}u_1, \ldots, t^{a_r}u_r$ and $r \geqslant 2$ so, $x = 1$ and $\Psi = w(\Psi'_1, \ldots, \Psi'_\ell) \in \langle \Psi'_1, \ldots, \Psi'_\ell \rangle$.

Suppose now that $r = \mathrm{r}(H\pi) = 1$. The argument in the previous paragraph tells us that $\mathrm{Aut}_H(G) = \langle \Psi'_1, \ldots, \Psi'_\ell, \Gamma_{\hat{u}_1} \rangle$, where $\hat{u}_1$ is the root of u_1 in F_n, i.e., the unique non-proper power in F_n such that $u_1 = \hat{u}_1^\alpha$ for $\alpha > 0$ (since now, in the last part of the argument, x only commutes with $u_1 \neq 1$).

Up to here we have proved that $\mathrm{Aut}_H(G)$ is finitely generated and a finite set of generators is algorithmically computable. Now we complete it by computing a finite set of defining relations for $\mathrm{Aut}_H(G)$.

To find the defining relations, we distinguish again the cases $r \geqslant 2$, and $r = 1$ (in increasing order of difficulty):

- *Case 1:* $r \geqslant 2$. In this case, we already know that $\mathrm{Aut}_H(G) = \langle \Psi'_1, \ldots, \Psi'_\ell \rangle$. Let us find a complete set of defining relations for this set of generators.

Observe first that, for every $\Psi \in \mathrm{Aut}_W(G)$, the decomposition $\Psi = \Psi'\Gamma_x$ mentioned in (4.6) is unique: if $\Psi'\Gamma_x = \Psi''\Gamma_y$, with $\Psi', \Psi'' \in \mathrm{Aut}_H(G)$ and $x, y \in F_n$, then $x^{-1}u_1x = y^{-1}u_1y$ and $x^{-1}u_2x = y^{-1}u_2y$, which implies that xy^{-1} commutes with the freely independent elements u_1, u_2 and so, $xy^{-1} = 1$; hence, $\Gamma_x = \Gamma_y$ and $\Psi' = \Psi''$. In other words, $\mathrm{Aut}_H(G) \cap \mathrm{Inn}(G) = \{\mathrm{id}_G\}$ and so,

$$\mathrm{Aut}_W(G)/\,\mathrm{Inn}(G) = \mathrm{Aut}_H(G)\,\mathrm{Inn}(G)/\,\mathrm{Inn}(G) \simeq$$

$$\simeq \mathrm{Aut}_H(G)/\big(\mathrm{Aut}_H(G) \cap \mathrm{Inn}(G)\big) = \mathrm{Aut}_H(G).$$

We have the following two sources of natural relations among the Ψ_i''s. From (4.7), for each $i = 1, \ldots, d$ we have $\mathrm{id}_G = R_i(\Psi_1, \ldots, \Psi_\ell) = R_i(\Psi_1'\Gamma_{x_1}, \ldots, \Psi_\ell'\Gamma_{x_\ell}) = R_i(\Psi_1', \ldots, \Psi_\ell')\Gamma_{y_i} = R_i(\Psi_1', \ldots, \Psi_\ell')$, where $y_i \in F_n$ must be 1, again, because $r \geqslant 2$. On the other hand, for each one of the n generating letters of F_n, say $z_1, \ldots, z_n$, compute an expression for the conjugation $\Gamma_{z_j} \in \mathrm{Inn}(G) \leqslant \mathrm{Aut}_W(G)$ in terms of $\Psi_1, \ldots, \Psi_\ell$, say $\Gamma_{z_j} = S_j(\Psi_1, \ldots, \Psi_\ell)$, and we have $\Gamma_{z_j} = S_j(\Psi_1, \ldots, \Psi_\ell) = S_j(\Psi_1'\Gamma_{x_1}, \ldots, \Psi_\ell'\Gamma_{x_\ell}) = S_j(\Psi_1', \ldots, \Psi_\ell')\Gamma_{y_j}$ for some $y_j \in F_n$; but then $\mathrm{id}_G = S_j(\Psi_1', \ldots, \Psi_\ell')\Gamma_{y_j z_j^{-1}} = S_j(\Psi_1', \ldots, \Psi_\ell')$, $j = 1, \ldots, n$, gives us a second set of relations for $\mathrm{Aut}_H(G)$ (here, again, $y_j z_j^{-1} = 1$ since $r \geqslant 2$). Therefore,

$$
\begin{aligned}
\mathrm{Aut}_H(G) &= \mathrm{Aut}_W(G)/\,\mathrm{Inn}(G) \\
&= \langle \Psi_1, \ldots, \Psi_\ell \mid R_1, \ldots, R_d \rangle/\,\mathrm{Inn}(G) \\
&= \langle \Psi_1', \ldots, \Psi_\ell' \mid R_1, \ldots, R_d, S_1, \ldots, S_n \rangle.
\end{aligned}
$$

(Note that $w(\Psi_1, \ldots, \Psi_\ell) \mapsto w(\Psi_1', \ldots, \Psi_\ell')$ or, equivalently, $\Psi \mapsto \Psi' = \Psi\Gamma_{x^{-1}}$ for the unique possible $x \in F_n$, is the canonical projection $\mathrm{Aut}_W(G) \twoheadrightarrow \mathrm{Aut}_H(G) \simeq \mathrm{Aut}_W(G)/\,\mathrm{Inn}(G)$.)

- *Case 2:* $r = 1$. Here, $H = \langle t^a u, t^{b_1}, \ldots, t^{b_s} \rangle \leqslant G$ with $1 \neq u \in F_n$ (for notational simplicity, we have deleted the subindex 1 from u and a). This case is a bit more complicated than Case 1 because the decomposition $\Psi = \Psi'\Gamma_x$ from (4.6) is not unique now; additionally, $\mathrm{Aut}_H(G)$ contains some non-trivial conjugation, namely $\Gamma_{\hat{u}}$, and so we cannot mod out $\mathrm{Inn}(G)$ from $\mathrm{Aut}_W(G)$ because this would kill part of $\mathrm{Aut}_H(G)$.

In the present case, we know that $\mathrm{Aut}_H(G) = \langle \Psi_1', \ldots, \Psi_\ell', \Gamma_{\hat{u}} \rangle$. Let us adapt the two previous sources of natural relations among them, and discover a third one. From (4.7), for each $i = 1, \ldots, d$ we have $\mathrm{id}_G = R_i(\Psi_1, \ldots, \Psi_\ell) = R_i(\Psi_1'\Gamma_{x_1}, \ldots, \Psi_\ell'\Gamma_{x_\ell}) = R_i(\Psi_1', \ldots, \Psi_\ell')\Gamma_{y_i}$, for some $y_i \in F_n$. But both id_G and $R_i(\Psi_1', \ldots, \Psi_\ell')$ fix $t^a u$ so, y_i

must equal $\hat{u}^{\alpha_i}$ for some $\alpha_i \in \mathbb{Z}$. Therefore, $\mathrm{id}_G = R_i(\Psi'_1, \ldots, \Psi'_\ell)\Gamma_{\hat{u}}^{\alpha_i}$, $i = 1, \ldots, d$, is a first set of relations for $\mathrm{Aut}_H(G)$.

On the other hand, for each generating letter, z_j, of F_n, $j = 1, \ldots, n$, we have the equality $\Gamma_{z_j} = S_j(\Psi_1, \ldots, \Psi_\ell) = S_j(\Psi'_1\Gamma_{x_1}, \ldots, \Psi'_\ell\Gamma_{x_\ell}) = S_j(\Psi'_1, \ldots, \Psi'_\ell)\Gamma_{y_j}$, for some $y_j \in F_n$. But then $\mathrm{id}_G = S_j(\Psi'_1, \ldots, \Psi'_\ell)\Gamma_{y_j z_j^{-1}}$, which implies $y_j z_j^{-1} = \hat{u}^{\beta_j}$ for some $\beta_j \in \mathbb{Z}$. Therefore, $\mathrm{id}_G = S_j(\Psi'_1, \ldots, \Psi'_\ell)\Gamma_{\hat{u}}^{\beta_j}$, $j = 1, \ldots, n$, is a second set of relations for $\mathrm{Aut}_H(G)$.

Finally, observe that for $k = 1, \ldots, \ell$, $\hat{u}\Psi'_k = t^{c_k}\hat{u}$ for some $c_k \in \mathbb{Z}^m$ and thus, $\Gamma_{\hat{u}}$ commutes with Ψ'_k. Therefore, $\Psi'_k\Gamma_{\hat{u}} = \Gamma_{\hat{u}}\Psi'_k$, $k = 1, \ldots, \ell$, is a third set of relations for $\mathrm{Aut}_H(G)$.

We are going to prove that

$$\mathrm{Aut}_H(G) \simeq \left\langle \Psi'_1, \ldots, \Psi'_\ell, \Gamma_{\hat{u}} \;\middle|\; \underset{i=1,\ldots,d}{R_i(\Psi'_1, \ldots, \Psi'_\ell)\Gamma_{\hat{u}}^{\alpha_i}}, \quad \underset{j=1,\ldots,n}{S_j(\Psi'_1, \ldots, \Psi'_\ell)\Gamma_{\hat{u}}^{\beta_j}}, \quad \underset{k=1,\ldots,\ell}{\Psi'_k\Gamma_{\hat{u}} = \Gamma_{\hat{u}}\Psi'_k} \right\rangle. \tag{4.8}$$

To this goal, denote by $\mathcal{G}$ the group presented by the presentation on the right hand side, where elements are formal words on the 'symbols' $\{\Psi'_1, \ldots, \Psi'_\ell, \Gamma_{\hat{u}}\}$ subject to the relations indicated (we abuse notation, denoting by $\Psi'_1, \ldots, \Psi'_\ell, \Gamma_{\hat{u}}$ both the corresponding symbols in $\mathcal{G}$, and the corresponding automorphisms in $\mathrm{Aut}_H(G)$, the real meaning being always clear from the context). Let us construct a map $f\colon \mathrm{Aut}_H(G) \to \mathcal{G}$, and a group homomorphism $G \leftarrow \mathcal{G} : g$ such that $fg = \mathrm{id}_{\mathrm{Aut}_H(G)}$ and $gf = \mathrm{id}_{\mathcal{G}}$. This will suffice to prove (4.8) and finish the argument.

