Commuting degree of infinite groups

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Outline

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Motivation

(Joint work with Y. Antolín and A. Martino.)

Theorem (Gustafson, 1973)

Let G be a finite group. If the probability that two elements from G commute is bigger than $5/8$, then G is abelian.

Proof. *Suppose G is not abelian. Then,*

$$dc(G) = \frac{|\{(u,v) \mid uv = vu\}|}{|G|^2} = \frac{1}{|G|^2}\sum_{u\in G}|C_G(u)| =$$

$$= \frac{1}{|G|^2}\Big(|Z(G)||G| + \sum_{u\in G\setminus Z(G)}|C_G(u)|\Big) \leqslant$$

$$\leqslant \frac{1}{|G|^2}\left(|Z(G)||G| + (|G| - |Z(G)|)\frac{|G|}{2}\right) \leqslant$$

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because $G/Z(G)$ cannot be cyclic and so, $|Z(G)| \leqslant |G|/4$. $\square$

Observation

The quaternion group has $dc(Q) = 5/8$.

"There is no live between $5/8$ and 1"

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Is there a version of dc for infinite groups ?

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Degree of commutativity

Definition

Let $G = \langle X \rangle$ be a f.g. group. The degree of commutativity of G w.r.t. X is

$$dc_X(G) = \limsup_{n\to\infty} \frac{|\{(u,v) \in \mathbb{B}_X(n) \times \mathbb{B}_X(n) \mid uv = vu\}|}{|\mathbb{B}_X(n)|^2} \in [0,1],$$

where $\mathbb{B}_X(n) = \{g \in G \mid |g|_X \leqslant n\}$.

Question

Is this a real lim ? Does it depend on X ?

About limsup we have no idea:

- *No example where lim doesn't exist;*
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Independence on X

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A f.g. group $G = \langle X \rangle$ is of

- *subexponential growth* if $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n+1)|}{|\mathbb{B}_X(n)|} = 1$;
- *polynomial growth* if $Cn^d \leqslant |\mathbb{B}_X(n)| \leqslant Dn^d$.

Definition

Let $G = \langle X \rangle$. A map $f\colon G \to \mathbb{N}$ is an estimation of the X-metric if $\exists$ $K > 0$ such that $\forall w \in G$

$$\frac{1}{K} f(w) \leqslant |w|_X \leqslant K\, f(w).$$

Example

It is well known that, for $G = \langle X \rangle = \langle Y \rangle$, $|\cdot|_X$ is an estimation of the Y-metric, and $|\cdot|_Y$ is an estimation of the X-metric.

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Define the f-ball and the f-dc:

$$\mathbb{B}_f(n) = \{w \in G \mid f(w) \leqslant n\},$$

$$dc_f(G) = \limsup_{n\to\infty} \frac{|\{(u,v) \in \mathbb{B}_f(n) \times \mathbb{B}_f(n) \mid uv = vu\}|}{|\mathbb{B}_f(n)|^2}.$$

Proposition

Let $G = \langle X \rangle$ be of polynomial growth, and $f\colon G \to \mathbb{N}$ be an estimation of the X-metric. Then,

$$dc_X(G) > 0 \iff dc_f(G) > 0.$$

Proof. *Clearly, $\mathbb{B}_f(n) \subseteq \mathbb{B}_X(Kn) \subseteq \mathbb{B}_f(K^2 n)$ so,*

$$|\{(u,v) \in (\mathbb{B}_f(n))^2 \mid uv = vu\}| \leqslant |\{(u,v) \in (\mathbb{B}_X(Kn))^2 \mid uv = vu\}|.$$

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So, $dc_X(G)=0 \Rightarrow dc_f(G)=0$, because

$$\frac{|\mathbb{B}_f(n)|}{|\mathbb{B}_X(Kn)|} \geqslant \frac{|\mathbb{B}_X(n/K)|}{|\mathbb{B}_X(Kn)|} \geqslant \frac{C(n/K)^d}{D(Kn)^d} = \frac{C}{DK^{2d}} > 0. \quad \square$$

Corollary

If $G=\langle X\rangle=\langle Y\rangle$ is of polynomial growth, then

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Finite index subgroups

Lemma (Burillo–Ventura, 2002)

If $H \leqslant_{f.i.} G = \langle X \rangle$ and G has subexponential growth then there exists $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n) \cap H|}{|\mathbb{B}_X(n)|} = \frac{1}{[G:H]}$.

