Computing endo-fixed closures in free groups

Enric Ventura

Departament de Matemàtica Aplicada III

Universitat Politècnica de Catalunya

Southampton, February 27, 2009

Outline

Outline

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{Mono}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\phi\colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \mathrm{End}\,(F_n)$ then
  $\mathrm{Fix}\,(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \cap_{\phi \in S} \mathrm{Fix}\,(\phi) \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{Mono}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\phi \colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \mathrm{End}(F_n)$ then
  $\mathrm{Fix}(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \cap_{\phi \in S} \mathrm{Fix}(\phi) \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{Mono}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\phi \colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \mathrm{End}(F_n)$ then
  $\mathrm{Fix}(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \cap_{\phi \in S} \mathrm{Fix}(\phi) \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{Mono}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\phi \colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \mathrm{End}\,(F_n)$ then
  $\mathrm{Fix}\,(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \cap_{\phi \in S} \mathrm{Fix}\,(\phi) \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{Mono}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\phi \colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \mathrm{End}(F_n)$ then $\mathrm{Fix}(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \cap_{\phi \in S} \mathrm{Fix}(\phi) \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{Mono}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\phi\colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \mathrm{End}\,(F_n)$ then $\mathrm{Fix}\,(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \cap_{\phi \in S} \mathrm{Fix}\,(\phi) \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{Mono}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\phi \colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \mathrm{End}(F_n)$ then
  $\mathrm{Fix}(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \cap_{\phi \in S} \mathrm{Fix}(\phi) \leqslant F_n$.

Fixed subgroups are complicated

$$\begin{aligned} \phi\colon F_3 &\to F_3 \\ a &\mapsto a \\ b &\mapsto ba \\ c &\mapsto ca^2 \end{aligned}$$

$$\mathrm{Fix}\,\phi = \langle a, bab^{-1}, cac^{-1} \rangle$$

$$\begin{aligned} \varphi\colon F_4 &\to F_4 \\ a &\mapsto dac \\ b &\mapsto c^{-1}a^{-1}d^{-1}ac \\ c &\mapsto c^{-1}a^{-1}b^{-1}ac \\ d &\mapsto c^{-1}a^{-1}bc \end{aligned}$$

$\mathrm{Fix}\,\varphi = \langle w \rangle$, where...

$$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1} a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}.$$

Fixed subgroups are complicated

$$\begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & ba \\ c & \mapsto & ca^2 \end{array}$$

$\mathrm{Fix}\,\phi = \langle a, bab^{-1}, cac^{-1} \rangle$

$$\begin{array}{rcl} \varphi\colon F_4 & \to & F_4 \\ a & \mapsto & dac \\ b & \mapsto & c^{-1}a^{-1}d^{-1}ac \\ c & \mapsto & c^{-1}a^{-1}b^{-1}ac \\ d & \mapsto & c^{-1}a^{-1}bc \end{array}$$

$\mathrm{Fix}\,\varphi = \langle w \rangle$, where...

$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1}$
$a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}.$

Fixed subgroups are complicated

$$\begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & ba \\ c & \mapsto & ca^2 \end{array}$$

$\mathrm{Fix}\,\phi = \langle a, bab^{-1}, cac^{-1} \rangle$

$$\begin{array}{rcl} \varphi\colon F_4 & \to & F_4 \\ a & \mapsto & dac \\ b & \mapsto & c^{-1}a^{-1}d^{-1}ac \\ c & \mapsto & c^{-1}a^{-1}b^{-1}ac \\ d & \mapsto & c^{-1}a^{-1}bc \end{array}$$

$\mathrm{Fix}\,\varphi = \langle w \rangle$, where...

$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1}a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}.$

Fixed subgroups are complicated

$$\begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & ba \\ c & \mapsto & ca^2 \end{array}$$

$\mathrm{Fix}\,\phi = \langle a, bab^{-1}, cac^{-1} \rangle$

$$\begin{array}{rcl} \varphi\colon F_4 & \to & F_4 \\ a & \mapsto & dac \\ b & \mapsto & c^{-1}a^{-1}d^{-1}ac \\ c & \mapsto & c^{-1}a^{-1}b^{-1}ac \\ d & \mapsto & c^{-1}a^{-1}bc \end{array}$$

$\mathrm{Fix}\,\varphi = \langle w \rangle$, where...

$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1}$
$a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}.$

Fixed subgroups are complicated

$$\begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & ba \\ c & \mapsto & ca^2 \end{array}$$

$\mathrm{Fix}\,\phi = \langle a, bab^{-1}, cac^{-1} \rangle$

$$\begin{array}{rcl} \varphi\colon F_4 & \to & F_4 \\ a & \mapsto & dac \\ b & \mapsto & c^{-1}a^{-1}d^{-1}ac \\ c & \mapsto & c^{-1}a^{-1}b^{-1}ac \\ d & \mapsto & c^{-1}a^{-1}bc \end{array}$$

$\mathrm{Fix}\,\varphi = \langle w \rangle$, where...

$$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1} a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}.$$

What is known about fixed subgroups ?