Define g by sending the symbol Ψ'_k to the automorphism Ψ'_k, $k = 1, \ldots, \ell$, and the symbol $\Gamma_{\hat{u}}$ to the automorphism $\Gamma_{\hat{u}}$; since, as we have proved in the three previous paragraphs, the relations from $\mathcal{G}$ are really satisfied in $\mathrm{Aut}_H(G)$, g determines a well defined homomorphism from $\mathcal{G}$ to $\mathrm{Aut}_H(G)$. (For later use, we emphasize the meaning of this: every equality holding symbolically in $\mathcal{G}$ holds also genuinely in $\mathrm{Aut}_H(G)$.) On the other hand, for $\Psi \in \mathrm{Aut}_H(G)$, define $\Psi f \in \mathcal{G}$ as follows: write $\Psi \in \mathrm{Aut}_H(G) \leqslant \mathrm{Aut}_W(G)$ as a word on $\Psi_1, \ldots, \Psi_\ell$, say $\Psi = v(\Psi_1, \ldots, \Psi_\ell)$, compute $\Psi = v(\Psi_1, \ldots, \Psi_\ell) = v(\Psi_1'\Gamma_{x_1}, \ldots, \Psi_\ell\Gamma_{x_\ell}) = v(\Psi_1', \ldots, \Psi_\ell')\Gamma_y = v(\Psi_1', \ldots, \Psi_\ell')\Gamma_{\hat{u}}^{\rho}$ (in $\mathrm{Aut}_H(G)$!), where $y = \hat{u}^\rho$ for some $\rho \in \mathbb{Z}$ since both Ψ and $v(\Psi_1', \ldots, \Psi_\ell')$ fix $t^a u$; and, finally, define Ψf to be the formal word $v(\Psi_1', \ldots, \Psi_\ell')\Gamma_{\hat{u}}^{\rho} \in \mathcal{G}$.

First, we have to see that f is well defined. That is, take $\Psi = w(\Psi_1, \ldots, \Psi_\ell)$ another expression for Ψ, write $\Psi = w(\Psi_1, \ldots, \Psi_\ell) = w(\Psi_1', \ldots, \Psi_\ell')\Gamma_{\hat{u}}^{\tau}$ (in $\mathrm{Aut}_H(G)$!) for the appropriate integer $\tau \in \mathbb{Z}$, and we have to prove that the equality $v(\Psi_1', \ldots, \Psi_\ell')\Gamma_{\hat{u}}^{\rho} = w(\Psi_1', \ldots, \Psi_\ell')\Gamma_{\hat{u}}^{\tau}$ holds, abstractly, in $\mathcal{G}$. From the fact $v(\Psi_1, \ldots, \Psi_\ell) = \Psi = w(\Psi_1, \ldots, \Psi_\ell)$ (equalities happening in the group (4.7)), we deduce that the word $v(\Psi_1, \ldots, \Psi_\ell)^{-1}w(\Psi_1, \ldots, \Psi_\ell)$ is formally a product of conjugates of $R_1(\Psi_1, \ldots, \Psi_\ell), \ldots, R_d(\Psi_1, \ldots, \Psi_\ell)$, say

$$v(\Psi_1, \ldots, \Psi_\ell)^{-1}w(\Psi_1, \ldots, \Psi_\ell) = \prod_{k=1}^{N} \left(R_{i_k}^{\epsilon_k}(\Psi_1, \ldots, \Psi_\ell)\right)^{c_k(\Psi_1, \ldots, \Psi_\ell)}.$$

Particularizing this identity on $\Psi_1', \ldots, \Psi_\ell' \in \mathcal{G}$, and working in $\mathcal{G}$ (i.e., *only* using symbolically the defining relations for $\mathcal{G}$), we have that

$$v(\Psi_1', \ldots, \Psi_\ell')^{-1}w(\Psi_1', \ldots, \Psi_\ell') = \prod_{k=1}^{N} \left(R_{i_k}^{\epsilon_k}(\Psi_1', \ldots, \Psi_\ell')\right)^{c_k(\Psi_1', \ldots, \Psi_\ell')} =$$

$$= \prod_{k=1}^{N} \left(\Gamma_{\hat{u}}^{-\epsilon_k \alpha_{i_k}}\right)^{c_k(\Psi_1', \ldots, \Psi_\ell')} = \prod_{k=1}^{N} \Gamma_{\hat{u}}^{-\epsilon_k \alpha_{i_k}} = \Gamma_{\hat{u}}^{-\sum_{k=1}^{N} \epsilon_k \alpha_{i_k}}.$$

But, applying g (i.e., reading the above equality in $\mathrm{Aut}_H(G)$), we have

$$\mathrm{id}_G = v(\Psi_1, \ldots, \Psi_\ell)^{-1}w(\Psi_1, \ldots, \Psi_\ell) =$$

$$= \Gamma_{\hat{u}}^{-\rho} v(\Psi_1', \ldots, \Psi_\ell')^{-1}w(\Psi_1', \ldots, \Psi_\ell')\Gamma_{\hat{u}}^{\tau} = \Gamma_{\hat{u}}^{\tau - \rho - \sum_{k=1}^{N} \epsilon_k \alpha_{i_k}}$$

and so, the exponent must be zero, $\tau - \rho - \sum_{k=1}^{N} \epsilon_k \alpha_{i_k} = 0$, because $N \geqslant 2$. Going again to $\mathcal{G}$, we conclude that $\Gamma_{\hat{u}}^{-\rho} v(\Psi'_1, \ldots, \Psi'_\ell)^{-1} w(\Psi'_1, \ldots, \Psi'_\ell) \Gamma_{\hat{u}}^{\tau} = \Gamma_{\hat{u}}^{\tau - \rho - \sum_{k=1}^{N} \epsilon_k \alpha_{i_k}} = 1$, showing that the map f is well defined.

Now consider the composition $fg\colon \operatorname{Aut}_H(G) \to \mathcal{G} \to \operatorname{Aut}_H(G)$: for every $\Psi \in \operatorname{Aut}_H(G)$, write (in $\operatorname{Aut}_H(G)$!) $\Psi = v(\Psi_1, \ldots, \Psi_\ell) = v(\Psi'_1, \ldots, \Psi'_\ell)\Gamma_{\hat{u}}^{\rho}$, $\rho \in \mathbb{Z}$, and we have $\Psi f = v(\Psi'_1, \ldots, \Psi'_\ell)\Gamma_{\hat{u}}^{\rho} \in \mathcal{G}$. But then, $\Psi fg = (v(\Psi'_1, \ldots, \Psi'_\ell)\Gamma_{\hat{u}}^{\rho})g = v(\Psi'_1, \ldots, \Psi'_\ell)\Gamma_{\hat{u}}^{\rho} = \Psi$ (in $\operatorname{Aut}_H(G)$!). Hence, $fg = \mathrm{id}_{\operatorname{Aut}_H(G)}$.

Finally, consider the composition $gf\colon \mathcal{G} \to \operatorname{Aut}_H(G) \to \mathcal{G}$. Take $k = 1, \ldots, \ell$ and, in order to compute $\Psi'_k gf = \Psi'_k f$, we have to express $\Psi'_k \in \operatorname{Aut}_H(G)$ as a word on $\Psi_1, \ldots, \Psi_\ell$; take, for example, $\Psi'_k = \Psi_k \Gamma_{x_k}^{-1} = \Psi_k \Gamma_{x_k(z_1,\ldots,z_n)}^{-1} = \Psi_k x_k(\Gamma_{z_1}, \ldots, \Gamma_{z_n})^{-1} = \Psi_k x_k(S_1(\Psi_1, \ldots, \Psi_\ell), \ldots, S_n(\Psi_1, \ldots, \Psi_\ell))^{-1}$; then, rewrite in terms of $\Psi'_1, \ldots, \Psi'_\ell$,

$$\Psi'_k = \Psi_k x_k(S_1(\Psi_1, \ldots, \Psi_\ell), \ldots, S_n(\Psi_1, \ldots, \Psi_\ell))^{-1} =$$

$$= \Psi'_k x_k(S_1(\Psi'_1, \ldots, \Psi'_\ell), \ldots, S_n(\Psi'_1, \ldots, \Psi'_\ell))^{-1}\Gamma_{\hat{u}}^{\rho},$$

for the appropriate integer $\rho \in \mathbb{Z}$; and we have, in $\mathcal{G}$,

$$\begin{aligned}
\Psi'_k gf = \Psi'_k f &= \Psi'_k x_k(S_1(\Psi'_1, \ldots, \Psi'_\ell), \ldots, S_n(\Psi'_1, \ldots, \Psi'_\ell))^{-1}\Gamma_{\hat{u}}^{\rho} \\
&= \Psi'_k x_k(\Gamma_{\hat{u}}^{-\beta_1}, \ldots, \Gamma_{\hat{u}}^{-\beta_n})^{-1}\Gamma_{\hat{u}}^{\rho} \\
&= \Psi'_k \Gamma_{\hat{u}}^{x_k^{\mathrm{ab}}\beta^T} \Gamma_{\hat{u}}^{\rho} \\
&= \Psi'_k \Gamma_{\hat{u}}^{x_k^{\mathrm{ab}}\beta^T + \rho},
\end{aligned}$$

where $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$. But, applying g, using $fg = \mathrm{id}_{\operatorname{Aut}_H(G)}$, and cancelling Ψ'_i from the left, we obtain $\mathrm{id}_G = \Gamma_{\hat{u}}^{x_k^{\mathrm{ab}}\beta^T + \rho}$ and so, $x_k^{\mathrm{ab}}\beta^T + \rho = 0$. Hence, back in $\mathcal{G}$, $\Psi'_k gf = \Psi'_k$, for $k = 1, \ldots, \ell$.

Similarly,

$$\begin{aligned}
\Gamma_{\hat{u}} gf = \Gamma_{\hat{u}} f &= (\hat{u}(\Gamma_{z_1}, \ldots, \Gamma_{z_n}))f \\
&= (\hat{u}(S_1(\Psi_1, \ldots, \Psi_\ell), \ldots, S_n(\Psi_1, \ldots, \Psi_\ell)))f \\
&= \hat{u}(S_1(\Psi'_1, \ldots, \Psi'_\ell), \ldots, S_n(\Psi'_1, \ldots, \Psi'_\ell))\Gamma_{\hat{u}}^{\rho} \\
&= \hat{u}(\Gamma_{\hat{u}}^{-\beta_1}, \ldots, \Gamma_{\hat{u}}^{-\beta_n})\Gamma_{\hat{u}}^{\rho} \\
&= \Gamma_{\hat{u}}^{-\hat{u}^{\mathrm{ab}}\beta^T + \rho},
\end{aligned}$$

for the appropriate integer $\rho \in Z$. But, applying g, we obtain $\Gamma_{\hat{u}} = \Gamma_{\hat{u}}^{-\hat{u}^{\mathrm{ab}}\beta^T + \rho}$ (in $\mathrm{Aut}_H(G)$!) and so, $-\hat{u}^{\mathrm{ab}}\beta^T + \rho = 1$. Hence, back in $\mathcal{G}$, $\Gamma_{\hat{u}} gf = \Gamma_{\hat{u}}$, finishing the proof that $gf = \mathrm{id}_{\mathcal{G}}$.

This completes the proof of the isomorphism (4.8) and so, the proof of the Theorem. □

The above proof that stabilizers of subgroups of $G = \mathbb{Z}^m \times F_n$ are finitely presented (and a finite presentation is computable) makes a strong use of the fact that the center of G is $\mathbb{Z}^m$, i.e., the elements of the form t^a commute with everybody in G. For this reason, this proof is far from generalizing to arbitrary right angled Artin groups, providing an analog of Day's Theorem 4.5.12 for real elements instead of conjugacy classes. This suggests the following question, which is open as far as we know.