Proposition

Let $\langle Y \rangle = H \leqslant_{f.i.} G = \langle X \rangle$ be of polynomial growth. Then, $dc_X(G) \geqslant \frac{1}{[G:H]^2} dc_Y(H)$.

Proposition (Gallagher, 1970)

Let G be a finite group and $H \trianglelefteq G$. Then, $dc(G) \leqslant dc(H) \cdot dc(G/H)$.

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Short exact sequences

Proposition

Let $G = \langle X \rangle$, $H \trianglelefteq G$, and let $\pi \colon G \twoheadrightarrow Q = G/H = \langle \overline{X} \rangle$. Put

$$0 \leqslant \lambda = \left(\liminf \frac{|\mathbb{B}_X(n)|}{|\mathbb{B}_{\overline{X}}(n)| \cdot |\mathbb{B}_X(2n) \cap H|} \right)^2.$$

Then, $\lambda \cdot dc_X(G) \leqslant dc_{\overline{X}}(Q) \cdot dc_X(H)$.

Proof. *Write $dc_X(G) = \limsup dc_X(G, n)$, where*

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$$|\mathbb{B}_X(n)|^2 dc_X(G,n) = |\{(u,v) \in (\mathbb{B}_X(n))^2 \mid uv = vu\}| =$$

$$= \sum_{u \in \mathbb{B}_X(n)} |C_G(u) \cap \mathbb{B}_X(n)| = \sum_{q \in \mathbb{B}_{\overline{X}}(n)} \sum_{\substack{u \in \mathbb{B}_X(n) \\ \pi(u)=q}} |C_G(u) \cap \mathbb{B}_X(n)| \leqslant$$

$$\leqslant \sum_{q \in \mathbb{B}_{\overline{X}}(n)} \sum_{\substack{u \in \mathbb{B}_X(n) \\ \pi(u)=q}} |C_Q(q) \cap \mathbb{B}_{\overline{X}}(n)| \cdot |C_H(u) \cap \mathbb{B}_X(2n)| =$$

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Short exact sequences

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It follows that

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Finally, taking limits, we get

$$\lambda \cdot dc_X(G) \leqslant dc_{\overline{X}}(Q) \cdot dc_X(H). \quad \square$$

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Short exact sequences

Proposition

There exists a positive $\lambda = \lambda(s) > 0$ s.t. $\forall$ f.g. G, $\forall$ X with $G = \langle X \rangle$ and $|\mathbb{B}_X(n)| \sim n^r$, $r \leqslant s$, and $\forall$ $H \trianglelefteq_{f.i.} G$, we have

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Short exact sequences

$$\geqslant \left(\frac{1}{[G:H]} - \epsilon\right) \cdot \frac{C \cdot (\lfloor \frac{n}{2} \rfloor)^r}{D \cdot (2n)^r} \cdot ([G:H] - \epsilon) \geqslant$$

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which is > 0 *for all* G, X *and* H *as above.* $\square$

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which is > 0 *for all* G, X *and* H *as above.* □

Outline

t.f. nilpotent groups

Proposition

Let $G = \langle X \rangle$ be t.f. nilpotent. Then, either G is abelian, or $dc_X(G) = 0$.

Proof. *Assume G is not abelian and $dc_X(G) > 0$ and let us find a contradiction.*

- *We have a uniform $\lambda > 0$ s.t., for every $H \trianglelefteq_{f.i.} G$,*

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- *Hence,*

$$1 < \lambda \cdot dc_X(G) \cdot (\frac{8}{5})^n \leqslant dc_X(H),$$

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A Gromov-like theorem

Theorem

Let G be a polynomially growing group. Then,

$$G \text{ is virtually abelian } \iff dc_X(G) > 0 \text{ for some (and hence all) } X.$$

Proof. *($\Rightarrow$) Ok.*

($\Leftarrow$)

- *By Gromov result, $\exists$ a nilpotent $H \leqslant_{f.i.} G$.*
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THANKS