The degree of commutativity of an infinite group

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Outline

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Motivation

(Joint work with Y. Antolín and A. Martino.)

Theorem (Gustafson, 1973)

Let G be a finite group. If the probability that two elements from G commute is bigger than 5/8, then G is abelian.

Proof. *Suppose G is not abelian. Then,*

$$dc(G) = \frac{|\{(u,v) \in G^2 \mid uv = vu\}|}{|G|^2} = \frac{1}{|G|^2} \sum_{u \in G} |C_G(u)| =$$

$$= \frac{1}{|G|^2} \Big(|Z(G)||G| + \sum_{u \in G \setminus Z(G)} |C_G(u)| \Big) \leqslant$$

$$\leqslant \frac{1}{|G|^2} \Big(|Z(G)||G| + (|G| - |Z(G)|) \frac{|G|}{2} \Big) =$$

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because $G/Z(G)$ cannot be cyclic and so, $|Z(G)| \leqslant |G|/4$. □

Observation

The quaternion group has $dc(Q) = 5/8$.

"There is no live between $5/8$ and 1"

(Goal)

Is there a version of dc for infinite groups ?

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Let $G = \langle X \rangle$ be a f.g. group. The degree of commutativity of G w.r.t. X is

$$dc_X(G) = \limsup_{n\to\infty} \frac{|\{(u,v) \in \mathbb{B}_X(n) \times \mathbb{B}_X(n) \mid uv = vu\}|}{|\mathbb{B}_X(n)|^2} \in [0,1],$$

where $\mathbb{B}_X(n) = \{g \in G \mid |g|_X \leqslant n\}$.

Question

Is this a real lim ? Does it depend on X ?

About limsup we have no idea:

- *No example where lim doesn't exist;*
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A f.g. group $G = \langle X \rangle$ is of

- *subexponential growth* if $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n+1)|}{|\mathbb{B}_X(n)|} = 1$;
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Let $G = \langle X \rangle$. A map $f\colon G \to \mathbb{N}$ is an estimation of the X-metric if $\exists K > 0$ such that $\forall w \in G$

$$\frac{1}{K} f(w) \leqslant |w|_X \leqslant K\, f(w).$$

Example

It is well known that, for $G = \langle X \rangle = \langle Y \rangle$, $|\cdot|_X$ is an estimation of the Y-metric, and $|\cdot|_Y$ is an estimation of the X-metric.

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Define the f-ball and the f-dc:

$$\mathbb{B}_f(n) = \{w \in G \mid f(w) \leqslant n\},$$

$$dc_f(G) = \limsup_{n\to\infty} \frac{|\{(u,v) \in \mathbb{B}_f(n) \times \mathbb{B}_f(n) \mid uv = vu\}|}{|\mathbb{B}_f(n)|^2}.$$

Proposition

Let $G = \langle X \rangle$ be of polynomial growth, and $f \colon G \to \mathbb{N}$ be an estimation of the X-metric. Then,

$$dc_X(G) > 0 \iff dc_f(G) > 0.$$

Proof. *Clearly,* $\mathbb{B}_f(n) \subseteq \mathbb{B}_X(Kn) \subseteq \mathbb{B}_f(K^2 n)$ *so,*

$$|\{(u,v) \in (\mathbb{B}_f(n))^2 \mid uv = vu\}| \leqslant |\{(u,v) \in (\mathbb{B}_X(Kn))^2 \mid uv = vu\}|.$$

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So, $dc_X(G) = 0 \quad \Rightarrow \quad dc_f(G) = 0$, because

$$\frac{|\mathbb{B}_f(n)|}{|\mathbb{B}_X(Kn)|} \geqslant \frac{|\mathbb{B}_X(n/K)|}{|\mathbb{B}_X(Kn)|} \geqslant \frac{C(n/K)^d}{D(Kn)^d} = \frac{C}{DK^{2d}} > 0. \quad \square$$

Corollary

If $G = \langle X \rangle = \langle Y \rangle$ is of polynomial growth, then

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If $G=\langle X\rangle=\langle Y\rangle$ *is of polynomial growth, then*

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Let $\langle Y\rangle = H \leqslant G = \langle X\rangle$. The subgroup H is undistorted if $\exists\ K > 0$ s.t. $\forall\, h \in H$, $|h|_Y/K \leqslant |h|_X \leqslant K|h|_Y$.
In this case, $|\cdot|_X$ restricted to H is an estimation of the Y-metric for H.

Corollary

Let $G = \langle X\rangle$ be of polynomial growth, and $\langle Y\rangle = H \leqslant G$ be a non-distorted subgroup. Then,

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Finite index subgroups

Lemma (Burillo–Ventura, 2002)

If $H \leqslant_{f.i.} G = \langle X \rangle$ and G has subexponential growth then, for every $g \in G$, there exists $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n) \cap gH|}{|\mathbb{B}_X(n)|} = \lim_{n\to\infty} \frac{|\mathbb{B}_X(n) \cap Hg|}{|\mathbb{B}_X(n)|} = \frac{1}{[G:H]}$.