Theorem (Dyer-Scott, 75)

Let $G \leqslant Aut(F_n)$ be a finite group of automorphisms of F_n. Then, $Fix(G) \leqslant_{\mathrm{ff}} F_n$; in particular, $r(Fix(G)) \leqslant n$.

Conjecture (Scott)

For every $\phi \in Aut(F_n)$, $r(Fix(\phi)) \leqslant n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) < \infty$.

Theorem (Thomas, 88)

Let $G \leqslant Aut(F_n)$ be an arbitrary group of automorphisms of F_n. Then, $r(Fix(G)) < \infty$.

What is known about fixed subgroups ?

Theorem (Dyer-Scott, 75)

Let $G \leqslant Aut(F_n)$ be a finite group of automorphisms of F_n. Then, $Fix(G) \leqslant_{\mathrm{ff}} F_n$; in particular, $r(Fix(G)) \leqslant n$.

Conjecture (Scott)

For every $\phi \in Aut(F_n)$, $r(Fix(\phi)) \leqslant n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) < \infty$.

Theorem (Thomas, 88)

Let $G \leqslant Aut(F_n)$ be an arbitrary group of automorphisms of F_n. Then, $r(Fix(G)) < \infty$.

What is known about fixed subgroups ?

Theorem (Dyer-Scott, 75)

Let $G \leqslant Aut(F_n)$ be a finite group of automorphisms of F_n. Then, $Fix(G) \leqslant_{\mathrm{ff}} F_n$; in particular, $r(Fix(G)) \leqslant n$.

Conjecture (Scott)

For every $\phi \in Aut(F_n)$, $r(Fix(\phi)) \leqslant n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) < \infty$.

Theorem (Thomas, 88)

Let $G \leqslant Aut(F_n)$ be an arbitrary group of automorphisms of F_n. Then, $r(Fix(G)) < \infty$.

What is known about fixed subgroups ?

Theorem (Dyer-Scott, 75)

Let $G \leqslant Aut(F_n)$ be a finite group of automorphisms of F_n. Then, $Fix(G) \leqslant_{\mathrm{ff}} F_n$; in particular, $r(Fix(G)) \leqslant n$.

Conjecture (Scott)

For every $\phi \in Aut(F_n)$, $r(Fix(\phi)) \leqslant n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) < \infty$.

Theorem (Thomas, 88)

Let $G \leqslant Aut(F_n)$ be an arbitrary group of automorphisms of F_n. Then, $r(Fix(G)) < \infty$.

Train-tracks

Main result in this story:

Theorem (Bestvina-Handel, 88 (published 92))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

introducing the theory of train-tracks for graphs.

After Bestvina-Handel, live continues ...

Theorem (Imrich-Turner, 89)

Let $\phi \in End(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

Theorem (Turner, 96)

Let $\phi \in End(F_n)$. If ϕ is not bijective then $r(Fix(\phi)) \leqslant n-1$.

Train-tracks

Main result in this story:

Theorem (Bestvina-Handel, 88 (published 92))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

introducing the theory of train-tracks for graphs.

After Bestvina-Handel, live continues ...

Theorem (Imrich-Turner, 89)

Let $\phi \in End(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

Theorem (Turner, 96)

Let $\phi \in End(F_n)$. If ϕ is not bijective then $r(Fix(\phi)) \leqslant n-1$.

Train-tracks

Main result in this story:

Theorem (Bestvina-Handel, 88 (published 92))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

introducing the theory of train-tracks for graphs.

After Bestvina-Handel, live continues ...

Theorem (Imrich-Turner, 89)

Let $\phi \in End(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

Theorem (Turner, 96)

Let $\phi \in End(F_n)$. If ϕ is not bijective then $r(Fix(\phi)) \leqslant n-1$.

Inertia

Definition

A subgroup $H \leqslant F_n$ is called ***inert*** *if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F_n$.*

Theorem (Dicks-V, 96)

Let $G \subseteq Mon(F_n)$ be an arbitrary set of monomorphisms of F_n. Then, $Fix(G)$ is inert; in particular, $r(Fix(G)) \leqslant n$.

Theorem (Bergman, 99)

Let $G \subseteq End(F_n)$ be an arbitrary set of endomorphisms of F_n. Then, $r(Fix(G)) \leqslant n$.

Conjecture (V.)

Let $\phi \in End(F_n)$. Then $Fix(\phi)$ is inert.

Inertia

Definition

A subgroup $H \leqslant F_n$ is called inert if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F_n$.

Theorem (Dicks-V, 96)

Let $G \subseteq Mon(F_n)$ be an arbitrary set of monomorphisms of F_n. Then, $Fix(G)$ is inert; in particular, $r(Fix(G)) \leqslant n$.

Theorem (Bergman, 99)

Let $G \subseteq End(F_n)$ be an arbitrary set of endomorphisms of F_n. Then, $r(Fix(G)) \leqslant n$.

Conjecture (V.)

Let $\phi \in End(F_n)$. Then $Fix(\phi)$ is inert.

Inertia

Definition

A subgroup $H \leqslant F_n$ is called inert if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F_n$.

Theorem (Dicks-V, 96)

Let $G \subseteq Mon(F_n)$ be an arbitrary set of monomorphisms of F_n. Then, $Fix(G)$ is inert; in particular, $r(Fix(G)) \leqslant n$.