Question 4.5.15. *Is it true that, for every finitely generated subgroup of a right angled Artin group, $H \leqslant_{fg} A(\Gamma)$, the stabilizer $\mathrm{Aut}_H(A(\Gamma))$ is finitely generated ? and finitely presented ? and a presentation is algorithmically computable from the given generators for H ?*

Now we turn to the computability of fixed points by a given collection of automorphisms.

Proposition 4.5.16. *Let $G = \mathbb{Z}^m \times F_n$. There is an algorithm which, given $\Psi_1, \ldots, \Psi_k \in \operatorname{Aut}(G)$, decides whether $\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$ is finitely generated or not and, in the affirmative case, computes a basis for it.*

Remark 4.5.17. Two related results are Theorem 4.5.2 above, and Theorem [10, Thm. 4.8]. With the first one we can decide whether each $\operatorname{Fix}\Psi_i$ is finitely generated and, in this case, compute a basis; and with the second, assuming $\operatorname{Fix}\Psi_i$ and $\operatorname{Fix}\Psi_j$ finitely generated, we can decide whether $\operatorname{Fix}\Psi_i \cap \operatorname{Fix}\Psi_j$ is finitely generated again and, in this case, compute a basis for it. However, these two results combined in an induction argument are not enough to prove Proposition 4.5.16 because it could very well happen that some of the individual $\operatorname{Fix}\Psi_i$'s (even a partial intersection of some of them) is not finitely generated while $\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$ is so. Thus, we are going to adapt the proof of Theorem 4.5.2 to compute directly the fixed subgroup of a finite tuple of automorphisms, without making reference to the fixed subgroup of each individual one.

Proof of Proposition 4.5.16. Write $\Psi_i = \Psi_{\phi_i, Q_i, P_i} \colon G \to G$, $t^a u \mapsto t^{aQ_i + u\rho P_i} u\phi_i$, for some $\phi_i \in \operatorname{Aut}(F_n)$, $Q_i \in \mathrm{GL}_m(\mathbb{Z})$, and $P_i \in \mathrm{M}_{n\times m}(\mathbb{Z})$, $i = 1, 2, \ldots, k$, where $\rho \colon F_n \twoheadrightarrow \mathbb{Z}^n$ is the abelianization map. We have

$$\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k = \left\{ t^a u \in G \mid u \in \cap_{i=1}^{k} \operatorname{Fix}\phi_i,\ a(I_m - Q_i) = u\rho P_i,\ i = 1, \ldots, k \right\} =$$

$$= \{ t^a u \in G \mid u \in \cap_{i=1}^{k} \operatorname{Fix}\phi_i,\ a(I_m - Q_1 | \cdots | I_m - Q_k) = u\rho(P_1 | \cdots | P_k) \},$$

where $(I_m - Q_1|\cdots|I_m - Q_k) \in \mathrm{M}_{m\times km}(\mathbb{Z})$ and $(P_1|\cdots|P_k) \in \mathrm{M}_{n\times km}(\mathbb{Z})$ are the indicated concatenated matrices, corresponding to linear maps $\tilde{Q} \colon \mathbb{Z}^m \to \mathbb{Z}^{km}$ and $\tilde{P} \colon \mathbb{Z}^n \to \mathbb{Z}^{km}$, respectively.

Let ρ' be the restriction of ρ to $\operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k$ (not to be confused with the abelianization map of the subgroup $\operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k$ itself), let $\tilde{P}'$ be the restriction of $\tilde{P}$ to $\operatorname{Im}\rho'$;

$$
\begin{array}{ccccc}
 F_n & \overset{\rho}{\twoheadrightarrow} & \mathbb{Z}^n & \xrightarrow{\tilde{P}} & \mathbb{Z}^{km} \geqslant M = \operatorname{Im}\tilde{Q} \\
 \rotatebox{90}{$\leqslant$} & & \rotatebox{90}{$\leqslant$} & & \rotatebox{90}{$\leqslant$} \\
 \operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k & \overset{\rho'}{\twoheadrightarrow} & \operatorname{Im}\rho' & \overset{\tilde{P}'}{\twoheadrightarrow} & \operatorname{Im}\tilde{P}' \\
 \rotatebox{90}{$\leqslant$} & & \rotatebox{90}{$\leqslant$} & & \rotatebox{90}{$\leqslant$} \\
 N\tilde{P}'^{-1}\rho'^{-1} & \longleftarrow\!\shortmid & N\tilde{P}'^{-1} & \longleftarrow\!\shortmid & N = M \cap \operatorname{Im}\tilde{P}'.
\end{array}
$$

Figure 4.1: Finite intersection of fixed point subgroups

let $M = \operatorname{Im}\tilde{Q} \leqslant \mathbb{Z}^{km}$, let $N = M \cap \operatorname{Im}\tilde{P}'$, and consider the preimages of N first by $\tilde{P}'$ and then by ρ', see the Fig. 4.1. We claim that $(\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k)\pi = N\tilde{P}'^{-1}\rho'^{-1}$. In fact, for $u \in (\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k)\pi$, there exists $a \in \mathbb{Z}^m$ such that $t^a u \in \operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$, i.e., $u \in \operatorname{Fix}\phi_i$ and $a(I_m - Q_i) = u\rho P_i$, $i = 1, \ldots, k$. So, $u \in \operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k$ and $u\rho'\tilde{P}' = a\tilde{Q} \in M \cap \operatorname{Im}\tilde{P}'$ and hence, $u \in N\tilde{P}'^{-1}\rho'^{-1}$. On the other hand, for $u \in N\tilde{P'}^{-1}\rho'^{-1}$, we have $u \in \operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k$ and $u\rho'\tilde{P}' \in N \leqslant M = \operatorname{Im}\tilde{Q}$ so, $u\rho\tilde{P}' = a\tilde{Q}$ for some $a \in \mathbb{Z}^m$; this means that $t^a u \in \operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$ and hence $u \in (\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k)\pi$. This proves the claim.

Now $\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k \leqslant G$ is finitely generated if and only if $(\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k)\pi = N\tilde{P}'^{-1}\rho'^{-1}$ is finitely generated, which (since it is a normal subgroup) happens if and only if $N\tilde{P}'^{-1}\rho'^{-1}$ is trivial (i.e., $\operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k = \langle u \rangle$ with $u\rho \neq 0$ and $N = \{0\}$) or of finite index in $\operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k$. That is, $\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$ is finitely generated if and only if

(i) $\operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k = \langle u \rangle$ with $u\rho \neq 0$ and $N = \{0\}$, or

(ii) $[\operatorname{Im}\tilde{P}' : N] = [\operatorname{Im}\rho' : N\tilde{P}'^{-1}] = [\operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k : N\tilde{P}'^{-1}\rho'^{-1}] < \infty$ or, equivalently, $\operatorname{r}(N) = \operatorname{r}(\operatorname{Im}\tilde{P}')$.

These conditions can effectively be checked by computing a free-basis for $\operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k$ with Theorem 4.5.1 and pull-backs of graphs, and then computing the ranks

$\mathrm{r}(\operatorname{Im}\tilde{P}')$ and $\mathrm{r}(N)$ with basic linear algebra techniques. So, we can effectively decide whether $\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$ is finitely generated or not.

Finally, let us assume it is so, and let us compute a basis for $\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$.

If we are in the situation (i) then $\operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k = \langle u \rangle$, $u\rho \neq 0$, and $M \cap \operatorname{Im}\tilde{P}' = N = \{0\}$ so, the only elements in $\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$ are those of the form $t^a u^r$ with $a(I_m - \tilde{Q}) = r \cdot u\rho\tilde{P} = 0$. That is, $\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k = \langle u,\, t^{d_1}, \ldots, t^{d_s} \rangle$ where $\langle d_1, \ldots, d_s \rangle = E_1(Q_1) \cap \cdots \cap E_1(Q_k) \leqslant \mathbb{Z}^m$.

If we are in situation (ii), then we can compute a set $\{c_1, \ldots, c_q\} \subset \mathbb{Z}^n$ of coset representatives of $N\tilde{P}'^{-1}$ in $\operatorname{Im}\rho'$, namely $\operatorname{Im}\rho' = (N\tilde{P}'^{-1})c_1 \sqcup \cdots \sqcup (N\tilde{P}'^{-1})c_q$. Having computed a free-basis $\{v_1, \ldots, v_p\}$ for $\operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k$, we can choose arbitrary preimages $y_1, \ldots, y_q$ of $c_1, \ldots, c_q$ up in $\operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k$, and we get a set of right coset representatives of $(\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k)\pi = N\tilde{P}'^{-1}\rho'^{-1}$ in $\operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k$,

$$\operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k = (N\tilde{P}'^{-1}\rho'^{-1})y_1 \sqcup \cdots \sqcup (N\tilde{P}'^{-1}\rho'^{-1})y_q. \tag{4.9}$$

Now, we build the Schreier graph for $N\tilde{P}'^{-1}\rho'^{-1} \leqslant_{fi} \operatorname{Fix}\phi_1 \cap \cdots \cap \operatorname{Fix}\phi_k$ with respect to $\{v_1, \ldots, v_p\}$ in the following way: (1) take the cosets from (4.9) as vertices, and with no edge; (2) for every vertex $(N\tilde{P}'^{-1}\rho'^{-1})y_i$ and every letter v_j, add an edge labeled v_j from $(N\tilde{P}'^{-1}\rho'^{-1})y_i$ to $(N\tilde{P}'^{-1}\rho'^{-1})y_i v_j$, algorithmically identified among the available vertices by repeatedly solving the membership problem for $N\tilde{P}'^{-1}\rho'^{-1}$ (note that we can easily do this by abelianizing the candidate and checking whether it belongs to $N\tilde{P}'^{-1}$). Once we have run over all $i = 1, \ldots, q$ and all $j = 1, \ldots, p$, we have computed the full (and finite!) Schreier graph, from which we can select a maximal tree and obtain a free-basis $\{u_1, \ldots, u_r\}$ for the subgroup corresponding to closed paths at the basepoint, i.e., for $N\tilde{P}'^{-1}\rho'^{-1} = (\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k)\pi$. Finally, solving linear systems of equations (which must be mandatorily compatible), we obtain vectors $e_1, \ldots, e_r \in \mathbb{Z}^m$ such that

$t^{e_1}u_1, \ldots, t^{e_r}u_r \in \operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$. We conclude that $\{t^{e_1}u_1, \ldots, t^{e_r}u_r,\, t^{d_1}, \ldots, t^{d_s}\}$ is a basis for $\operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$. □

