# Commuting degree of infinite groups 

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## Outline

(1) Motivation
(2) Main definition
(3) Finite index subgroups

4 Short exact sequences
(5) A Gromov-like theorem

6 Other results

## Outline

Main definition(3) Finite index subgroups

4 Short exact sequences
(5) A Gromov-like theorem

6 Other results

## Motivation

(Joint work with Y. Antolín and A. Martino.)
Theorem (Gustaison, 1973)
Let $G$ be a finite group. If the probability that two elements from $G$ commute is bigger than $5 / 8$, then $G$ is abelian.

Proof. Suppose G is not abelian. Then,

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\begin{aligned}
d c(G) & =\frac{\mid\{(u, v)|u v-v u|}{|G|^{2}}=\frac{1}{|G|^{2}} \sum_{u \in G}\left|C_{G}(u)\right|= \\
& =\frac{1}{|G|^{2}}\left(|Z(G)||G|+\sum_{u \in G \backslash Z(G)}\left|C_{G}(u)\right|\right) \leqslant \\
& \leqslant \frac{1}{|G|^{2}}\left(|Z(G)||G|+(|G|-|Z(G)|) \frac{|G|}{2}\right) \leqslant
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because $G / Z(G)$ cannot be cyclic and so, $|Z(G)| \leqslant|G| / 4$.

## Observation

The quaternion group has $d c(Q)=5 / 8$.
"There is no live between 5/8 and 1"

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Is there a version of do for infinite groups?

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## (2) Main definition

3 Finite index subgroups

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## Degree of commutativity

## Definition

Let $G=\langle X\rangle$ be a f.g. group. The degree of commutativity of $G$ w.r.t. $X$ is

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d c_{X}(G)=\limsup _{n \rightarrow \infty} \frac{\left|\left\{(u, v) \in \mathbb{B}_{X}(n) \times \mathbb{B}_{X}(n) \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(n)\right|^{2}} \in[0,1],
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where $\mathbb{B}_{X}(n)=\left\{\left.g \in G| | g\right|_{X} \leqslant n\right\}$.
Question
Is this a real lim ? Does it depend on $X$ ?

About limsup we have no idea:

- No example where lim doesn't exist;
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## Independence on $X$

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A f.g. group $G=\langle X\rangle$ is of

- subexponential growth if $\lim _{n \rightarrow \infty} \frac{\left|\mathbb{B}_{x}(n+1)\right|}{\left|\mathbb{B}_{x}(n)\right|}=1$;
- polynomial growth if $\mathrm{Cn}^{\text {d }}$


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Let $G=\langle X\rangle$. A map $f: G \rightarrow \mathbb{N}$ is an estimation of the $X$-metric if $\exists$
$K>0$ such that $\forall w \in G$


## Example

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\begin{gathered}
\mathbb{B}_{f}(n)=\{w \in G \mid f(w) \leqslant n\} \\
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## Proposition

Let $G=\langle X\rangle$ be of polynomial growth, and $f: G \rightarrow \mathbb{N}$ be an estimation of the $X$-metric. Then,
$d c_{X}(G)>0 \quad \Longleftrightarrow \quad d c_{f}(G)>0$.

Proof. Clearly, $\mathbb{B}_{f}(n) \subseteq \mathbb{B}_{x}(K n) \subseteq \mathbb{B}_{f}\left(K^{2} n\right)$ so,

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\| \frac{\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(K n)\right)^{2} \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(K n)\right|^{2}} \\
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\text { So, } d C_{X}(G)=0 \Rightarrow d C_{f}(G)=0 \text {, because } \\
\frac{\left|\mathbb{B}_{f}(n)\right|}{\left|\mathbb{B}_{X}(K n)\right|} \geqslant \frac{|\mathbb{B} X(n / K)|}{\left|\mathbb{B}_{X}(K n)\right|} \geqslant \frac{C(n / K)^{d}}{D(K n)^{d}}=\frac{C}{D K^{2 d}}>0
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## Corollary

If $G=\langle X\rangle=\langle Y\rangle$ is of polynomial growth, then

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So, $d c_{x}(G)=0 \quad \Rightarrow \quad d c_{f}(G)=0$, because

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d c_{X}(G)=0 \quad \Longleftrightarrow \quad d c_{Y}(G)=0
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4 Short exact sequences
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6 Other results

## Finite index subgroups

Lemma (Burillo-Ventura, 2002)
If $H \leqslant$ f.i. $G=\langle X\rangle$ and $G$ has subexponential growth then there exists $\lim _{n \rightarrow \infty} \frac{\left|\mathbb{B}_{X}(n) \cap H\right|}{\left|\mathbb{B}_{X}(n)\right|}=\frac{1}{[G: H]}$.