Theorem (Bergman, 99)

Let $G \subseteq End(F_n)$ be an arbitrary set of endomorphisms of F_n. Then, $r(Fix(G)) \leqslant n$.

Conjecture (V.)

Let $\phi \in End(F_n)$. Then $Fix(\phi)$ is inert.

Inertia

Definition

A subgroup $H \leqslant F_n$ is called inert if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F_n$.

Theorem (Dicks-V, 96)

Let $G \subseteq Mon(F_n)$ be an arbitrary set of monomorphisms of F_n. Then, $Fix(G)$ is inert; in particular, $r(Fix(G)) \leqslant n$.

Theorem (Bergman, 99)

Let $G \subseteq End(F_n)$ be an arbitrary set of endomorphisms of F_n. Then, $r(Fix(G)) \leqslant n$.

Conjecture (V.)

Let $\phi \in End(F_n)$. Then $Fix(\phi)$ is inert.

The four families

Definition

A subgroup $H \leqslant F_n$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_n)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_n)$,*
- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
- *endo-fixed if $H = Fix(S)$ for some $S \subseteq End(F_n)$,*

Easy to see that 1-mono-fixed = 1-auto-fixed.

The four families

Definition

A subgroup $H \leqslant F_n$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_n)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_n)$,*
- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
- *endo-fixed if $H = Fix(S)$ for some $S \subseteq End(F_n)$,*

Easy to see that 1-mono-fixed = 1-auto-fixed.

The four families

Definition

A subgroup $H \leqslant F_n$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_n)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_n)$,*
- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
- *endo-fixed if $H = Fix(S)$ for some $S \subseteq End(F_n)$,*

Easy to see that 1-mono-fixed = 1-auto-fixed.

The four families

Definition

A subgroup $H \leqslant F_n$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_n)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_n)$,*
- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
- *endo-fixed if $H = Fix(S)$ for some $S \subseteq End(F_n)$,*

Easy to see that 1-mono-fixed = 1-auto-fixed.

The four families

Definition

A subgroup $H \leqslant F_n$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_n)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_n)$,*
- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
- *endo-fixed if $H = Fix(S)$ for some $S \subseteq End(F_n)$,*

Easy to see that 1-mono-fixed = 1-auto-fixed.

Relations between them

$$\begin{array}{ccc} \boxed{1-auto-fixed} & \subseteq & \boxed{1-endo-fixed} \\ \cap| & & \cap| \\ \boxed{auto-fixed} & \subseteq & \boxed{endo-fixed} \end{array}$$

Relations between them

$$\begin{array}{ccc} \boxed{1-auto-fixed} & \subsetneqq & \boxed{1-endo-fixed} \\ \cap| & & \cap| \\ \boxed{auto-fixed} & \subsetneqq & \boxed{endo-fixed} \end{array}$$

Example (Martino-V., 03; Ciobanu-Dicks, 06)

Let $F_3 = \langle a, b, c \rangle$ *and* $H = \langle b, cacbab^{-1}c^{-1} \rangle \leqslant F_3$. *Then,* $H = Fix(a \mapsto 1,\ b \mapsto b,\ c \mapsto cacbab^{-1}c^{-1})$, *but* H *is NOT the fixed subgroup of any set of automorphism of* F_3.

Relations between them

$$\boxed{1-auto-fixed} \quad \overset{\subseteq}{\neq} \quad \boxed{1-endo-fixed}$$

$$\cap| \quad \|? \qquad\qquad \cap| \quad \|?$$

$$\boxed{auto-fixed} \quad \overset{\subseteq}{\neq} \quad \boxed{endo-fixed}$$

Theorem (Martino-V., 00)

Let $S \subseteq End(F_n)$. Then, $\exists \phi \in \langle S \rangle$ such that $Fix(S) \leqslant_{ff} Fix(\phi)$.

But... free factors of 1-endo-fixed (1-auto-fixed) subgroups need not be even endo-fixed (auto-fixed).

Outline

Computing fixed subgroups

Proposition (Turner, 86)

There exists a pseudo-algorithm to compute fix of an endo.

Easy but is not an algorithm...

Theorem (Maslakova, 03)

Fixed subgroups of automorphisms of F_n are computable.

Difficult but it is an algorithm!

Conjecture

Fixed subgroups of endomorphisms of F_n are computable.

Computing fixed subgroups

Proposition (Turner, 86)

There exists a pseudo-algorithm to compute fix of an endo.

Easy but is not an algorithm...

Theorem (Maslakova, 03)

Fixed subgroups of automorphisms of F_n are computable.

Difficult but it is an algorithm!

Conjecture

Fixed subgroups of endomorphisms of F_n are computable.

Computing fixed subgroups

Proposition (Turner, 86)

There exists a pseudo-algorithm to compute fix of an endo.

Easy but is not an algorithm...

Theorem (Maslakova, 03)

Fixed subgroups of automorphisms of F_n are computable.

Difficult but it is an algorithm!

Conjecture

Fixed subgroups of endomorphisms of F_n are computable.