Theorem 4.5.18. *Let $G = \mathbb{Z}^m \times F_n$. There is an algorithm which, given a finite set of generators for a subgroup $H \leqslant_{fg} G$, outputs a finite set of automorphisms $\Psi_1, \ldots, \Psi_k \in \operatorname{Aut}(G)$ such that $\text{a-Cl}_G(H) = \operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$, decides whether this is finitely generated or not and, in case it is, computes a basis for it.*

Proof. From the given generators, compute a basis for H, say $\{t^{a_1}u_1, \ldots, t^{a_r}u_r, t^{b_1}, \ldots, t^{b_s}\}$. Now, using Theorem 4.5.14, we can compute finitely many automorphisms $\Psi_1, \ldots, \Psi_k \in \operatorname{Aut}(G)$ such that $\operatorname{Aut}_H(G) = \langle \Psi_1, \ldots, \Psi_k \rangle$. So, we have that $\text{a-Cl}_G(H) = \operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$. Finally, using Proposition 4.5.16, we can decide whether this intersection is finitely generated or not and, in the affirmative case, compute a basis for it. □

Corollary 4.5.19. *One can algorithmically decide whether a given $H \leqslant_{fg} G$ is auto-fixed or not, and in case it is, compute a finite set of automorphisms $\Psi_1, \ldots, \Psi_k \in \operatorname{Aut}(G)$ such that $H = \operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$.*

Proof. Given generators for $H \leqslant_{fg} G$, apply Theorem 4.5.18. If $\text{a-Cl}_G(H)$ is not finitely generated then conclude that H is not auto-fixed. Otherwise, we get a finite set of automorphisms $\Psi_1, \ldots, \Psi_k \in \operatorname{Aut}(G)$ such that $\text{a-Cl}_G(H) = \operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$, and a basis for $\text{a-Cl}_G(H) \geqslant H$. Now H is auto-fixed if and only if this last inclusion is an equality, which can be algorithmically checked by using a solution to the membership problem in G; see [10, Prop. 1.11]; and in this case, $\Psi_1, \ldots, \Psi_k$ are the automorphisms such that $H = \operatorname{Fix}\Psi_1 \cap \cdots \cap \operatorname{Fix}\Psi_k$. □

5 Intersection and application to cryptography

"Cryptography shifts the balance of power from those with monopoly on violence to those who comprehend mathematics and security design."

— Jacob Appelbaum

We develop a secret-sharing scheme based on the fact that $\mathbb{Z}^m \times F_n$ does not satisfy Howson property. In our scheme, the shares are k finitely generated subgroups $H_1, \ldots, H_k \leqslant \mathbb{Z}^m \times F_n$ such that every intersection of shares is not finitely generated, except the total one $\bigcap_{i=1}^k H_i$ which is taken as the secret. In the present chapter we claim that for any integer $k \geq 3$ we can always build such a family $\mathcal{F} = \{H_1, H_2, \ldots, H_k\}$. As any finite intersection (except the total one) of the shares is not finitely generated, in this way we protect the scheme from any illegal coalition of players to extract any practical additional information about the secret.

5.1 Algorithm to compute finite intersection of subgroups of $\mathbb{Z}^{\mathrm{m}} \times \mathbf{F}_{\mathrm{n}}$

In [10] Delgado–Ventura gave an algorithm to decide if the intersection of two finitely generated subgroups H_1, H_2 of $\mathbb{Z}^m \times F_n$ is again finitely generated or not and in the affirmative case, the generators of $H_1 \cap H_2$ is computable; but for our scheme, we need to compute any finite (not only two) intersection of finitely generated subgroups of $\mathbb{Z}^m \times F_n$ without computing the smaller ones.

Lemma 5.1.1. *Let* $p_1, \ldots, p_k \in \mathbb{Z}^m$*, and* $L_1, \ldots, L_k \leqslant \mathbb{Z}^m$*. Then,*

$$\bigcap_{i=1}^{k}(p_i + L_i) \neq \emptyset \Leftrightarrow (p_2 - p_1 \mid p_3 - p_2 \mid \ldots \mid p_k - p_{k-1}) \in \operatorname{Im} \mathbf{L} \tag{5.1}$$

where

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_1 & & & & \\ -\mathbf{L}_2 & \mathbf{L}_2 & & & \\ & -\mathbf{L}_3 & \mathbf{L}_3 & & \\ & & \ddots & \ddots & \\ & & & -\mathbf{L}_{k-2} & \mathbf{L}_{k-1} \\ & & & & -\mathbf{L}_k \end{bmatrix}$$

and for each $i \in [1,k]$*,* $\mathbf{L}_i$ *is a* $d_i \times m$ *integer matrix with row space* L_i*.*

Proof. It is enough to check that the following statements are equivalent:

(a) $\bigcap_{i=1}^{k}(p_i + L_i) \neq \varnothing$.

(b) $\exists q \in \mathbb{Z}^m$ such that $\begin{cases} q = p_1 + l_1, \text{ where } l_1 \in L_1 \\ q = p_2 + l_2, \text{ where } l_2 \in L_2 \\ \quad \cdots \\ q = p_k + l_k, \text{ where } l_k \in L_k \end{cases}$

(c) $\forall i \in [1,k]\ \exists l_i \in L_i$ such that $\begin{cases} p_2 - p_1 = l_1 - l_2 \\ p_3 - p_2 = l_2 - l_3 \\ \quad \cdots \\ p_k - p_{k-1} = l_{k-1} - l_k \end{cases}$

(d) $\forall i \in [1,r]\ \exists a_i \in \mathbb{Z}^{d_i}$ such that $\begin{cases} p_2 - p_1 = a_1 L_1 - a_2 L_2 = [a_1 \mid a_2] \left[\begin{smallmatrix} L_1 \\ -L_2 \end{smallmatrix}\right] \\ p_3 - p_2 = a_2 L_2 - a_3 L_3 = [a_2 \mid a_3] \left[\begin{smallmatrix} L_2 \\ -L_3 \end{smallmatrix}\right] \\ \qquad \cdots \\ p_k - p_{k-1} = a_{k-1} L_{k-1} - a_k L_k = [a_{k-1} \mid a_k] \left[\begin{smallmatrix} L_{k-1} \\ -L_k \end{smallmatrix}\right] \end{cases}$

(e) $\exists a \in \mathbb{Z}^{\sum_{i=1}^k d_i}$ such that $[p_2 - p_1 \mid p_3 - p_2 \mid \ldots \mid p_k - p_{k-1}] = aL$.

$\square$

Theorem 5.1.2. *Let $H_1, \ldots, H_k \leqslant_{fg} \mathbb{Z}^m \times F_n$, where k is finite and each H_i is given by a finite set of generators. Then it is algorithmically decidable if $\bigcap_{i=1}^k H_i$ is finitely generated or not and in the affirmative case we can compute generators for $\bigcap_{i=1}^k H_i$.*

Proof. Let each $H_i = \langle t^{a_1^i} u_1^i, \ldots, t^{a_{n_i}^i} u_{n_i}^i, t^{b_1^i}, \ldots, t^{b_{d_i}^i} \rangle$ where $\{u_1^i, \ldots, u_{n_i}^i\}$ is a free-basis of $H_i\pi$ and $\{b_1^i, \ldots, b_{d_i}^i\}$ is an abelian-basis of $L_i = H_i \cap \mathbb{Z}^m$. An easy observation is $(H_1 \cap \cdots \cap H_k)\pi \trianglelefteq H_1\pi \cap \cdots H_k\pi$. So we can deduce that $\bigcap_{i=1}^k H_i$ is finitely generated if and only if the index of $(H_1 \cap \cdots \cap H_k)\pi$ is finite in $H_1\pi \cap \cdots \cap H_k\pi$. Let,

$$A^i = \begin{pmatrix} a_1^i \\ a_2^i \\ \vdots \\ \vdots \\ a_{n_i}^i \end{pmatrix} \in \mathrm{M}_{n_i \times m}(\mathbb{Z}), \quad L = \begin{bmatrix} L_1 & & & & \\ -L_2 & L_2 & & & \\ & -L_3 & L_3 & & \\ & & \ddots & \ddots & \\ & & & -L_{k-1} & L_{k-1} \\ & & & & -L_k \end{bmatrix}$$

where for each $i \in [k]$, L_i is a $d_i \times m$ integer matrix with row space $H_i \cap \mathbb{Z}^m$. Consider ρ as the abelianization map of $\bigcap_{i=1}^k H_i\pi$ abelianizing $\bigcap_{i=1}^k H_i\pi$ onto $\mathbb{Z}^q$ and P^i is describing the abelianization of the inclusion map $\iota_i \colon \bigcap_{i=1}^k H_i\pi \longrightarrow H_i\pi$, i.e., $P^i \in \mathrm{M}_{q \times n_i}(\mathbb{Z})$ (see Fig. 5.1). Let $R = (P^2A^2 - P^1A^1 \mid \cdots \mid P^kA^k - P^{k-1}A^{k-1}) \in \mathrm{M}_{q \times (k-1)m}(\mathbb{Z})$ be a concatenated matrix.

$$
\begin{tikzcd}
H_1\pi \cap \cdots \cap H_k\pi \arrow[r, hook, "\iota"] \arrow[d, "\rho"] \arrow[dr, phantom, "///"] & H_i\pi \arrow[d, "\rho_i"] \\
\mathbb{Z}^q \arrow[r, "P^i"] & \mathbb{Z}^{n_i} \arrow[dl, "A^i"'] \\
\mathbb{Z}^m &
\end{tikzcd}
$$

Figure 5.1: Diagram of finite intersection

Now,

$$
\begin{aligned}
(H_1 \cap \cdots \cap H_k)\pi \quad &= \\
&= \{w \in \textstyle\bigcap_{i=1}^k H_i\pi \mid w \text{ has a common completion } \} \\
&= \{w \in \textstyle\bigcap_{i=1}^k H_i\pi \mid (w\rho P^1 A^1 + L_1) \cap \cdots \cap (w\rho P^k A^k + L_k) \neq \emptyset\} \\
&= \{w \in \textstyle\bigcap_{i=1}^k H_i\pi \mid w\rho(P^2A^2 - P^1A^1 \mid \cdots \mid P^kA^k - P^{k-1}A^{k-1}) \in \operatorname{Im} L\} \\
&= \{w \in \textstyle\bigcap_{i=1}^k H_i\pi \mid w\rho R \in \operatorname{Im} L\} \\
&= (\operatorname{Im} L)R^{-1}\rho^{-1}.
\end{aligned}
$$

As ρ is onto, using the above mentioned argument, we can reduce the algorithm which decides if $\bigcap_{i=1}^k H_i$ is finitely generated into just deciding the fact $[\mathbb{Z}^q \colon (\operatorname{Im} L)R^{-1}] < \infty$.

