# The degree of commutativity/nilpotency of an infinite group

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# Outline

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# Motivation

(Joint work with Y. Antolín and A. Martino.)

**Theorem (Gustafson, 1973)**

*Let $G$ be a finite group. If the probability that two elements from $G$ commute is bigger than $5/8$, then $G$ is abelian.*

***Proof.*** *Suppose $G$ is not abelian. Then,*

$$dc(G) = \frac{|\{(u,v) \in G^2 \mid uv = vu\}|}{|G|^2} = \frac{1}{|G|^2} \sum_{u \in G} |C_G(u)| =$$

$$= \frac{1}{|G|^2} \Big( |Z(G)||G| + \sum_{u \in G \setminus Z(G)} |C_G(u)| \Big) \leqslant$$

$$\leqslant \frac{1}{|G|^2} \Big( |Z(G)||G| + (|G| - |Z(G)|) \frac{|G|}{2} \Big) =$$

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*because $G/Z(G)$ cannot be cyclic and so, $|Z(G)| \leqslant |G|/4$.* □

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*The quaternion group has $dc(Q) = 5/8$.*

"There is no live between $5/8$ and 1"

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*Is there a version of dc for infinite groups ?*

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*Is this a real lim ? Does it depend on $X$ ?*

*About limsup we have no idea:*

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- *subexponential growth* if $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n+1)|}{|\mathbb{B}_X(n)|} = 1$;
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*Let $G = \langle X \rangle$. A map $f : G \to \mathbb{N}$ is an estimation of the $X$-metric if $\exists$ $K > 0$ such that $\forall w \in G$*

$$\frac{1}{K} f(w) \leqslant |w|_X \leqslant K f(w).$$

**Example**

*It is well known that, for $G = \langle X \rangle = \langle Y \rangle$, $|\cdot|_X$ is an estimation of the $Y$-metric, and $|\cdot|_Y$ is an estimation of the $X$-metric.*

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**Proposition**

*Let $G = \langle X \rangle$ be of polynomial growth, and $f\colon G \to \mathbb{N}$ be an estimation of the $X$-metric. Then,*

$$dc_X(G) > 0 \iff dc_f(G) > 0.$$

***Proof.*** *Clearly, $\mathbb{B}_f(n) \subseteq \mathbb{B}_X(Kn) \subseteq \mathbb{B}_f(K^2 n)$ so,*

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So, $dc_X(G)=0 \quad\Rightarrow\quad dc_f(G)=0$, because

$$\frac{|\mathbb{B}_f(n)|}{|\mathbb{B}_X(Kn)|} \geqslant \frac{|\mathbb{B}_X(n/K)|}{|\mathbb{B}_X(Kn)|} \geqslant \frac{C(n/K)^d}{D(Kn)^d} = \frac{C}{DK^{2d}} > 0. \quad \square$$

**Corollary**

*If $G=\langle X\rangle=\langle Y\rangle$ is of polynomial growth, then*

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*Let $\langle Y\rangle = H \leqslant G = \langle X\rangle$. The subgroup $H$ is undistorted if $\exists\ K > 0$ s.t. $\forall\, h \in H,\ \ |h|_Y/K \leqslant |h|_X \leqslant K|h|_Y$.*
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## Corollary

*Let $G = \langle X\rangle$ be of polynomial growth, and $\langle Y\rangle = H \leqslant G$ be a non-distorted subgroup. Then,*

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**Lemma (Burillo–Ventura, 2002)**

*If $H \leqslant_{f.i.} G = \langle X \rangle$ and $G$ has subexponential growth then, for every $g \in G$, there exists $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n) \cap gH|}{|\mathbb{B}_X(n)|} = \lim_{n\to\infty} \frac{|\mathbb{B}_X(n) \cap Hg|}{|\mathbb{B}_X(n)|} = \frac{1}{[G:H]}$.*

**Remark**

*This is false in the free group: $H = \{\text{even words}\} \leqslant_2 F_r$.*

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*Let $\langle Y \rangle = H \leqslant_{f.i.} G = \langle X \rangle$ be of polynomial growth. Then,*

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*In particular, $dc_Y(H) > 0 \Rightarrow dc_X(H) > 0 \Rightarrow dc_X(G) > 0$.*

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***Proof.*** *Clearly,*

$|\{(u,v) \in (\mathbb{B}_X(n))^2 \mid uv = vu\}| \geqslant |\{(u,v) \in (H \cap \mathbb{B}_X(n))^2 \mid uv = vu\}|$.