Deciding fixedness

In this talk, I'll solve the two dual problems:

Theorem

Given $H \leqslant_{\mathrm{fg}} F_n$, one can algorithmically decide whether

i) *H is auto-fixed or not,*

ii) *H is endo-fixed or not,*

and in the affirmative case, find a finite family, $S = \{\phi_1, \ldots, \phi_m\}$, of automorphisms (endomorphisms) of F_n such that $\mathrm{Fix}(S) = H$.

Conjecture

Given $H \leqslant_{\mathrm{fg}} F_n$, one can algorithmically decide whether

i) *H is 1-auto-fixed or not,*

ii) *H is 1-endo-fixed or not,*

and in the affirmative case, find one automorphism (endomorphism) ϕ of F_n such that $\mathrm{Fix}(\phi) = H$.

Deciding fixedness

In this talk, I'll solve the two dual problems:

Theorem

Given $H \leqslant_{\mathrm{fg}} F_n$, one can algorithmically decide whether

i) *H is auto-fixed or not,*

ii) *H is endo-fixed or not,*

and in the affirmative case, find a finite family, $S = \{\phi_1, \ldots, \phi_m\}$, of automorphisms (endomorphisms) of F_n such that $Fix(S) = H$.

Conjecture

Given $H \leqslant_{\mathrm{fg}} F_n$, one can algorithmically decide whether

i) *H is 1-auto-fixed or not,*

ii) *H is 1-endo-fixed or not,*

and in the affirmative case, find one automorphism (endomorphism) ϕ of F_n such that $Fix(\phi) = H$.

Deciding fixedness

In this talk, I'll solve the two dual problems:

Theorem

Given $H \leqslant_{\mathrm{fg}} F_n$, one can algorithmically decide whether

i) *H is auto-fixed or not,*

ii) *H is endo-fixed or not,*

and in the affirmative case, find a finite family, $S = \{\phi_1, \ldots, \phi_m\}$, of automorphisms (endomorphisms) of F_n such that $\mathrm{Fix}(S) = H$.

Conjecture

Given $H \leqslant_{\mathrm{fg}} F_n$, one can algorithmically decide whether

i) *H is 1-auto-fixed or not,*

ii) *H is 1-endo-fixed or not,*

and in the affirmative case, find one automorphism (endomorphism) ϕ of F_n such that $\mathrm{Fix}(\phi) = H$.

Outline

Fixed closures

Definition

Given $H \leqslant_{\mathrm{fg}} F_n$, we define the (auto- and endo-) stabilizer of H, respectively, as

$$Aut_H(F_n) = \{\phi \in Aut(F_n) \mid H \leqslant Fix(\phi)\} \leqslant Aut(F_n)$$

and

$$End_H(F_n) = \{\phi \in End(F_n) \mid H \leqslant Fix(\phi)\} \leqslant End(F_n)$$

Definition

Given $H \leqslant F_n$, we define the auto-closure and endo-closure of H as

$$a\text{-}Cl(H) = Fix(Aut_H(F_n)) \geqslant H$$

and

$$e\text{-}Cl(H) = Fix(End_H(F_n)) \geqslant H$$

Fixed closures

Definition

Given $H \leqslant_{\mathrm{fg}} F_n$, we define the (auto- and endo-) stabilizer of H, respectively, as

$$Aut_H(F_n) = \{\phi \in Aut(F_n) \mid H \leqslant Fix(\phi)\} \leqslant Aut(F_n)$$

and

$$End_H(F_n) = \{\phi \in End(F_n) \mid H \leqslant Fix(\phi)\} \leqslant End(F_n)$$

Definition

Given $H \leqslant F_n$, we define the auto-closure and endo-closure of H as

$$a\text{-}Cl(H) = Fix(Aut_H(F_n)) \geqslant H$$

and

$$e\text{-}Cl(H) = Fix(End_H(F_n)) \geqslant H$$

Main result

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, a-$Cl(H)$ and e-$Cl(H)$ are finitely generated and one can algorithmically compute bases for them.

Corollary

Auto-fixedness and endo-fixedness are decidable.

Observe that $e\text{-}Cl(H) \leqslant a\text{-}Cl(H)$ but, in general, they are not equal.

Main result

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, a-$Cl(H)$ and e-$Cl(H)$ are finitely generated and one can algorithmically compute bases for them.

Corollary

Auto-fixedness and endo-fixedness are decidable.

Observe that e-$Cl(H) \leqslant$ a-$Cl(H)$ but, in general, they are not equal.

Retracts

Definition

A subgroup $H \leqslant F_n$ is a retract if there exists a retraction, i.e. a morphism $\rho\colon F_n \to H$ which restricts to the identity of H.

Free factors are retracts, but there are more.

Observation

If $H \leqslant F_n$ is a retract then $r(H) \leqslant n$ (and, $r(H) = n \Leftrightarrow H = F_n$).

Observation (Turner)

It is algorithmically decidable whether a given $H \leqslant F_n$ is a retract or not.

Retracts

Definition

A subgroup $H \leqslant F_n$ is a retract if there exists a retraction, i.e. a morphism $\rho\colon F_n \to H$ which restricts to the identity of H.

Free factors are retracts, but there are more.