Finally, let us assume $\bigcap_{i=1}^k H_i$ is finitely generated, and let us compute a basis for $\bigcap_{i=1}^k H_i$. We can compute a set $\{c_1, \ldots, c_r\} \subset \mathbb{Z}^q$ of coset representatives of $(\operatorname{Im} L)R^{-1}$ in $\mathbb{Z}^q$, namely $\mathbb{Z}^q = ((\operatorname{Im} L)R^{-1}) + c_1 \sqcup \cdots \sqcup ((\operatorname{Im} L)R^{-1}) + c_r$. By Stalling's graph technique having computed a free basis $\{v_1, \ldots, v_q\}$ for $\bigcap_{i=1}^k H_i\pi$ we can choose arbitrary preimages

$z_1, \dots, z_r$ of $c_1, \dots, c_r$ up in $H_1\pi \cap \cdots \cap H_k\pi$, and we get a set of right coset representatives of $(H_1 \cap \cdots \cap H_k)\pi = (\operatorname{Im} L)R^{-1}\rho^{-1}$ in $H_1\pi \cap \cdots \cap H_k\pi$:

$$\bigcap_{i=1}^{k} H_i\pi = ((\operatorname{Im} L)R^{-1}\rho^{-1})z_1 \sqcup \cdots \sqcup ((\operatorname{Im} L)R^{-1}\rho^{-1})z_r. \tag{5.2}$$

Now, we build the Schreier graph for $(\operatorname{Im} L)R^{-1}\rho^{-1} \leqslant \bigcap_{i=1}^{k} H_i\pi$ with respect to $\{v_1, \dots, v_q\}$ in the following way:

1. take the cosets from (5.2) as vertices, and with no edge;
2. for every vertex $((\operatorname{Im} L)R^{-1}\rho^{-1})z_i$ and every letter v_j, add an edge labeled v_j from $((\operatorname{Im} L)R^{-1}\rho^{-1})z_i$ to $((\operatorname{Im} L)R^{-1}\rho^{-1})z_i v_j$, algorithmically identified among the available vertices by repeatedly solving the membership problem for $(\operatorname{Im} L)R^{-1}\rho^{-1}$ (note that we can easily do this by abelianizing the candidate and checking whether it belongs to $(\operatorname{Im} L)R^{-1}$).

Once we have run over all $i = 1, \dots, r$ and all $j = 1, \dots, q$, we have computed the full Schreier graph, from which we can select a maximal tree and obtain a free-basis $\{w_1, \dots, w_s\}$ for $(\operatorname{Im} L)R^{-1}\rho^{-1} = (H_1 \cap \cdots \cap H_k)\pi$. Finally, solving (compatible) linear systems of equations, we obtain vectors $d_1, \dots, d_s \in \mathbb{Z}^m$ such that $t^{d_1}w_1, \dots, t^{d_s}w_s \in H_1 \cap \cdots \cap H_k$ and $\bigcap_{i=1}^{k} L_i = \langle t^{y_1}, \dots t^{y_l} \rangle$. We conclude that $\{t^{d_1}w_1, \dots, t^{d_s}w_s, t^{y_1}, \dots, t^{y_l}\}$ is a basis for $H_1 \cap \cdots \cap H_k$. □

5.2 Secret sharing scheme

In our scheme the shares for the k players are going to be k finitely generated subgroups $H_1, \dots, H_k$ of $\mathbb{Z}^m \times F_n$ such that every finite intersection of shares is not finitely generated, except for the total one $\bigcap_{i=1}^{k} H_i$, which is taken as the secret. This way we significantly increase the difficulty for an illegal coalition of players to extract any practical additional information about the secret (since the intersection of their shares illegitimately shared is not finitely generated so they can only hope to compute a finite truncation of it). In this section we will prove our main theorem (5.2.2); in other words we want to prove that, for any integer k, one can effectively built such a family of subgroups of $\mathbb{Z}^m \times F_n$.

First we want to give overview of the notations which will be used repeatedly throughout the proof of the theorem.

Notation: We have k players and $2^k - 1$ tokens to distribute among them. By convention, $[2^k - 1] = \{n \in \mathbb{N} \mid 1 \leqslant n \leqslant 2^k - 1\}$. As, there are k many players, we will do the binary decomposition of each $n \in [2^k - 1]$ into k many positions; in other words $[2^k - 1] = \{n \in \mathbb{N} \mid 1 \leqslant n \leqslant 2^k - 1\} = \{n_k \dots n_i \dots n_1 \mid n_i \in \{0, 1\}\} \setminus \{0, \dots, 0\}$. For $i \in [k]$, we denote $S_i = \{n \in [2^k - 1] \mid n_i = 1\}$, where n_i is the i-th position of the binary decomposition of n. And we denote, $S_{\mathcal{N}} = \bigcap_{i \in \mathcal{N}} S_i$ for $\mathcal{N} \subseteq [k]$.

Observation 5.2.1. *These sets $S_{\mathcal{N}}$ satisfy:*

(i) $\#S_i = 2^{k-1}$*;*

(ii) $\#S_{\{}i, j\} = 2^{k-2}$*; and so*

(iii) $\mathcal{N} \subseteq [k]$, $\#S_{\mathcal{N}} = 2^{k - \#\mathcal{N}}$.

Theorem 5.2.2. *For every $k \geq 3$ we can always build a family $\mathcal{F} = \{H_1, H_2, \dots, H_k\}$ of finitely generated subgroups of $\mathbb{Z}^m \times F_n$ such that, for each nonempty subfamily $\mathcal{S} \subseteq \mathcal{F}$,*

$$\bigcap_{i \in \mathcal{S}} H_i \text{ is finitely generated} \Leftrightarrow \#\mathcal{S} \in \{1, k\}. \tag{5.3}$$

Proof. We choose $2^k - 1$ freely independent elements $\{v_1, \dots, v_{2^k - 1}\} \in F_n$ and use them as tokens to distribute among players. And we will choose $k \times (2^k - 1)$ many vectors $a^i_j \in \mathbb{Z}^m$, $\forall i \in [k], j \in [2^k - 1]$ with the following conditions:

- for $j = 2^k - 1$, and $\forall i_1, i_2 \in [k]$,

$$a_j^{i_1} = a_j^{i_2}, \tag{5.4}$$

- for $j_1, j_2 \neq 2^k - 1$, and $\forall i_1, i_2 \in [k]$,

$$a_{j_1}^{i_1} = a_{j_2}^{i_2} \Leftrightarrow i_1 = i_2 \text{ and } j_1 = j_2. \tag{5.5}$$

Choosing the vectors a_j^i, with the above mentioned conditions, the k shares are defined as follows:

$$\begin{aligned} H_1 &= \langle t^{a_j^1} v_j \mid j \in S_1 \rangle \\ &\vdots \\ H_i &= \langle t^{a_j^i} v_j \mid j \in S_i \rangle \\ &\vdots \\ H_k &= \langle t^{a_j^k} v_j \mid j \in S_k \rangle. \end{aligned}$$

Note that, some of the chosen vectors do not show up and are never used. Note also that, for every $i \in [k]$, $L_i = \{0\}$. From the construction and considering the fact that $v_1, \ldots, v_{2^k-1}$ are freely independent, it is easy to observe that $H_{\mathcal{N}}\pi = \bigcap_{i \in \mathcal{N}} H_i\pi = \langle v_j \mid j \in S_{\mathcal{N}} \rangle$. We define the matrix A^i in the following way:

$$A^i = \begin{pmatrix} a_1^i \\ a_2^i \\ \vdots \\ \vdots \\ a_{2^k-1}^i \end{pmatrix} \in \mathrm{M}_{(2^k-1)\times m}(\mathbb{Z}),$$

and let $A^i{\downarrow}$ be the matrix A^i after deleting all rows corresponding to indices j not in S_i; in other words, A^i is a linear mapping $A^i \colon \mathbb{Z}^{2^k-1} \to \mathbb{Z}^m$ while $A^i{\downarrow}$ is its restriction to the subspace generated by coordinates in S_i, say $A^i{\downarrow} \colon \mathbb{Z}^{2^{k-1}} \to \mathbb{Z}^m$.

Let $\mathcal{N} = \{i_1, i_2, \ldots, i_\ell\}$ be a set of two or more indices, with $i_1 < i_2 < \cdots < i_\ell$ and $2 \leqslant \ell \leqslant k$. We have to prove that $H_\mathcal{N}$ is finitely generated if and only if $\#\mathcal{N} = \ell = k$. In fact, choose $i \in \{i_1, \ldots, i_\ell\}$ and consider the diagram

$$
\begin{tikzcd}
H_{i_1}\pi \cap \cdots \cap H_{i_\ell}\pi \arrow[r, hook, "\iota"] \arrow[d, "\rho"] \arrow[dr, phantom, "///"] & H_i\pi \arrow[d, "\rho_i"] \\
\mathbb{Z}^{2^{k-\ell}} \arrow[r, "P_{\mathcal{N}\to i}"] & \mathbb{Z}^{2^{k-1}} \arrow[dl, "A^i\downarrow"'] \\
\mathbb{Z}^m &
\end{tikzcd}
\tag{5.6}
$$

where $P_{\mathcal{N}\to i}$ is describing the abelianization of the inclusion map and so $P_{\mathcal{N}\to i} \in \mathrm{M}_{(2^{k-\ell})\times(2^{k-1})}(\mathbb{Z})$. As v_j's are freely independent for all $j \in [2^k - 1]$ and $v_{2^k-1} \in H_i\pi$ for all $i \in [k]$, the matrix $P_{\mathcal{N}\to i}$ contains only 0 and 1 as entries. Also note that every row of the matrix $P_{\mathcal{N}\to i}$ has one and only one 1 and the last row of the matrix $P_{\mathcal{N}\to i}$ is $(0, \ldots, 0, 1)$. Now we have two cases,

1. $\ell < k$;
2. $\ell = k$.

If we consider the first case,

$$R_\mathcal{N} = (P_{\mathcal{N}\to 2}A^2\downarrow - P_{\mathcal{N}\to 1}A^1\downarrow \mid \cdots \mid P_{\mathcal{N}\to l}A^\ell\downarrow - P_{\mathcal{N}\to(\ell-1)}A^{\ell-1}\downarrow), \tag{5.7}$$

i.e., $R_\mathcal{N} \in \mathrm{M}_{2^{k-\ell}\times(\ell-1)m}(\mathbb{Z})$ is the concatenated matrix, corresponding to the linear map $R_\mathcal{N} \colon \mathbb{Z}^{2^{(k-\ell)}} \to \mathbb{Z}^{(\ell-1)m}$ and

$$\Big(\bigcap_{i\in\mathcal{N}} H_i\Big)\pi = (\{0\}R_\mathcal{N}{}^{-1})\rho^{-1} = (\mathrm{Ker}\, R_\mathcal{N})\rho^{-1}. \tag{5.8}$$

From assumptions (5.5) about the vectors $a_j^i \in \mathbb{Z}^m$, $i \in [k]$ and $j \in [2^k - 2]$ and from the construction of the matrices $P_{\mathcal{N}\to i}$, it is clear that at least one row of the matrix $R_{\mathcal{N}}$ is non-zero, i.e., $\mathrm{r}(\mathrm{Im}\, R_{\mathcal{N}}) \geqslant 1$ so, $\mathrm{r}(\mathrm{Ker}\, R_{\mathcal{N}}) \leqslant 2^{(k-\ell)} - 1$ hence $[\mathbb{Z}^{2^{k-\ell}} : (\mathrm{Ker}\, R_{\mathcal{N}})] = \infty$. As ρ is onto, $[\bigcap_{i\in\mathcal{N}} H_i\pi : (\mathrm{Ker}\, R_{\mathcal{N}})\rho^{-1}] = [\bigcap_{i\in\mathcal{N}} H_i\pi : (\bigcap_{i\in\mathcal{N}} H_i)\pi] = \infty$ which in turn implies that $(\bigcap_{i\in\mathcal{N}} H_i)\pi$ is not finitely generated as $(\bigcap_{i\in\mathcal{N}} H_i)\pi \trianglelefteq \bigcap_{i\in\mathcal{N}} H_i\pi \leqslant F_n$. Hence, $\bigcap_{i\in\mathcal{N}} H_i$ is not finitely generated when $\#\mathcal{N} = \ell < k$.