## Proposition



## Proposition (Gallagher, 1970)

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## Corollary

Let $\langle Y\rangle=H \quad G=\langle X\rangle$ be of polynomial growth. Then, $d c x(G)>0$
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## Short exact sequences

## Proposition

Let $G=\langle X\rangle, H \unlhd G$, and let $\pi: G \rightarrow Q=G / H=\langle\bar{X}\rangle$. Put

$$
0 \leqslant \lambda=\left(\lim \inf \frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}\right)^{2}
$$

Then, $\lambda \cdot d c_{x}(G) \leqslant d c_{\bar{x}}(Q) \cdot d c_{x}(H)$.

## Proof. Write $d c_{X}(G)=\lim \sup d c_{X}(G, n)$, where



We have,

## Short exact sequences

## Proposition

Let $G=\langle X\rangle, H \unlhd G$, and let $\pi: G \rightarrow Q=G / H=\langle\bar{X}\rangle$. Put

$$
0 \leqslant \lambda=\left(\lim \inf \frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}\right)^{2} .
$$

Then, $\lambda \cdot d c_{X}(G) \leqslant d c_{\bar{x}}(Q) \cdot d c_{\chi}(H)$.

Proof. Write $d c_{X}(G)=\lim \sup d c_{X}(G, n)$, where

$$
d c_{X}(G, n)=\frac{\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(n)\right)^{2} \mid u v=v u\right\}\right|}{\left|\mathbb{B}_{X}(n)\right|^{2}} .
$$

We have,

## Short exact sequences

$$
\left|\mathbb{B}_{X}(n)\right|^{2} d c_{X}(G, n)=\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(n)\right)^{2} \mid u v=v u\right\}\right|=
$$



## Short exact sequences

$$
\begin{gathered}
\left|\mathbb{B}_{X}(n)\right|^{2} d c_{X}(G, n)=\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(n)\right)^{2} \mid u v=v u\right\}\right|= \\
=\sum_{u \in \mathbb{B}_{X}(n)}\left|C_{G}(u) \cap \mathbb{B}_{X}(n)\right|=\sum_{q \in \mathbb{B}_{X}(n)} \sum_{\substack{u \in \mathbb{B}_{\mathbb{X}^{\prime}}(n) \\
\pi(u)=q}}\left|C_{G}(u) \cap \mathbb{B}_{X}(n)\right| \leqslant
\end{gathered}
$$

## Short exact sequences

$$
\begin{aligned}
& \quad\left|\mathbb{B}_{X}(n)\right|^{2} d c_{X}(G, n)=\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(n)\right)^{2} \mid u v=v u\right\}\right|= \\
& =\sum_{u \in \mathbb{B}_{X}(n)}\left|C_{G}(u) \cap \mathbb{B}_{X}(n)\right|=\sum_{q \in \mathbb{B}_{\bar{X}}(n)} \sum_{\substack{u \in \mathbb{B}_{X}(n) \\
\pi(u)=q}}\left|C_{G}(u) \cap \mathbb{B}_{X}(n)\right| \leqslant \\
& \leqslant \sum_{q \in \mathbb{B}_{\bar{X}}(n)} \sum_{\substack{u \in \mathbb{B}_{X}(n) \\
\pi(u)=q}}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \cdot\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|= \\
& =\sum_{q \in \mathbb{B}_{X}(n)}\left|C_{Q}(q) \cap \mathbb{B}_{X}(n)\right| \sum_{\substack{u \in)^{\prime}(n)}}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|=(1) \mid
\end{aligned}
$$