Observation

If $H \leqslant F_n$ is a retract then $r(H) \leqslant n$ (and, $r(H) = n \Leftrightarrow H = F_n$).

Observation (Turner)

It is algorithmically decidable whether a given $H \leqslant F_n$ is a retract or not.

Retracts

Definition

A subgroup $H \leqslant F_n$ is a retract if there exists a retraction, i.e. a morphism $\rho\colon F_n \to H$ which restricts to the identity of H.

Free factors are retracts, but there are more.

Observation

If $H \leqslant F_n$ is a retract then $r(H) \leqslant n$ (and, $r(H) = n \Leftrightarrow H = F_n$).

Observation (Turner)

It is algorithmically decidable whether a given $H \leqslant F_n$ is a retract or not.

The stable image

Definition

Let $\phi \in End(F_n)$. The stable image of ϕ is $F_n\phi^\infty = \cap_{i=1}^{\infty} F_n\phi^i$.

Theorem (Imrich-Turner, 89)

For every endomorphism $\phi\colon F_n \to F_n$,

i) *$F_n\phi^\infty$ is ϕ-invariant,*
ii) *the restriction $\phi\colon F_n\phi^\infty \to F_n\phi^\infty$ is an isomorphism,*
iii) *$F_n\phi^\infty$ is a retract.*

Example: For $\phi\colon F_2 \to F_2$, $a \mapsto a$, $b \mapsto b^2$, we have $F_2\phi = \langle a, b^2\rangle$, $F_2\phi^2 = \langle a, b^4\rangle$, $F_2\phi^3 = \langle a, b^8\rangle, \ldots$. So, $F_2\phi^\infty = \langle a\rangle \leqslant_{\mathrm{ff}} F_2$.

The stable image

Definition

Let $\phi \in End(F_n)$. The ***stable image*** *of ϕ is $F_n\phi^\infty = \cap_{i=1}^{\infty} F_n\phi^i$.*

Theorem (Imrich-Turner, 89)

For every endomorphism $\phi \colon F_n \to F_n$,

i) *$F_n\phi^\infty$ is ϕ-invariant,*
ii) *the restriction $\phi \colon F_n\phi^\infty \to F_n\phi^\infty$ is an isomorphism,*
iii) *$F_n\phi^\infty$ is a retract.*

Example: For $\phi \colon F_2 \to F_2$, $a \mapsto a$, $b \mapsto b^2$, we have $F_2\phi = \langle a, b^2\rangle$, $F_2\phi^2 = \langle a, b^4\rangle$, $F_2\phi^3 = \langle a, b^8\rangle$, ... So, $F_2\phi^\infty = \langle a\rangle \leqslant_{\mathrm{ff}} F_2$.

The stable image

Definition

Let $\phi \in End(F_n)$. The stable image of ϕ is $F_n\phi^\infty = \cap_{i=1}^{\infty} F_n\phi^i$.

Theorem (Imrich-Turner, 89)

For every endomorphism $\phi \colon F_n \to F_n$,

i) *$F_n\phi^\infty$ is ϕ-invariant,*
ii) *the restriction $\phi \colon F_n\phi^\infty \to F_n\phi^\infty$ is an isomorphism,*
iii) *$F_n\phi^\infty$ is a retract.*

Example: For $\phi \colon F_2 \to F_2$, $a \mapsto a$, $b \mapsto b^2$, we have $F_2\phi = \langle a, b^2 \rangle$, $F_2\phi^2 = \langle a, b^4 \rangle$, $F_2\phi^3 = \langle a, b^8 \rangle$, ... So, $F_2\phi^\infty = \langle a \rangle \leqslant_{\text{ff}} F_2$.

Stallings' graphs and intersections

Theorem (Stallings, 83)

For any free group $F_n = F(A)$, there is an effectively computable bijection

$$\{\textit{f.g. subgroups of } F_n\} \longleftrightarrow \{\textit{finite A-labeled core graphs}\}$$

Theorem

Given sets of generators for $H, K \leqslant_{\mathrm{fg}} F_n$, one can algorithmically compute a basis for $H \cap K$.

Stallings' graphs and intersections

Theorem (Stallings, 83)

For any free group $F_n = F(A)$, there is an effectively computable bijection

$$\{\text{f.g. subgroups of } F_n\} \longleftrightarrow \{\text{finite } A\text{-labeled core graphs}\}$$

Theorem

Given sets of generators for $H, K \leqslant_{\mathrm{fg}} F_n$, one can algorithmically compute a basis for $H \cap K$.

Algebraic extensions

Definition

An extension of subgroups $H \leqslant K \leqslant F_n$ is called algebraic, denoted $H \leqslant_{\mathrm{alg}} K$, if H is not contained in any proper free factor of K. Write

$$\mathcal{AE}(H) = \{K \leqslant F_n \mid H \leqslant_{\mathrm{alg}} K\}.$$

Theorem (Takahasi, 51; V., 97; Margolis-Sapir-Weil, 01; Kapovich-Miasnikov, 02)

If $H \leqslant_{\mathrm{fg}} F_n$ then $\mathcal{AE}(H)$ is finite and computable (i.e. H has finitely many algebraic extensions, all of them are finitely generated, and bases are computable from H).