If we consider the second case, all the matrices $P_{\mathcal{N}\to i}$ have a single row; more concretely $P_{\mathcal{N}\to i} = (0, \ldots, 0, 1) \in \mathrm{M}_{1\times 2^{k-1}}(\mathbb{Z})$. From the construction of the H_i's and condition (5.4) of the vectors a_j^i, we will have,

$$R_{\mathcal{N}} = (a_{2^k-1}^2 - a_{2^k-1}^1 \mid \cdots \mid a_{2^k-1}^k - a_{2^k-1}^{k-1}) = (0 \mid \cdots \mid 0) = 0$$

Thus we have $\mathrm{Ker}\, R_{\mathcal{N}} = (\bigcap_{i\in\mathcal{N}} H_i\pi)\rho$ and again using the fact that ρ is onto, $\bigcap_{i\in\mathcal{N}} H_i\pi = (\bigcap_{i\in\mathcal{N}} H_i)\pi$. With our assumption that all $H_i\pi$s are finitely generated and the Howson property of the subgroups of F_n, we have $\bigcap_{i\in\mathcal{N}} H_i\pi$ is finitely generated when $\#\mathcal{N} = \ell = k$. In fact, in this case, $(\bigcap_{i\in\mathcal{N}} H_i)\pi = \langle v_{2^k-1}\rangle$. Hence, we come to the conclusion that $\bigcap_{i\in\mathcal{N}} H_i$ is finitely generated when $\#\mathcal{N} = k$. $\square$

Remark 5.2.3. The obvious two disadvantages of this secret sharing scheme as it is written are:

(1) the *secret* is cyclic,

(2) the *secret* is the very last element of the generating set of every group H_i.

To overcome the first disadvantage we will consider another one or finitely many free words $w_1, \ldots, w_p \in F_n$, such that the whole set of elements $\{v_1, \ldots, v_{2^k-1}, w_1, \ldots, w_p\}$ is freely independent. Then, add $t^{b_\ell} w_\ell$ for $\ell = 1, \ldots, p$ to the set of generators for each H_i and the secret will be $\bigcap_{i=1}^k H_i = \langle t^{a_{2^k-1}} v_{2^k-1}, t^{b_1} w_1, \ldots, t^{b_p} w_p\rangle$.

To overcome the second disadvantage, we can consider k different automorphisms, $\Psi_i \in \mathrm{Aut}(H_i)$, for each $i \in [k]$. $H_i\Psi = \langle (t^{a^i_j} v_j)\Psi_i \mid j \in S_i \rangle$ and as the Ψ_i's are automorphisms, it is obvious that $\langle (t^{a^i_j} v_j)\Psi_i \mid j \in S_i \rangle = \langle t^{a^i_j} v_j \mid j \in S_i \rangle$. This way the k shares remain the same, $H_1\Psi_1 = H_1, \ldots, H_k\Psi_k = H_k$, but the secret is no longer visible as the last positions of the generators given to each player; instead, it is hidden as a subgroup of each H_i.

6 Future work

This final chapter is intended to summarize and discuss briefly some possible continuations, extensions, and possible future applications arising from the work which is comprehended in this dissertation.

6.1 Rationality, computability of $\mathrm{di}_{\mathrm{F_n}}$ and di_{G} vs. $\mathrm{di}'_{\mathrm{G}}$

The supremum in the definition of degree of compression is always a maximum, since the numerator has a fixed value and the denominator takes only natural values. Although we do not have any particular example, the supremum in the definition of degree of inertia could (in principle) not be attained at any particular subgroup K. In this sense, the following is an intriguing question for which, at the time of writing, we have no idea how to answer. I would like to investigate more in this direction in the future, either in the free group or in any other context:

Project 6.1.1. To answer the question: is there a (finitely generated) group G and a subgroup $H \leqslant_{fg} G$ such that $\mathrm{di}_G(H)$ is irrational? Or such that the supremum in $\mathrm{di}_G(H)$ is not a maximum? What about the free group $G = F_n$?

For free groups, we also do not know if the supremum in the definition of di_{F_n} is an actual maximum or not; in fact, we were not able to compute di_{F_n}. In this aspect, it is worth to mention that S. Ivanov [20] already considered and studied the strengthened version of this notion of degree of inertia for free groups. He defined the *Walter Neumann coefficient* of $H \leqslant_{fg} F_n$ as $\sigma(H) := \sup_{K \leqslant_{fg} F_n} \{\tilde{\mathrm{r}}(H,K)/\,\tilde{\mathrm{r}}(H)\,\tilde{\mathrm{r}}(K)\}$, where $\tilde{\mathrm{r}}(H,K) = \sum_{s \in H \backslash F_n / K} \tilde{\mathrm{r}}(H \cap K^s)$ (understanding $0/0 = 1$). In other words, $\sigma(H)$ is the smallest possible constant $\alpha \in \mathbb{R}$ such that $\tilde{r}(H, K) \leqslant \alpha\, \tilde{r}(H)\, \tilde{r}(K)$, for every $K \leqslant_{fg} F_n$. Using linear programming techniques, Ivanov was able to prove the following remarkable result:

Theorem 6.1.2 (Ivanov, [20]). *For any finitely generated free group F_n, the function σ is computable and the supremum is a maximum; more precisely, there is an algorithm which, on input $h_1, \ldots, h_r \in F_n$, it computes the value of $\sigma(\langle h_1, \ldots, h_r \rangle)$ and outputs a free basis of a subgroup $K \leqslant_{fg} F_n$ where that supremum is attained.*

Ivanov's proof is involved and technical. And a crucial point is that his arguments are global about the entire pullback, even if it is disconnected; this is why he gets his result Walter Neumann coefficient, the disconnected analog of our degree of inertia. As Ivanov himself recognizes in his paper, it seems hard to adapt this arguments to the connected component containing the basepoint and get information about the degree of inertia. Although they look quite similar goals, we also have been unable to adapt Ivanov's arguments to answer any of the following problems which, as far as we know, remain open. They can also be considered as suitable projects for the near future:

Project 6.1.3. To compute the function di_{F_n}. And to answer: Is that supremum always a maximum? is there and algorithm which, on input $h_1, \ldots, h_r \in F_n$, it computes the value of $\mathrm{di}_{F_n}(\langle h_1, \ldots, h_r \rangle)$? or even more, it outputs a free basis of a subgroup $K \leqslant_{fg} F_n$ where it is attained?

As we discussed before, an arbitrary group G may not be Howson and in that case we are interested only in *finitary inert* subroups of G. But we do not know if there really exists a group G which posses a finitely generated *finitary inert* subgroup H such that H is *not inert*. Hence the following problem is open, as far as we know:

Project 6.1.4. To prove or disprove the existence of a (finitely generated) group G with a subgroup $H \leqslant_{fg} G$ being finitary inert but not inert (i.e., satisfying $\tilde{r}(H \cap K) \leqslant \tilde{r}(K)$

for every $K \leqslant_{fg} G$ with $H \cap K \leqslant_{fg} G$, but simultaneously admitting some $K_0 \leqslant_{fg} G$ with $\tilde{\mathrm{r}}(H \cap K_0) = \infty$).

Restricting our attention to free-abelian times free groups, and going down to more technical questions, there is a natural continuation of the work done in Chapter 3: in Theorem 3.4.2(iii) we proved that, for any subgroup $H \leqslant \mathbb{Z}^m \times F_n$ with $r(H\pi) \geqslant 2$ and $[\mathbb{Z}^m : H \cap \mathbb{Z}^m] = \ell < \infty$, we have that $\mathrm{di}_G(H) \leqslant \ell\, \mathrm{di}_{F_n}(H\pi)$. We conjecture that this is, in reality, an equality. We did not succeed proving completely the other inequality, but our effort was partially successful in the sense that we could prove Theorem 3.5.14: the equality holds under a couple of extra assumptions on the subgroups K intersecting the given H. Formally, we introduced these conditions into the definition of degree of inertia, getting this way the so-called *restricted degree of inertia*, where the supremum is restricted to those subgroups K satisfying that $H\pi \cap K\pi$ is not contained in the commutator $[F_n, F_n]$ and has infinite index in $H\pi$; and with this restricted definition, we proved that $\mathrm{di}'_G(H) = \ell\, \mathrm{di}'_{F_n}(H\pi)$. In pages 74 and 75 we gave an intuitive idea about a possible way to sort out the use of these technical conditions and get the equality in full. At the time of writing this did not crystallized into a solid argument yet, but we hope it can be done in the near future (probably, developing several other technical Lemmas and Proposition like 3.5.3, 3.5.4 and 3.5.8). This is a reasonable plan to follow in the immediate future:

Project 6.1.5. To proof that, for any finitely generated subgroup H of $G = \mathbb{Z}^m \times F_n$, $\mathrm{di}_G(H) = \mathrm{di}'_G(H)$. Or, at least, to show that, for any $H \leqslant \mathbb{Z}^m \times F_n$ with $r(H\pi) \geqslant 2$ and $[\mathbb{Z}^m : H \cap \mathbb{Z}^m] = \ell < \infty$, the equality $\mathrm{di}_G(H) = \ell\, \mathrm{di}_{F_n}(H\pi)$ holds.

6.2 Computability of endo-fixed closures

Before discussing about the endo-fixed closure of a finitely generated subgroup H in $\mathbb{Z}^m \times F_n$, we want to emphasize that we did not succeed in the task of constructing an example of a finitely generated subgroup $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$ such that $\mathrm{a\text{-}Cl}_G(H)$ is *not* finitely generated; it could be that such examples do not exist so the following is an interesting open problem:

Project 6.2.1. To decide if for every $H \leqslant_{fg} G = \mathbb{Z}^m \times F_n$, the auto-fixed closure $\mathrm{a\text{-}Cl}_G(H)$ is again finitely generated or not. What about the endo-fixed closure $\mathrm{e\text{-}Cl}_G(H)$?