## Short exact sequences

$$
\begin{aligned}
& \left|\mathbb{B}_{X}(n)\right|^{2} d c_{X}(G, n)=\left|\left\{(u, v) \in\left(\mathbb{B}_{X}(n)\right)^{2} \mid u v=v u\right\}\right|= \\
= & \sum_{u \in \mathbb{B}_{X}(n)}\left|C_{G}(u) \cap \mathbb{B}_{X}(n)\right|=\sum_{\substack{q \in \mathbb{B}_{X}(n)}} \sum_{\substack{u \in \mathbb{B}_{X}(n) \\
\pi(u)=q}}\left|C_{G}(u) \cap \mathbb{B}_{X}(n)\right| \leqslant \\
\leqslant & \sum_{q \in \mathbb{B}_{X}(n)} \sum_{\substack{u \in \mathbb{B}_{X}(n) \\
\pi(u)=q}}\left|C_{Q}(q) \cap \mathbb{B}_{X}(n)\right| \cdot\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|= \\
= & \sum_{q \in \mathbb{B}_{X}(n)}\left(\left|C_{Q}(q) \cap \mathbb{B}_{X}(n)\right| \sum_{\substack{u \in \mathbb{B}_{X}(n) \\
\pi(u)=q}}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|\right)=(1)
\end{aligned}
$$

## Short exact sequences

But, fixing $u_{0} \in \mathbb{B}_{X}(n)$ with $\pi\left(u_{0}\right)=q$,

$$
\sum_{\mathbb{X}_{x}(n), \pi(u)=q}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|=
$$



## Short exact sequences

But, fixing $u_{0} \in \mathbb{B}_{X}(n)$ with $\pi\left(u_{0}\right)=q$,

$$
\begin{gathered}
\sum_{u \in \mathbb{B}_{X}(n), \pi(u)=q}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|= \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right)\left|v \in C_{H}\left(u_{0} h\right),\left|u_{0} h\right|_{X} \leq n\right\} \mid=\right.\right.
\end{gathered}
$$



## Short exact sequences

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$$
\begin{gathered}
\sum_{u \in \mathbb{B}_{X}(n), \pi(u)=q}\left|C_{H}(u) \cap \mathbb{B}_{X}(2 n)\right|= \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right)\left|v \in C_{H}\left(u_{0} h\right),\left|u_{0} h\right|_{X} \leq n\right\} \mid=\right.\right. \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right) \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right|=
\end{gathered}
$$

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=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right)\left|v \in C_{H}\left(u_{0} h\right),\left|u_{0} h\right| x \leq n\right\} \mid=\right.\right. \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right) \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right|= \\
=\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|\left\{h \in H \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right| \leqslant
\end{gathered}
$$

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$$
\begin{gathered}
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=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right)\left|v \in C_{H}\left(u_{0} h\right),\left|u_{0} h\right| X \leq n\right\} \mid=\right.\right. \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right) \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right|= \\
=\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|\left\{h \in H \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right| \leqslant \\
\leqslant \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right| .
\end{gathered}
$$

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\begin{gathered}
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=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right)\left|v \in C_{H}\left(u_{0} h\right),\left|u_{0} h\right| X \leq n\right\} \mid=\right.\right. \\
=\left|\left\{(h, v) \in H \times\left(\mathbb{B}_{X}(2 n) \cap H\right) \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right|= \\
=\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|\left\{h \in H \mid u_{0} h \in C_{G}(v) \cap \mathbb{B}_{X}(n)\right\}\right| \leqslant \\
\leqslant \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right| .
\end{gathered}
$$

Hence,

## Short exact sequences

$$
(1) \leqslant \sum_{q \in \mathbb{B}_{X}(n)}\left(\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)=
$$



$$
\left|\mathbb{B}_{X}(n)\right|^{2} \cdot d c_{X}(Q, n) \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|^{2} \cdot d c_{X}(H, 2 n) .
$$

## It follows that



Finally, taking limits, we get

## Short exact sequences

$$
\begin{aligned}
& (1) \leqslant \sum_{q \in \mathbb{B}_{X}(n)}\left(\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
& =\left(\sum_{q \in \mathbb{B}_{X}(n)}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right|\right)\left(\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)=
\end{aligned}
$$

$$
\left|\mathbb{B}_{X}(n)\right|^{2} \cdot d c_{X}(Q, n) \cdot\left|\mathbb{B}_{x}(2 n) \cap H\right|^{2} \cdot d c_{X}(H, 2 n) .
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=\left(\sum_{q \in \mathbb{B}_{X}(n)}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right|\right)\left(\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
\left|\mathbb{B}_{\bar{X}}(n)\right|^{2} \cdot d c_{X}(Q, n) \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|^{2} \cdot d c_{X}(H, 2 n) .
\end{gathered}
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=\left(\sum_{q \in \mathbb{B}_{\bar{X}}(n)}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right|\right)\left(\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
\left|\mathbb{B}_{\bar{X}}(n)\right|^{2} \cdot d c_{\bar{X}}(Q, n) \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|^{2} \cdot d c_{X}(H, 2 n) .
\end{gathered}
$$