Theorem (Kapovich-Miasnikov, 02)

Every extension $H \leqslant K$ of f.g. subgroups of F splits in a unique way as $H \leqslant_{alg} L \leqslant_{ff} K$ (L is called the K-algebraic closure of H).

Algebraic extensions

Definition

An extension of subgroups $H \leqslant K \leqslant F_n$ is called algebraic, denoted $H \leqslant_{\mathrm{alg}} K$, if H is not contained in any proper free factor of K. Write

$$\mathcal{AE}(H) = \{K \leqslant F_n \mid H \leqslant_{\mathrm{alg}} K\}.$$

Theorem (Takahasi, 51; V., 97; Margolis-Sapir-Weil, 01; Kapovich-Miasnikov, 02)

If $H \leqslant_{\mathrm{fg}} F_n$ then $\mathcal{AE}(H)$ is finite and computable (i.e. H has finitely many algebraic extensions, all of them are finitely generated, and bases are computable from H).

Theorem (Kapovich-Miasnikov, 02)

Every extension $H \leqslant K$ of f.g. subgroups of F splits in a unique way as $H \leqslant_{alg} L \leqslant_{ff} K$ (L is called the K-algebraic closure of H).

Algebraic extensions

Definition

An extension of subgroups $H \leqslant K \leqslant F_n$ is called algebraic, denoted $H \leqslant_{\mathrm{alg}} K$, if H is not contained in any proper free factor of K. Write

$$\mathcal{AE}(H) = \{K \leqslant F_n \mid H \leqslant_{\mathrm{alg}} K\}.$$

Theorem (Takahasi, 51; V., 97; Margolis-Sapir-Weil, 01; Kapovich-Miasnikov, 02)

If $H \leqslant_{\mathrm{fg}} F_n$ then $\mathcal{AE}(H)$ is finite and computable (i.e. H has finitely many algebraic extensions, all of them are finitely generated, and bases are computable from H).

Theorem (Kapovich-Miasnikov, 02)

Every extension $H \leqslant K$ of f.g. subgroups of F splits in a unique way as $H \leqslant_{alg} L \leqslant_{ff} K$ (L is called the K-algebraic closure of H).

Outline

The automorphism case

Theorem (McCool, 70's)

Let $H \leqslant_{\mathrm{fg}} F_n$. Then $Aut_H(F_n)$ is finitely generated (in fact, finitely presented) and a finite set of generators (and relations) is algorithmically computable from H.

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, a-$Cl(H)$ is finitely generated and algorithmically computable.

Proof.

$$\begin{aligned} a\text{-}Cl(H) &= \mathrm{Fix}\,(\mathrm{Aut}_H(F_n)) \\ &= \mathrm{Fix}\,(\langle \phi_1, \ldots, \phi_m \rangle) \\ &= \mathrm{Fix}\,(\phi_1) \cap \cdots \cap \mathrm{Fix}\,(\phi_m). \quad \square \end{aligned}$$

The automorphism case

Theorem (McCool, 70's)

Let $H \leqslant_{\mathrm{fg}} F_n$. Then $\mathrm{Aut}_H(F_n)$ is finitely generated (in fact, finitely presented) and a finite set of generators (and relations) is algorithmically computable from H.

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, a-$Cl(H)$ is finitely generated and algorithmically computable.

Proof.

$$\begin{aligned} a\text{-}Cl(H) &= \mathrm{Fix}\,(\mathrm{Aut}_H(F_n)) \\ &= \mathrm{Fix}\,(\langle \phi_1, \ldots, \phi_m \rangle) \\ &= \mathrm{Fix}\,(\phi_1) \cap \cdots \cap \mathrm{Fix}\,(\phi_m). \quad \square \end{aligned}$$

The automorphism case

Theorem (McCool, 70's)

Let $H \leqslant_{\mathrm{fg}} F_n$. Then $Aut_H(F_n)$ is finitely generated (in fact, finitely presented) and a finite set of generators (and relations) is algorithmically computable from H.

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, a-$Cl(H)$ is finitely generated and algorithmically computable.

Proof.

$$\begin{aligned} a\text{-}Cl(H) &= \mathrm{Fix}\,(\mathrm{Aut}_H(F_n)) \\ &= \mathrm{Fix}\,(\langle \phi_1, \ldots, \phi_m \rangle) \\ &= \mathrm{Fix}\,(\phi_1) \cap \cdots \cap \mathrm{Fix}\,(\phi_m). \quad \square \end{aligned}$$

The endomorphism case

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, e-$Cl(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

The endomorphism case

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, e-$Cl(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

The endomorphism case

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, e-$Cl(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

The endomorphism case

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, e-$Cl(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

The endomorphism case

Theorem

For every $H \leqslant_{\text{fg}} F_n$, e-$Cl(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

The endomorphism case

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, e-$Cl(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

The endomorphism case

Theorem

For every $H \leqslant_{\text{fg}} F_n$, $e\text{-}Cl(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