The versions of Theorem 4.5.18 and Corollary 4.5.19 for endomorphisms seem to be much more tricky and remain open: their versions for the free group, contained in Theorem 4.5.8 and Corollary 4.5.9, are already much more complicated because the monoid $\mathrm{End}_{F_n}(H)$ is not necessarily finitely generated (even with H being so) and also because computability of fixed subgroups is not known for endomorphisms. In the free context these two obstacles were overcome using algebraic extensions and the Takahasi theorem, where the finiteness of the set of algebraic extensions of a finitely generated subgroup plays a crucial role. We got a first version of Takahasi's theorem in the context of free-abelian times free groups (see Theorem 4.2.5), but relaxing that finiteness condition to an infinity of subgroups following finitely many patterns, with vectorial parameters. This is by the moment not enough to translate here the arguments from free group endo-closures, but constitutes a first building block to do so in the future: if we manage to work with this parametric finiteness there is hope to extend the known results about endo-closures from the free to the free-abelian times free context.

Project 6.2.2. To answer: Let $G = \mathbb{Z}^m \times F_n$. Is there an algorithm which, given a finite set of generators for a subgroup $H \leqslant_{fg} G$, decides whether

(i) the monoid $\mathrm{End}_H(G)$ is finitely generated or not and, in case it is, computes a set of endomorphisms $\Psi_1, \ldots, \Psi_k \in \mathrm{End}(G)$ such that $\mathrm{End}_H(G) = \langle \Psi_1, \ldots, \Psi_k \rangle$?

(ii) $\mathrm{e\text{-}Cl}_G(H)$ is finitely generated or not and, in case it is, computes a basis for it ?

(iii) H is endo-fixed or not ?

6.3 Takahasi theorem for $\mathbb{Z}^{\mathrm{m}} \times \mathbf{F}_{\mathrm{n}}$ and its possible applications

One of the particular interests to our discussion is the result given by Takahasi [39] in 1951 for free groups. The original proof, due to M. Takahasi was combinatorial, using words and their lengths with respect to different sets of generators. And the more geometrical proof was done later independently by Kapovich–Miasnikov in [22], by Ventura in [41], and by Margolis–Sapir–Weil in [24], with a later unification by Miasnikov–Ventura–Weil in [31]. Takahasi theorem is an important tool in free groups as there has been several research works where it played a crucial role in proving of them. Here are some of these applications:

- Computation of the *endo- fixed closure* of a subgroup $H \leqslant_{fg} F_n$, namely,

  $$\text{e-Cl}_G(H) = \bigcap_{\substack{\varphi \in \mathrm{End}(F_n) \\ H \leqslant \mathrm{Fix}(\varphi)}} \mathrm{Fix}(\varphi)$$

  This was done by E. Ventura in [42] where, additionally, an algorithm is given to decide if a given subgroup is the fixed subgroup of a finite family of endos (or autos) or not, and in the affirmative case, computing such a family of endos (or autos).

- A. Martino and E. Ventura [26] also proved that, for every autos (or endos) f, g there is another one being a word on them, say $h = w(f, g)$, such that $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is a free factor of $\mathrm{Fix}(h)$. And in the same paper [26] the authors conjectured that the family of fixed subgroups is closed by intersections (i.e., one can always avoid the free complement). In other words, is $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ always equal to $\mathrm{Fix}(h)$ for some h? This is still an open problem even in free groups.

- Computation of pro-$\mathcal{V}$ closures (like pro-p, pro-solvable, pro-nilpotent, etc) of finitely generated subgroups of a free group F_n. Consider a variety $\mathcal{V}$ of finite groups, i.e., a family of finite groups closed under taking subgroups, quotients, and direct products). Given such a variety $\mathcal{V}$ and an arbitrary group G, one can put the pro-$\mathcal{V}$ topology in G defined (metrically) in the following way: given two elements $g, g' \in G$ define the distance between them as $d(g, g') = 2^{-v(g,g')}$, where $v(g, g')$ is the smallest cardinal of a group $H \in \mathcal{V}$ for which there is a homomorphism $\varphi \colon G \to H$ separating g and g', i.e., such that $g\varphi \neq g'\varphi$ (take $d(g, g') = 0$ if $v(g, g') = \infty$ meaning that there is no such finite group $H \in \mathcal{V}$). This is a pseudo-metric in G which induces a topology called the pro-$\mathcal{V}$ topology (in case the group G is residually-$\mathcal{V}$ it is then a real metric and the topology becomes Hausdorff). Typical examples are the pro-finite topology (take $\mathcal{V}$ to be all finite groups), the pro-p topology (take $\mathcal{V}$ to be all finite p-groups), the pro-nilpotent topology (take $\mathcal{V}$ to be all finite nilpotent groups), the pro-solvable topology (take $\mathcal{V}$ to be all finite solvable groups), etc.

  Let us particularize the situation to the free group, $G = F_n$. In [24], Margolis–Sapir–Weil proved among other results that, when the variety $\mathcal{V}$ is extension-closed (i.e., for any short exact sequence $1 \to A \to B \to C \to 1$ of finite groups, if $A, C \in \mathcal{V}$ then $B \in \mathcal{V}$) then free factors of closed subgroups of F_n are again closed subgroups. This automatically connects with Takahasi theorem because it implies that, for any subgroup $H \leqslant_{fg} F_n$, its pro-$\mathcal{V}$ closure $\overline{H}$ must be one of the finitely many algebraic extensions of H, $\overline{H} \in \mathcal{AE}(H)$. Using this idea the authors of [24] gave algorithms to compute the pro-finite, pro-p, and pro-nilpotent closures of finitely generated subgroups of F_n (the computation of the pro-solvable closure is still an open problem).

As mentioned above, we adapted appropriately the notions of "free factor" and "algebraic extension" from free groups to $\mathbb{Z}^m \times F_n$, and we gave a version of Takahasi's theorem for free-abelian times free groups in Theorem 4.2.5, paying the price that the finiteness of algebraic extensions in the classical context gets replaced here with a parametric finiteness (finiteness modulo finitely many vectorial parameters). This is yet not enough, but a first step towards the generalization of the above applications of Takahasi theorem for free groups to the context of free-abelian times free groups:

Project 6.3.1. Is the family of fixed subgroups $\mathbb{Z}^m \times F_n$ in some sense closed under intersections ? Maybe up to "factors"? Can such an intersection be not finitely generated? (remind that $\mathbb{Z}^m \times F_n$ is not Howson and so, all questions related to intersections tend in general to be more tricky).

Project 6.3.2. Consider the pro-$\mathcal{V}$ topology in $\mathbb{Z}^m \times F_n$, given by an extension closed variety $\mathcal{V}$. Reprove here the fact that, in the extension closed case, "factors" of closed subgroups are closed again, and then extend the algorithms for computing finite, p-, and nilpotent closures, from the free group to $\mathbb{Z}^m \times F_n$.

6.4 Finite presentation of $\mathrm{Aut}_{\mathbf{H}}(\mathbf{A}(\mathbf{\Gamma}))$

McCool's Theorem 4.5.7 was a variation and an extension of a much earlier result: back in the 1930's, Whitehead already solved the orbit problem for conjugacy classes in the free group: given two tuples of conjugacy classes $V = ([v_1], \ldots, [v_k])$ and $W = ([w_1], \ldots, [w_k])$ in F_n, one can algorithmically decide whether there is an automorphism $\phi \in \mathrm{Aut}(F_n)$ such that $v_i\phi \sim w_i$, for every $i = 1, \ldots, k$; see [23, Prop. 4.21] or [43]; this was based on the so-called Whitehead automorphisms and the peak reduction technique. McCool's work 40 years later consisted of (1) deducing as a corollary that $\mathrm{Aut}_W(F_n)$ if finitely presented and a finite presentation is computable from the given W; and (2) extending everything to real elements instead of conjugacy classes and so, getting a solution to the orbit problem for tuples of elements, and the finite presentability (and computability) for stabilizers of subgroups, stated in Theorem 4.5.7.

Much more recently, a new version of these peak reduction techniques has been developed by M. Day [9] for right-angled Artin groups, extending McCool result (1) above to this bigger class of groups.

And in our Theorem 4.5.14 we generalized Day's theorem from conjugacy classes to real elements, in the case of $\mathbb{Z}^m \times F_n$, a very particular subclass of right-angled Artin groups. A natural project here is to try to export this generalization to all right-angled Artin groups. We have to say that our arguments worked in $\mathbb{Z}^m \times F_n$ thanks to the very special fact that all the commutativity of the group is concentrated in its center, a fact that is far from true in general for arbitrary right-angled Artin groups. This makes us think that the general situation could be much more complicated than just the free-abelian times free case.

Project 6.4.1. Extending Day's theorem (see 4.5.12) to real elements instead of conjugacy classes for any arbitrary right-angled Artin group $A(\Gamma)$. Or at least for a less wild subclass of such groups containing free-abelian times free groups, like for example Droms groups.

6.5 Semidirect products of the form $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$

Definition 6.5.1. Let $A_1, \ldots, A_n \in \mathrm{GL}_m(\mathbb{Z})$ be n invertible integral $m \times m$ matrices (acting on the right of vectors, $A_i \colon \mathbb{Z}^m \to \mathbb{Z}^m, a \mapsto aA_i$), and consider the semidirect product

$$G = \mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n = \langle z_1, \ldots, z_n, t_1, \ldots, t_m \mid [t_i, t_j] = 1,\ z_i^{-1} t^a z_i = t^{aA_i} \rangle.$$

Of course, the particular case corresponding to the identity matrices, $A_1 = \cdots = A_n = Id$, gives our standard free-abelian times free group, $\mathbb{Z}^m \rtimes_{Id,\ldots,Id} F_n = \mathbb{Z}^m \times F_n$. Furthermore, the following are easy observations for these semidirect products, generalizing precisely the same results for free-abelian times free groups:

Observation 6.5.2. *We have the natural split short exact sequence*

$$1 \to \mathbb{Z}^m \to \mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n \to F_n \to 1,$$

and computable normal forms $t^a w(\vec{z})$ for the elements of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$, where $a \in \mathbb{Z}^m$ and $w \in F_n = F(\{z_1, \ldots, z_n\})$.

Proposition 6.5.3. *For every subgroup $H \leqslant G = \mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$, the sub-short exact sequence*

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z}^m & \to & G & \xrightarrow{\pi} & F_n & \to & 1 \\ & & \vee & & \vee & & \vee & & \\ 1 & \to & L_H = H \cap \mathbb{Z}^m & \to & H & \xrightarrow{\pi} & H\pi & \to & 1 \end{array}$$

also splits and so, $H \simeq L_H \rtimes_{\mathcal{A}} H\pi$, where $\mathcal{A}$ is the restriction of the defining action $F_n \to \operatorname{Aut}(\mathbb{Z}^m)$ to $\mathcal{A}\colon H\pi \to \operatorname{Aut}(L_H)$.

In particular, every $H \leqslant \mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$, $n \geqslant 2$, is again of the form $H \simeq \mathbb{Z}^{m'} \rtimes_{A'_1,\ldots,A'_{n'}} F_{n'}$, for some $n' \in \mathbb{N} \cup \{\infty\}$ and $m' \leqslant m$.