It follows that

$$
\left(\frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}\right)^{2} \cdot d c_{X}(G, n) \leqslant d c_{X}(Q, n) \cdot d c_{X}(H, 2 n) .
$$

## Short exact sequences

$$
\begin{gathered}
(1) \leqslant \sum_{q \in \mathbb{B}_{\bar{X}}(n)}\left(\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
=\left(\sum_{q \in \mathbb{B}_{\bar{X}}(n)}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right|\right)\left(\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
\left|\mathbb{B}_{\bar{X}}(n)\right|^{2} \cdot d c_{\bar{X}}(Q, n) \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|^{2} \cdot d c_{X}(H, 2 n) .
\end{gathered}
$$

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\left(\frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}\right)^{2} \cdot d c_{X}(G, n) \leqslant d c_{\bar{X}}(Q, n) \cdot d c_{X}(H, 2 n)
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Finally, taking limits, we get

## Short exact sequences

$$
\begin{gathered}
(1) \leqslant \sum_{q \in \mathbb{B}_{\bar{X}}(n)}\left(\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right| \sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
=\left(\sum_{q \in \mathbb{B}_{\bar{X}}(n)}\left|C_{Q}(q) \cap \mathbb{B}_{\bar{X}}(n)\right|\right)\left(\sum_{v \in \mathbb{B}_{X}(2 n) \cap H}\left|C_{H}(v) \cap \mathbb{B}_{X}(2 n)\right|\right)= \\
\left|\mathbb{B}_{\bar{X}}(n)\right|^{2} \cdot d c_{\bar{X}}(Q, n) \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|^{2} \cdot d c_{X}(H, 2 n) .
\end{gathered}
$$

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$$
\left(\frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}\right)^{2} \cdot d c_{X}(G, n) \leqslant d c_{\bar{X}}(Q, n) \cdot d c_{X}(H, 2 n)
$$

Finally, taking limits, we get

$$
\lambda \cdot d c_{X}(G) \leqslant d c_{\bar{X}}(Q) \cdot d c_{X}(H)
$$

## Short exact sequences

## Proposition

There exists a positive $\lambda=\lambda(s)>0$ s.t. $\forall$ f.g. $G, \forall X$ with $G=\langle X\rangle$ and $\left|\mathbb{B}_{X}(n)\right| \sim n^{r}, r \leqslant s$, and $\forall H \unlhd_{\text {f.i. }} G$, we have

$$
\lambda \cdot d c_{X}(G) \leq d c_{\bar{x}}(G / H) \cdot d c_{X}(H)
$$

## Proof. Clearly, $\left|\mathbb{B}_{x}(2 n)\right| \geqslant\left|\mathbb{B}_{\bar{x}}(n)\right| \cdot\left|\mathbb{B}_{X}(n) \cap H\right|$. Thus, for $n \gg 0$, and

 small enough $\epsilon>0$,

$$
=\frac{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{X}(2 n) \cap H\right|}=\frac{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right|}
$$




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$$
\frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|} \geqslant \frac{\left|\mathbb{B}_{\bar{X}}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right| \cdot\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}=
$$

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$$
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Proof. Clearly, $\left|\mathbb{B}_{X}(2 n)\right| \geqslant\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(n) \cap H\right|$. Thus, for $n \gg 0$, and small enough $\epsilon>0$,

$$
\begin{gathered}
\frac{\left|\mathbb{B}_{X}(n)\right|}{\left|\mathbb{B}_{X}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|} \geqslant \frac{\left|\mathbb{B}_{\bar{X}}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right| \cdot\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{\bar{X}}(n)\right| \cdot\left|\mathbb{B}_{X}(2 n) \cap H\right|}= \\
=\frac{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{X}(2 n) \cap H\right|}=\frac{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right) \cap H\right|}{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right|} \cdot \frac{\left|\mathbb{B}_{X}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right|}{\left|\mathbb{B}_{X}(2 n)\right|} \cdot \frac{\left|\mathbb{B}_{X}(2 n)\right|}{\left|\mathbb{B}_{X}(2 n) \cap H\right|} \geqslant
\end{gathered}
$$