Remark

This is false in the free group: $H = \{\text{even words}\} \leqslant_2 F_r$.

Proposition

Let $\langle Y \rangle = H \leqslant_{f.i.} G = \langle X \rangle$ be of polynomial growth. Then,

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In particular, $dc_Y(H) > 0 \Rightarrow dc_X(H) > 0 \Rightarrow dc_X(G) > 0$.

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Proof. *Clearly,*
$|\{(u,v) \in (\mathbb{B}_X(n))^2 \mid uv = vu\}| \geqslant |\{(u,v) \in (H \cap \mathbb{B}_X(n))^2 \mid uv = vu\}|.$
Therefore, given $\varepsilon > 0$*, we have for* $n \gg 0$

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$$\frac{|\{(u,v) \in (H \cap \mathbb{B}_X(n))^2 \mid uv = vu\}|}{|H \cap \mathbb{B}_X(n)|^2} \left(\frac{1}{[G:H]} - \varepsilon\right)^2.$$

Taking limsups, $dc_X(G) \geqslant dc_X(H)\left(\frac{1}{[G:H]} - \varepsilon\right)^2$*. And this is true* $\forall \varepsilon > 0$ *so,* $dc_X(G) \geqslant \frac{1}{[G:H]^2} dc_X(H).$ □

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Finite index subgroups

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Proposition (Gallagher, 1970)

Let G be a finite group and $H \trianglelefteq G$. Then, $dc(G) \leqslant dc(H) \cdot dc(G/H)$.

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Let $G = \langle X \rangle$ be subexponentially growing. Then, for any finite quotient G/N, we have $dc_X(G) \leqslant dc(G/N)$.

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Outline

The main result

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Let $G = \langle X \rangle$ be of subexponential growth and residually finite. Then,

(i) $dc_X(G) > 5/8 \Leftrightarrow G$ is abelian;

(ii) $dc_X(G) > 0 \Leftrightarrow G$ is virtually abelian.

In particular, (i) and (ii) is true for polynomially growing groups.

Corollary

Let $G = \langle X \rangle = \langle Y \rangle$ be of subexponential growth and residually finite. Then,

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Then $\forall i$, $\quad K_i \trianglelefteq G, \quad (G/K_i)/(K_{i-1}/K_i) = G/K_{i-1}$ and, by Gallagher,

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By induction, $dc(G/K_i) \leqslant (5/8)^i$ and so,

$$dc_X(G) \leqslant dc(G/K_i) \leqslant (5/8)^i,$$

for every i. Therefore, $dc_X(G) = 0$. $\square$

Outline

Generalizations

- *We can replace $xy = yx$ by any system of equations.*
- *We can replace the uniform measures on balls to any sequence of measures (random walks, etc).*

Definition

Let $\{X_1, \ldots, X_k\}$ be a set of abstract variables and $\mathcal{F}$ the free group on it. Think elements $w \in \mathcal{F}$ as equations, $w = 1$, and subsets $\mathcal{E} \subseteq \mathcal{F}$ as systems of equations. Define solutions on a group G in the obvious way.

Definition

Given $G = \langle X \rangle$ and a system of equations $\mathcal{E} \subseteq \mathcal{F}$, we define the degree of satisfiability of $\mathcal{E}$ in G w.r.t X as

$$ds_X(G, \mathcal{E}) = \limsup_{n\to\infty} \frac{|\{(g_1, \ldots, g_k) \in (\mathbb{B}_X(n))^k \mid (g_1, \ldots, g_k) \text{ sol. } \mathcal{E}\}|}{|\mathbb{B}_X(n)|^k} \in [0, 1].$$

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Generalizations

Definition

Let G and $\mathcal{E}$ be as before. Fix a collection of measures μ_n in G. We define the degree of satisfiability of $\mathcal{E}$ in G w.r.t. μ_n *as*

$$ds_X(G, \mathcal{E}, \{\mu_n\}_n) =$$

$$\limsup_{n\to\infty} \mu_n^{\times k}(\{(g_1, \ldots, g_k) \in G^k \mid (g_1, \ldots, g_k) \text{ sol. } \mathcal{E}\}) \in [0, 1].$$

Generalizations

Conjecture

Let G, $\mathcal{E}$, and $\{\mu_n\}_n$ be as above, with μ_n "reasonable". Then,

$$ds(G, \mathcal{E}, \{\mu_n\}_n) > 0 \iff \mathcal{E} \text{ is a virtual law in } G.$$

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$\mathcal{E}$ is a law in G if every $(g_1, \ldots, g_k) \in G^k$ is a solution of $\mathcal{E}$ in G.

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THANKS