The endomorphism case

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

Proof. Let us see that

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathrm{Aut}(H_i) \\ H \leqslant \mathrm{Fix}(\alpha)}} \mathrm{Fix}(\alpha) = \bigcap_{\substack{\beta \in \mathrm{End}(F_n) \\ H \leqslant \mathrm{Fix}(\beta)}} \mathrm{Fix}(\beta).$$

- Take $\beta \in \mathrm{End}(F_n)$ with $H \leqslant \mathrm{Fix}(\beta)$.
- $\exists i = 1, \ldots, r$ such that $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^{\infty} \leqslant_{ret} F$ (so, $H_i \leqslant_{ret} F$).
- Now, β restricts to an automorphism $\alpha \colon H_i \to H_i$.
- And, clearly, $H \leqslant \mathrm{Fix}(\alpha) \leqslant \mathrm{Fix}(\beta)$.
- Hence, we have "$\leqslant$".

The endomorphism case

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H).$

Proof. Let us see that

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathrm{Aut}(H_i) \\ H \leqslant \mathrm{Fix}(\alpha)}} \mathrm{Fix}(\alpha) = \bigcap_{\substack{\beta \in \mathrm{End}(F_n) \\ H \leqslant \mathrm{Fix}(\beta)}} \mathrm{Fix}(\beta).$$

- Take $\beta \in \mathrm{End}(F_n)$ with $H \leqslant \mathrm{Fix}(\beta)$.
- $\exists i = 1, \ldots, r$ such that $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^{\infty} \leqslant_{ret} F$ (so, $H_i \leqslant_{ret} F$).
- Now, β restricts to an automorphism $\alpha \colon H_i \to H_i$.
- And, clearly, $H \leqslant \mathrm{Fix}(\alpha) \leqslant \mathrm{Fix}(\beta)$.
- Hence, we have "$\leqslant$".

The endomorphism case

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H)$.

Proof. Let us see that

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathrm{Aut}(H_i) \\ H \leqslant \mathrm{Fix}(\alpha)}} \mathrm{Fix}(\alpha) = \bigcap_{\substack{\beta \in \mathrm{End}(F_n) \\ H \leqslant \mathrm{Fix}(\beta)}} \mathrm{Fix}(\beta).$$

- Take $\beta \in \mathrm{End}(F_n)$ with $H \leqslant \mathrm{Fix}(\beta)$.
- $\exists i = 1, \ldots, r$ such that $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^{\infty} \leqslant_{ret} F$ (so, $H_i \leqslant_{ret} F$).
- Now, β restricts to an automorphism $\alpha \colon H_i \to H_i$.
- And, clearly, $H \leqslant \mathrm{Fix}(\alpha) \leqslant \mathrm{Fix}(\beta)$.
- Hence, we have "$\leqslant$".

The endomorphism case

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H).$

Proof. Let us see that

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathrm{Aut}(H_i) \\ H \leqslant \mathrm{Fix}(\alpha)}} \mathrm{Fix}(\alpha) = \bigcap_{\substack{\beta \in \mathrm{End}(F_n) \\ H \leqslant \mathrm{Fix}(\beta)}} \mathrm{Fix}(\beta).$$

- Take $\beta \in \mathrm{End}(F_n)$ with $H \leqslant \mathrm{Fix}(\beta)$.
- $\exists i = 1, \ldots, r$ such that $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^{\infty} \leqslant_{ret} F$ (so, $H_i \leqslant_{ret} F$).
- Now, β restricts to an automorphism $\alpha \colon H_i \to H_i$.
- And, clearly, $H \leqslant \mathrm{Fix}(\alpha) \leqslant \mathrm{Fix}(\beta)$.
- Hence, we have "$\leqslant$".

The endomorphism case

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H).$

Proof. Let us see that

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathrm{Aut}(H_i) \\ H \leqslant \mathrm{Fix}(\alpha)}} \mathrm{Fix}(\alpha) = \bigcap_{\substack{\beta \in \mathrm{End}(F_n) \\ H \leqslant \mathrm{Fix}(\beta)}} \mathrm{Fix}(\beta).$$

- Take $\beta \in \mathrm{End}(F_n)$ with $H \leqslant \mathrm{Fix}(\beta)$.
- $\exists i = 1, \ldots, r$ such that $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^{\infty} \leqslant_{ret} F$ (so, $H_i \leqslant_{ret} F$).
- Now, β restricts to an automorphism $\alpha \colon H_i \to H_i$.
- And, clearly, $H \leqslant \mathrm{Fix}(\alpha) \leqslant \mathrm{Fix}(\beta)$.
- Hence, we have "$\leqslant$".

The endomorphism case

Claim

$a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H) = e\text{-}Cl(H).$

Proof. Let us see that

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathrm{Aut}(H_i) \\ H \leqslant \mathrm{Fix}(\alpha)}} \mathrm{Fix}(\alpha) = \bigcap_{\substack{\beta \in \mathrm{End}(F_n) \\ H \leqslant \mathrm{Fix}(\beta)}} \mathrm{Fix}(\beta).$$

- Take $\beta \in \mathrm{End}(F_n)$ with $H \leqslant \mathrm{Fix}(\beta)$.
- $\exists i = 1, \ldots, r$ such that $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^{\infty} \leqslant_{ret} F$ (so, $H_i \leqslant_{ret} F$).
- Now, β restricts to an automorphism $\alpha\colon H_i \to H_i$.
- And, clearly, $H \leqslant \mathrm{Fix}(\alpha) \leqslant \mathrm{Fix}(\beta)$.
- Hence, we have "$\leqslant$".