*Therefore, given $\varepsilon > 0$, we have for $n \gg 0$*

$$\frac{|\{(u,v) \in (\mathbb{B}_X(n))^2 \mid uv = vu\}|}{|\mathbb{B}_X(n)|^2} \geqslant$$

$$\frac{|\{(u,v) \in (H \cap \mathbb{B}_X(n))^2 \mid uv = vu\}|}{|H \cap \mathbb{B}_X(n)|^2} \cdot \frac{|H \cap \mathbb{B}_X(n)|^2}{|\mathbb{B}_X(n)|^2} \geqslant$$

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*Taking limsups, $dc_X(G) \geqslant dc_X(H) \left(\frac{1}{[G:H]} - \varepsilon\right)^2$. And this is true $\forall \varepsilon > 0$ so, $dc_X(G) \geqslant \frac{1}{[G:H]^2} dc_X(H)$.* □

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**Proposition* (Gallagher, 1970)**

*Let $G$ be a finite group and $H \trianglelefteq G$. Then, $dc(G) \leqslant dc(H) \cdot dc(G/H)$.*

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*Let $G = \langle X \rangle$ be subexponentially growing. Then, for any finite quotient $G/N$, we have $dc_X(G) \leqslant dc(G/N)$.*

***Proof.*** *Let $N \trianglelefteq G$ with $[G : N] = d$.*
*By B–V, $\forall g \in G$ $\lim_{n\to\infty} |gN \cap \mathbb{B}_X(n)|/|\mathbb{B}_X(n)| = 1/d$, indep. $X$ and $g$.*
*But $|G/N| < \infty$, so this lim is uniform on $g$, i.e.,*
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$\exists\delta > 0$ *s.t.* $|\{(u,v)\in(\mathbb{B}_X(n))^2 \mid uv = vu\}|/|\mathbb{B}_X(n)|^2 > dc(G/N) + \delta$ *for infinitely many $n$'s.*

*In the above inequality, take $\varepsilon > 0$ small enough so that $2\varepsilon d + \varepsilon^2 d^2 < \delta$, and $\exists n \gg 0$ such that*

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# Outline

# The main result

**Theorem**

*Let $G = \langle X \rangle$ be of subexponential growth and residually finite. Then,*

*(i) $dc_X(G) > 5/8 \Leftrightarrow G$ is abelian;*

*(ii) $dc_X(G) > 0 \Leftrightarrow G$ is virtually abelian.*

*In particular, (i) and (ii) is true for polynomially growing groups.*

**Corollary**

*Let $G = \langle X \rangle = \langle Y \rangle$ be of subexponential growth and residually finite. Then,*

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***Proof.*** *(i). Suppose $dc_X(G) > 5/8$. Then, $dc(G/N) > 5/8$ for every $N \trianglelefteq_{f.i.} G$. Hence, by Gustafson's thm, every finite quotient of $G$ is abelian. Residual finiteness implies $G$ abelian.*

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# Proof of the main result

**Theorem**

*Let $G = \langle X \rangle$ be of subexponential growth and residually finite. Then,*
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***Obs.*** *If $H$ is f.g., r.f., not virtually abelian then $\exists K \trianglelefteq_{\substack{ch.,\\ f.i.}} H$ such that $H/K$ is (finite) not abelian.*

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*Then $\forall i, \quad K_i \trianglelefteq G, \quad (G/K_i)/(K_{i-1}/K_i) = G/K_{i-1}$ and, by Gallagher,*

$$dc(G/K_i) \leqslant dc(K_{i-1}/K_i) \cdot dc(G/K_{i-1}) \leqslant 5/8 \cdot dc(G/K_{i-1}).$$

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*By induction, $dc(G/K_i) \leqslant (5/8)^i$ and so,*

$$dc_X(G) \leqslant dc(G/K_i) \leqslant (5/8)^i,$$

*for every $i$. Therefore, $dc_X(G) = 0$.* $\square$

# Outline

# Generalizations

- *We can replace $xy = yx$ by any system of equations.*
- *We can replace the uniform measures on balls to any sequence of measures (random walks, etc).*