## Short exact sequences

$$
\geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot \frac{C \cdot\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^{r}}{D \cdot(2 n)^{r}} \cdot([G: H]-\epsilon) \geqslant
$$


which is $>0$ for all $G, X$ and $H$ as above. $\square$

## Short exact sequences

$$
\begin{aligned}
& \geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot \frac{C \cdot\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^{r}}{D \cdot(2 n)^{r}} \cdot([G: H]-\epsilon) \geqslant \\
& \geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot\left(\frac{C}{D \cdot 2^{r}}-\epsilon\right) \cdot([G: H]-\epsilon) \geqslant
\end{aligned}
$$

$$
\text { which is }>0 \text { for all } G, X \text { and } H \text { as above. } \square
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& \geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot\left(\frac{C}{D \cdot 2^{r}}-\epsilon\right) \cdot([G: H]-\epsilon) \geqslant \\
& \geqslant\left(\frac{1}{[G: H]}-\epsilon\right) \cdot\left(\frac{C}{D \cdot 2^{s}}-\epsilon\right) \cdot([G: H]-\epsilon),
\end{aligned}
$$

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\end{aligned}
$$

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## Outline

Main definition(3) Finite index subgroups

4 Short exact sequences
(5) A Gromov-like theorem

6 Other results

## t.f. nilpotent groups

## Proposition

Let $G=\langle X\rangle$ be t.f. nilpotent. Then, either $G$ is abelian, or $d c_{X}(G)=0$.

## Proof. Assume $G$ is not abelian and $d_{X}(G)>0$ and let us find a

 contradiction.- We have a uniform $\lambda>0$ s.t., for every $H \unlhd_{\text {f.i. }} G$,

$$
\lambda \cdot d c_{X}(G) \leqslant d c_{\bar{X}}(G / H) \cdot d c_{X}(H)
$$

- Choose $n$ s.t. $\lambda \cdot d c_{X}(G) \cdot\left(\frac{8}{5}\right)^{n}>1$.
- Take $\left\{p_{1}, \ldots, p_{n}\right\}$ be n pairwise different primes.
- By Grumbergs' classical result, $G$ is residually- $p_{i}$.
- Hence, $G$ has a non-abelian, finite $p_{i}$-quotient $\pi_{i}: G \rightarrow Q_{i}$; in particular, $d c\left(Q_{i}\right) \leqslant \frac{5}{8}$.


## t.f. nilpotent groups

Proposition
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Proof. Assume $G$ is not abelian and $d c_{X}(G)>0$ and let us find a contradiction.
> - We have a uniform $\lambda>0$ s.t., for every $H \unlhd_{\text {f.i. }} G$, $\lambda \cdot d c_{X}(G) \leqslant d c_{\bar{x}}(G / H) \cdot d c_{X}(H)$
> - Choose $n$ s.t. $\lambda \cdot d c_{X}(G) \cdot\left(\frac{8}{5}\right)^{n}>1$
> - Take $\left\{p_{1}, \ldots, p_{n}\right\}$ be n pairwise different primes.
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- Choose $n$ s.t. $\lambda \cdot d c_{X}(G) \cdot\left(\frac{8}{5}\right)^{n}>1$.
- Take $\left\{p_{1}, \ldots, p_{n}\right\}$ be $n$ pairwise different primes.
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- By Grumbergs' classical result, $G$ is residually- $p_{i}$
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## A Gromov-like theorem

## Theorem

Let $G$ be a polynomially growing group. Then,
$G$ is virtually abelian $\Longleftrightarrow d c_{x}(G)>0$ for some (and hence all) $X$.

## Proof. ( $\Rightarrow$ ) Ok.

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- By Gromov result, $\exists$ a nilpotent $H \leqslant$ fi. G.
- So, $\exists$ a t.f. nilpotent $K \leqslant_{\text {fi. }} H \leqslant_{\text {fi. } i .} G$.
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Every f.g. group $G$ with super-polinomial growth has $d c_{X}(G)=0$ for every $X$.

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## Outline

MotivationMain definition（3）Finite index subgroups

4 Short exact sequences
（5）A Gromov－like theorem
（6）Other results

## Other results

## Theorem <br> Let $G$ be non-elementary hyperbolic. Then $d c_{X}(G)=0$ for every $X$.

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## THANKS