The endomorphism case

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathrm{Aut}(H_i) \\ H \leqslant \mathrm{Fix}(\alpha)}} \mathrm{Fix}(\alpha) = \bigcap_{\substack{\beta \in \mathrm{End}(F_n) \\ H \leqslant \mathrm{Fix}(\beta)}} \mathrm{Fix}(\beta).$$

- Take $H_i \in \mathcal{AE}_{ret}(H)$, and $\alpha \in \mathrm{Aut}(H_i)$ with $H \leqslant \mathrm{Fix}(\alpha)$.
- Let $\rho\colon F \to H_i$ be a retraction, and consider the endomorphism, $\beta\colon F_n \xrightarrow{\rho} H_i \xrightarrow{\alpha} H_i \xhookrightarrow{\iota} F_n$.
- Clearly, $H \leqslant \mathrm{Fix}(\alpha) = \mathrm{Fix}(\beta)$.
- Hence, we have "$\geqslant$". $\square$

The endomorphism case

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathrm{Aut}(H_i) \\ H \leqslant \mathrm{Fix}(\alpha)}} \mathrm{Fix}(\alpha) = \bigcap_{\substack{\beta \in \mathrm{End}(F_n) \\ H \leqslant \mathrm{Fix}(\beta)}} \mathrm{Fix}(\beta).$$

- Take $H_i \in \mathcal{AE}_{ret}(H)$, and $\alpha \in \mathrm{Aut}(H_i)$ with $H \leqslant \mathrm{Fix}(\alpha)$.
- Let $\rho\colon F \to H_i$ be a retraction, and consider the endomorphism, $\beta\colon F_n \xrightarrow{\rho} H_i \xrightarrow{\alpha} H_i \xhookrightarrow{\iota} F_n$.
- Clearly, $H \leqslant \mathrm{Fix}(\alpha) = \mathrm{Fix}(\beta)$.
- Hence, we have "$\geqslant$". $\square$

The endomorphism case

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathrm{Aut}(H_i) \\ H \leqslant \mathrm{Fix}(\alpha)}} \mathrm{Fix}(\alpha) = \bigcap_{\substack{\beta \in \mathrm{End}(F_n) \\ H \leqslant \mathrm{Fix}(\beta)}} \mathrm{Fix}(\beta).$$

- Take $H_i \in \mathcal{AE}_{ret}(H)$, and $\alpha \in \mathrm{Aut}(H_i)$ with $H \leqslant \mathrm{Fix}(\alpha)$.
- Let $\rho \colon F \to H_i$ be a retraction, and consider the endomorphism, $\beta \colon F_n \xrightarrow{\rho} H_i \xrightarrow{\alpha} H_i \xhookrightarrow{\iota} F_n$.
- Clearly, $H \leqslant \mathrm{Fix}(\alpha) = \mathrm{Fix}(\beta)$.
- Hence, we have "$\geqslant$". $\square$

The endomorphism case

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathrm{Aut}(H_i) \\ H \leqslant \mathrm{Fix}(\alpha)}} \mathrm{Fix}(\alpha) = \bigcap_{\substack{\beta \in \mathrm{End}(F_n) \\ H \leqslant \mathrm{Fix}(\beta)}} \mathrm{Fix}(\beta).$$

- Take $H_i \in \mathcal{AE}_{ret}(H)$, and $\alpha \in \mathrm{Aut}(H_i)$ with $H \leqslant \mathrm{Fix}(\alpha)$.
- Let $\rho \colon F \to H_i$ be a retraction, and consider the endomorphism, $\beta \colon F_n \xrightarrow{\rho} H_i \xrightarrow{\alpha} H_i \xhookrightarrow{\iota} F_n$.
- Clearly, $H \leqslant \mathrm{Fix}(\alpha) = \mathrm{Fix}(\beta)$.
- Hence, we have "$\geqslant$". $\square$

The endomorphism case

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathrm{Aut}(H_i) \\ H \leqslant \mathrm{Fix}(\alpha)}} \mathrm{Fix}(\alpha) = \bigcap_{\substack{\beta \in \mathrm{End}(F_n) \\ H \leqslant \mathrm{Fix}(\beta)}} \mathrm{Fix}(\beta).$$

- Take $H_i \in \mathcal{AE}_{ret}(H)$, and $\alpha \in \mathrm{Aut}(H_i)$ with $H \leqslant \mathrm{Fix}(\alpha)$.
- Let $\rho\colon F \to H_i$ be a retraction, and consider the endomorphism, $\beta\colon F_n \xrightarrow{\rho} H_i \xrightarrow{\alpha} H_i \overset{\iota}{\hookrightarrow} F_n$.
- Clearly, $H \leqslant \mathrm{Fix}(\alpha) = \mathrm{Fix}(\beta)$.
- Hence, we have "$\geqslant$". $\square$

THANKS