**Definition**

*Let $\{X_1, \ldots, X_k\}$ be a set of abstract variables and $\mathcal{F}$ the free group on it. Think elements $w \in \mathcal{F}$ as equations, $w = 1$, and subsets $\mathcal{E} \subseteq \mathcal{F}$ as systems of equations. Define solutions on a group $G$ in the obvious way.*

**Definition**

*Given $G = \langle X \rangle$ and a system of equations $\mathcal{E} \subseteq \mathcal{F}$, we define the degree of satisfiability of $\mathcal{E}$ in $G$ w.r.t $X$ as*

$$ds_X(G, \mathcal{E}) = \limsup_{n\to\infty} \frac{|\{(g_1, \ldots, g_k) \in (\mathbb{B}_X(n))^k \mid (g_1, \ldots, g_k) \text{ sol. } \mathcal{E}\}|}{|\mathbb{B}_X(n)|^k} \in [0, 1].$$

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# Generalizations

## Definition

*Let G and $\mathcal{E}$ be as before. Fix a collection of measures $\mu_n$ in G. We define the degree of satisfiability of $\mathcal{E}$ in G w.r.t. $\mu_n$ as*

$$ds_X(G, \mathcal{E}, \{\mu_n\}_n) =$$

$$\limsup_{n\to\infty} \mu_n^{\times k}(\{(g_1, \ldots, g_k) \in G^k \mid (g_1, \ldots, g_k) \text{ sol. } \mathcal{E}\}) \in [0, 1].$$

# Generalizations

## Meta-conjecture

*Let G, $\mathcal{E}$, and $\{\mu_n\}_n$ be as above, with $\mu_n$ "reasonable". Then,*

$$ds(G, \mathcal{E}, \{\mu_n\}_n) > 0 \iff \mathcal{E} \text{ is a virtual law in } G.$$

## Definition

*$\mathcal{E}$ is a law in G if every $(g_1, \ldots, g_k) \in G^k$ is a solution of $\mathcal{E}$ in G.*

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# Outline

# Degree of $r$-nilpotency

*Let us consider the $r$-equation:* $[[\ [[x_0, x_1], x_2] \cdots ], x_r]$.

*Notation:* $\mathbf{u} = (u_0, \dots, u_r)$, $[\mathbf{u}] = [u_0, \dots, u_r] = [[\ [[u_0, u_1], u_2] \cdots ], u_r]$.

**Definition**

*For a finite group $G$, the degree of $r$-nilpotency is*

$$dn_r(G) = \frac{|\{\mathbf{u} \in G^{r+1} \mid [[\ [[u_0, u_1], u_2] \cdots ], u_r] = 1\}|}{|G|^{r+1}}.$$

**Observation**

- *If $dn_r(G) = 1$ then $dn_{r+1}(G) = 1$.*
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# The infinite setting

## Definition

*Let $G = \langle X \rangle$ be f.g. The degree of $r$-nilpotency of $G$ w.r.t. $X$ is*

$$dn_{r,X}(G) = \limsup_{n\to\infty} \frac{|\{\mathbf{u} \in \mathbb{B}_X(n)^{r+1} \mid [[\,[[u_0, u_1], u_2] \cdots], u_r] = 1\}|}{|\mathbb{B}_X(n)|^{r+1}},$$

*where $\mathbb{B}_X(n) = \{g \in G \mid |g|_X \leqslant n\}$.*

## Question

- *Is this a real limit ?*
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# The gap for finite groups

## Proposition

*Let $G$ be a finite group. Then, for $r \geqslant 1$,*

$$dn_r(G) \leqslant dn_{r+1}(G) \leqslant \frac{1 + dn_r(G/Z(G))}{2}.$$

## Proof.

$$\begin{aligned}
dn_{r+1}(G) &= \frac{|\{\mathbf{u} \in G^{r+2} \mid [u_0, \ldots, u_{r+1}] = 1\}|}{|G|^{r+2}} \\
&= \frac{1}{|G|^{r+2}} \sum_{\mathbf{u} \in G^{r+1}} |C([u_0, \ldots, u_r])| \\
&\leqslant \frac{1}{|G|^{r+2}} \Bigg( \sum_{\substack{\mathbf{u} \in G^{r+1} \\ [u_0, \ldots, u_r] \in Z(G)}} |G| + \sum_{\substack{\mathbf{u} \in G^{r+1} \\ [u_0, \ldots, u_r] \notin Z(G)}} \frac{|G|}{2} \Bigg)
\end{aligned}$$