The first reasonable step in this family is to study the degree of compression. The arguments involved in the study and computability of the degree of compression for a subgroup of $\mathbb{Z}^m \times F_n$ are purely about the free group (Stallings graphs, fringe, algebraic extensions, etc) or about linear algebra (PAQ-reduction of integral matrices, linear systems of equations, manipulation of direct summands, etc). It seems reasonable to think that these arguments will extend and work in a semidirect product $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$, just with the matrices $A_1, \ldots, A_n$ twisting the calculations and making the arguments more involved. An interesting point here is the fact that, while the rank of $\mathbb{Z}^m \times F_n$ (i.e., the minimal number of generators) is $m + n$, the rank of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$ could easily be less than this because of the effect of the action matrices. I do not see yet a clear way to compute/understand ranks of (free-abelian)-by-free groups; maybe the notion of degree of compression will have to be considered with respect to the invariant $dim(\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n) = m + n$ instead of the usual rank. In whatever sense it needs to be considered, the project here is the following:

Project 6.5.4. Find formulas and algorithms to compute the degree of compression of finitely generated subgroups of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$.

Other possibilities for future investigation are the search of a Takahasi theorem for semidirect products (subject to finding a good enough notion of "factor", which is not clear at the moment, and needs more detailed thinking), and the study of particular properties of fixed subgroups of automorphisms (subject to being able to obtain a more or less explicit description of all automorphisms of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$, similar to what we have in the free-abelian times free case):

Project 6.5.5. Find a good enough notion of "factor" for subgroups of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$ and prove a Takahasi-like theorem for this family of groups. Obtain similar applications as those done for the free case (see above).

Project 6.5.6. Find a good enough description of the automorphisms of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$ and, from it, analyze special properties of fixed point subgroups of automorphisms in this family of groups (bounding the rank, compression, inertia, etc, following again the guide of what happens in free groups).

About degree of inertia we are much more skeptical: our understanding of the degree of inertia for subgroups of $\mathbb{Z}^m \times F_n$ strongly relies on the diagram (3.6), invented by Delgado–Ventura in [10], to understand arbitrary intersections of finitely generated subgroups. As far as we know, these arguments do not extend to semidirect products, where the control of intersections seems to be much more involved, and unknown at the present time. Without a way of understanding intersections, it does not seem plausible to try to understand the degree of inertia in semidirect products.

Delgado-Ventura [12] have built an adaptation of the Stallings' automata theory to work with subgroups of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$; they are essentially classical Stallings graphs decorated with vectors in a clever enough way to keep all the information of the subgroup in a finite geometric object. It is very possible that this nice construction helps us in our goals within this family of groups.

6.6 Direct products involving surface groups

Free-abelian times free groups are direct products of one (or non) free group F_n with several copies of $\mathbb{Z}$. Another possibility to extend our scope to a more general class of groups is to allow several factors being free groups, and to allow other building blocks apart from F_n and $\mathbb{Z}$. Surface groups have similar properties to free groups, and have interesting connections to them, so they seem good candidates to be new building blocks.

Let us consider then as building blocks all surface groups, i.e., fundamental groups of connected compact surfaces, both orientable and non orientable, and with and without finitely many punctures (note that this already includes $\mathbb{Z}$ and all the free groups).

Definition 6.6.1. A surface group is the fundamental group, $G = \pi_1(X)$, of a connected compact (possibly non-orientable) surface X. To fix the notation, we shall denote by Σ_g the closed orientable surface of genus $g \geqslant 0$, and by

$$S_g = \pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$$

its fundamental group (by convention, $S_0 = \langle \mid \rangle$ stands for the trivial group, the fundamental group of the sphere Σ_0). And for the non-orientable case, we shall denote by $N\Sigma_k$ the connected sum of $k \geqslant 1$ projective planes, and by

$$NS_k = \pi(N\Sigma_k) = \langle a_1, a_2 \ldots, a_k, \mid {a_1}^2 \cdots {a_k}^2 \rangle$$

its fundamental group. Note that, among surface groups, the only abelian ones are $S_0 = 1$ (for the sphere), $S_1 = \mathbb{Z}^2$ (for the torus), and $NS_1 = \mathbb{Z}/2\mathbb{Z}$ (for the projective plane).

It is well known that the Euler characteristic of an orientable surface is $\chi(\Sigma_g) = 2 - 2g$, and of the non-orientable ones is $\chi(\Sigma_k) = 2 - k$. Hence, all surfaces have negative Euler characteristic (these are said to be of hyperbolic type) except for the sphere Σ_0, the torus Σ_1, the projective plain $N\Sigma_1$, and the Klein bottle $N\Sigma_2$, homeomorphic to the connected sum of two projective plains (these exceptional ones are said to be of Euclidean type).

These surface groups have some interesting properties making them very similar to what happens in free groups:

- Any subgroup H of a surface group G either has finite index in G or it is free; and if H has index d in G, then it is again a surface group with $\chi(H) = d \cdot \chi(G)$.
- The fundamental group of a compact surface with punctures is free.
- For a surface group G with negative Euler characteristic, $\chi(G) < 0$, its center is trivial, $Z(G) = 1$, and the centralizer of any non-trivial element $1 \neq g \in G$ is infinite cyclic, $Cen_G(g) \simeq \mathbb{Z}$.

Moreover, some results about automorphisms and endomorphisms for free groups (specially those concerning compression or inertia) will work in a similar way for surface groups with negative Euler characteristic; S_0, S_1, NS_1, and NS_2 will usually present special and exceptional behaviour (in part, due to the structure of the center and centralizers in these cases). In this direction the first results were given by Jiang-Wang-Zhang [21] in 2011.

Theorem 6.6.2 (Jiang-Wang-Zhang, [21]). *Let G be a surface group with $\chi(G) < 0$. Then $\mathrm{r}(\mathrm{Fix}(\phi)) \leqslant \mathrm{r}(G)$, $\forall \phi \in \mathrm{End}(G)$.*

And this result was later extended to the following results:

Theorem 6.6.3 (Wu-Zhang, [45]). *Let G be a surface group with $\chi(G) < 0$, and $B \subseteq \mathrm{End}(G)$. Then,*

(i) $\mathrm{r}(\mathrm{Fix}\, B) \leqslant \mathrm{r}(G)$*, with equality if and only if* $B = \{id\}$*;*

(ii) $\mathrm{r}(\mathrm{Fix}\, B) \leqslant \frac{1}{2}\,\mathrm{r}(G)$*, if B contains a non-epimorphic endomorphism;*

(iii) if $B \subseteq \mathrm{Aut}(G)$*, then* $\mathrm{Fix}\, B$ *is inert in G.*

And then, recent results are also given in the inertia direction:

Theorem 6.6.4 (Zhang–Ventura–Wu, [46]). *(i) Let F_n be a finitely generated free group, let $B \subset \mathrm{End}(F_n)$ and let $\beta_0 \in \langle B \rangle \leqslant \mathrm{End}(F_n)$ be with $\mathrm{r}(\beta_0(F_n))$ minimal. Then, $\mathrm{Fix}(B)$ is inert in $\beta_0(F_n)$. Moreover, if $\beta_0(F_n)$ is inert in F_n then $\mathrm{Fix}(B)$ is inert in F_n.*

(ii) Let G be a surface group, let $B \subseteq \mathrm{End}(G)$ be an arbitrary family of endomorphisms, let $\langle B \rangle \leqslant \mathrm{End}(G)$ be the submonoid generated by B, and let $\beta_0 \in \langle B \rangle \leqslant \mathrm{End}(G)$ with image of minimal rank. Then, for every subgroup $K \leqslant G$ such that $\beta_0(K) \cap \mathrm{Fix}(B) \leqslant K$, we have that $\mathrm{r}(K \cap \mathrm{Fix}(B)) \leqslant \mathrm{r}(K)$.

In the paper [46], Zhang–Ventura–Wu introduced the family of groups $\mathcal{P}$ consisting in direct products of finitely many surface groups in this broad sense, i.e., groups of the form $G = G_1 \times G_2 \times \cdots \times G_n$, where $n \geqslant 1$ and each G_i is either $\mathbb{Z}$, or F_n with $n \geqslant 2$, or S_g with $g \geqslant 2$, or NS_k with $k \geqslant 1$. Such a group G was called of *hyperbolic* type if all its factors are hyperbolic, of *Euclidean* type if all its factors are Euclidean, and of *mixed* type otherwise.

Zhang–Ventura–Wu already studied automorphism φ of such a group $G = G_1 \times G_2 \times \cdots \times G_n \in \mathcal{P}$ of hyperbolic type and proved it is always equal to the direct product of automorphisms of each component, $\varphi = \varphi_1, \times \cdots \times \varphi_n$, $\varphi_i \in \mathrm{Aut}(G_i)$, just modulo permutations of the possibly repeated factors ($G_i = G_j$), if any; see Proposition 4.4 from [46] for the exact statement. This result allowed them to connect properties of the automorphisms of G with the corresponding properties about automorphisms of the factors G_i's. In this sense, [46] contains the following nice characterization:

Theorem 6.6.5 (Zhang–Ventura–Wu, [46])**.** *Let $G = G_1 \times G_2 \times \cdots \times G_n \in \mathcal{P}$. Then, $\mathrm{r}(Fix(\varphi)) \leqslant \mathrm{r}(G)$ for every $\varphi \in \mathrm{Aut}(G)$, if and only if G is either of hyperbolic or of Euclidean type.*

In fact, in the case of mixed type, and copying the idea from $\mathbb{Z} \times F_2$, one can easily construct an automorphism $\varphi \in \mathrm{Aut}(G)$ whose fixed subgroup is even not finitely generated. Additionally, [46] also contains partial results in the direction of characterizing which $G \in \mathcal{P}$ satisfy that $\mathrm{Fix}(\varphi)$ is compressed, or $\mathrm{Fix}(\varphi)$ is inert, for every $\varphi \in \mathrm{Aut}(G)$. Would be nice to complete this characterization in the spirit of the above theorem:

Project 6.6.6. Give an explicit characterization of those $G \in \mathcal{P}$ which satisfy: (i) $\mathrm{Fix}(\varphi)$ is compressed for every $\varphi \in \mathrm{Aut}(G)$; or (ii) $\mathrm{Fix}(\varphi)$ is inert for every $\varphi \in \mathrm{Aut}(G)$. Study the similar questions about endomorphisms.

Finally, going into the direction of the degrees of compression/inertia, an interesting project would be to study the degree of compression of subgroups in this family of groups; for similar reasons as those given in the case of semidirect products, the study of the degree of inertia seems to be much more tricky and out of reach, at least before understanding intersections in this more general class of groups:

Project 6.6.7. Find formulas and algorithms to compute the degree of compression of finitely generated subgroups of a group G in $\mathcal{P}$ (maybe under technical restrictions, if necessary, on the subgroup and/or on the factors of G).

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