# The gap for finite groups

**Proof.**

$$= \frac{|Z(G)|^{r+1}}{|G|^{r+2}} \Big( \sum_{\substack{\mathbf{u} \in (G/Z(G))^{r+1} \\ [\overline{u_0}, \ldots, \overline{u_r}] = 1}} |G| + \sum_{\substack{\mathbf{u} \in (G/Z(G))^{r+1} \\ [\overline{u_0}, \ldots, \overline{u_r}] \neq 1}} \frac{|G|}{2} \Big)$$

$$= \frac{|Z(G)|^{r+1}}{|G|^{r+1}} \left( \frac{|G|^{r+1}}{|Z(G)|^{r+1}} dn_r(G/Z(G)) + \frac{|G|^{r+1}}{|Z(G)|^{r+1}} \frac{1 - dn_r(G/Z(G))}{2} \right)$$

$$= \frac{1 + dn_r(G/Z(G))}{2}.$$

□

**Corollary**

*For $r \geqslant 1$, any finite group $G$, if $dn_r(G) < 1$ then $dn_r(G) \leqslant 1 - \frac{3}{2^{r+2}}$.*

# The gap for finite groups

**Proof.**

$$= \frac{|Z(G)|^{r+1}}{|G|^{r+2}} \left( \sum_{\substack{\mathbf{u} \in (G/Z(G))^{r+1} \\ [\overline{u_0}, \ldots, \overline{u_r}] = 1}} |G| + \sum_{\substack{\mathbf{u} \in (G/Z(G))^{r+1} \\ [\overline{u_0}, \ldots, \overline{u_r}] \neq 1}} \frac{|G|}{2} \right)$$

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**Proof.**

By induction on $r$. For $r = 1$ is ok.

Assume it true for $r$, and suppose $dn_{r+1}(G) < 1$. Then, $dn_r(G/Z(G)) < 1$ and, by induction, $dn_r(G/Z(G)) \leqslant 1 - \frac{3}{2^{r+2}}$. So,

$$dn_{r+1}(G) \leqslant \frac{1 + dn_r(G/Z(G))}{2} \leqslant \frac{1 + 1 - \frac{3}{2^{r+2}}}{2} = 1 - \frac{3}{2^{r+3}}.$$

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Assume it true for $r$, and suppose $dn_{r+1}(G) < 1$. Then, $dn_r(G/Z(G)) < 1$ and, by induction, $dn_r(G/Z(G)) \leqslant 1 - \frac{3}{2^{r+2}}$. So,

$$dn_{r+1}(G) \leqslant \frac{1 + dn_r(G/Z(G))}{2} \leqslant \frac{1 + 1 - \frac{3}{2^{r+2}}}{2} = 1 - \frac{3}{2^{r+3}}.$$

□

# The gap for finite groups

## Corollary

*For $r \geqslant 1$, any finite group $G$, if $dn_r(G) < 1$ then $dn_r(G) \leqslant 1 - \frac{3}{2^{r+2}}$.*

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# The meta-conjecture for $r$-nilpotence

**Theorem**

*Let $G = \langle X \rangle$ be of subexponential growth and residually-p for infinitely many primes. Then,*

*(i) $dn_{r,X}(G) > 1 - \frac{3}{2^{r+2}} \Leftrightarrow G$ is $r$-nilpotent;*

*(ii) $dn_{r,X}(G) > 0 \Leftrightarrow G$ is virtually $r$-nilpotent.*

*The problem here is that we still don't have the analogous to Gallagher's result:*
*If $G$ is finite and $H \trianglelefteq G$, then is it true that*

$$dn_r(G) \leqslant dn_r(H) \cdot dn_r(G/H) \quad ?$$

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# Connection to character theory

**Corollary**

*Let $G$ be a finite group. Then,*

(i) $dn_1(G) = dc(G) = \frac{k(G)}{|G|}$;

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THANKS