Automaton groups with unsolvable conjugacy problem

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Outline

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Main result

Theorem (Sunic-V.)

There exist automaton groups (i.e. self-similar groups generated by finite self-similar sets) with unsolvable conjugacy problem.

Related results:

- Grigorchuk-Nekrashevych-Sushchanskiĭ (00): Is CP solvable for automaton groups ?
- WP is solvable for all such groups (straightforward, at most exponential time).
- WP is solvable in polynomial time, for the subclass of f.g. contracting groups.

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- Wilson-Zaleskii (97): CP for the Gupta-Sidki groups.
- Grigorchuk-Wilson (00): CP for all subgroups of finite index in the first Grigorchuk group.
- Bondarenko-Bondarenko-Sidki-Zapata (10): CP for groups generated by bounded automata (i.e. Pol(0) groups).
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Strategy of the proof

Will use results from Bogopolski-Martino-Ventura:

Observation (B-M-V, 08)

Let H be f.g., and $\Gamma \leqslant \mathrm{Aut}(H)$ f.g. If $\Gamma \leqslant \mathrm{Aut}(H)$ is orbit undecidable then $H \rtimes \Gamma$ has unsolvable CP.

and

Proposition (B-M-V, 08)

For $d \geqslant 4$, there exist f.g., orbit undecidable, subgroups $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$.

and then show that

Theorem (Sunic-V.)

Let $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$ be f.g. Then, $\mathbb{Z}^d \rtimes \Gamma$ is an automaton group.

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For $d \geqslant 6$, $\mathrm{GL}_d(\mathbb{Z})$ contains f.g., orbit undecidable, free, subgroups.

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For $d \geqslant 6$, there exists a f.p. group G simultaneously satisfying the following three conditions:

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Orbit decidability

(joint work with O. Bogopolski and A. Martino)

Definition

Let H be f.g. A subgroup $\Gamma \leqslant \mathrm{Aut}(H)$ is said to be orbit decidable (O.D.) if there is an algorithm s.t., given $u, v \in H$, it decides whether v and $\alpha(u)$ are conjugate, for some $\alpha \in \Gamma$.

First examples: $H = \mathbb{Z}^d$

Observation (folklore)

The full group $\mathrm{Aut}(\mathbb{Z}^d) = \mathrm{GL}_d(\mathbb{Z})$ is orbit decidable.

Proof. *For $u, v \in \mathbb{Z}^d$, there exists $A \in \mathrm{GL}_d(\mathbb{Z})$ such that $v = Au$ if and only if $gcd(u_1, \ldots, u_d) = gcd(v_1, \ldots, v_d)$.*

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For $A \in \mathrm{GL}_d(\mathbb{Z})$, the subgroup $\langle A \rangle \leqslant GL_d(\mathbb{Z})$ is O.D.

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OD subgroups in $\mathrm{Aut}(F_r)$

Examples over the free group: $H = F_r$

Theorem (Whitehead'30)

The full group $\mathrm{Aut}(F_r)$ is orbit decidable. That is, given $u, v \in F_r$ one can decide whether $v = \alpha(u)$ for some $\alpha \in \mathrm{Aut}(F_r)$.

Proof. *This is a classical and very influential result.*

Theorem (Brinkmann, 06)

Cyclic groups of $\mathrm{Aut}(F_r)$ are orbit decidable. That is, given $\varphi \in \mathrm{Aut}(F_r)$ and $u, v \in F_r$, one can decide whether $v = \varphi^n(u)$, up to conjugacy, for some $n \in \mathbb{Z}$.

Proof. *A difficult result using train-tracks.*

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Connection to semidirect products

Observation (B-M-V)

Let H be f.g., and $\Gamma \leqslant \mathrm{Aut}(H)$ f.g. If $H \rtimes \Gamma$ has solvable CP, then $\Gamma \leqslant \mathrm{Aut}(H)$ is orbit decidable.

Proof. $G = H \rtimes \Gamma$ contains elements $(h, \gamma) \in H \times \Gamma$ operated like

$$(h_1, \gamma_1) \cdot (h_2, \gamma_2) = (h_1 \gamma_1(h_2), \gamma_1 \gamma_2)$$

$$(h, \gamma)^{-1} = (\gamma^{-1}(h^{-1}), \gamma^{-1}).$$

For $h_1, h_2 \in H \leqslant G$, we have $h_1 \sim_G h_2 \iff \exists (h, \gamma) \in H \rtimes \Gamma$ s.t.

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In fact, for the free and free abelian cases (among others), the convers is also true, after "erasing the relations from Γ":

Theorem (B-M-V, 08)

Let H be $\mathbb{Z}^d$ or F_r, and $\Gamma \leqslant \mathrm{Aut}(H)$ generated by $\alpha_1, \ldots, \alpha_m$. Then, $H \rtimes_{\alpha_1,\ldots,\alpha_m} F_m$ has solvable CP if and only if $\Gamma = \langle \alpha_1, \ldots, \alpha_m \rangle \leqslant \mathrm{Aut}(H)$ is orbit decidable.

Corollary

$\mathbb{Z}^d$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

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If $\Gamma = \langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_d(\mathbb{Z})$ then $\mathbb{Z}^d \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

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Corollary (Bogopolski-Martino-Maslakova-V., 06)

Free-by-cyclic groups have solvable conjugacy problem.

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If $\Gamma = \langle \varphi_1, \ldots, \varphi_m \rangle$ has finite index in $\mathrm{Aut}(F_r)$ then $F_r \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

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Every F_2-by-free group has solvable conjugacy problem.

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Theorem (Miller, 70's)

There are free-by-free groups with unsolvable conjugacy problem.

So, there must be orbit undecidable subgroups in $\mathrm{Aut}(F_r)$, for $r \geqslant 3$.
Where are them ?

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Let H be a group, and let $A \leqslant B \leqslant \mathrm{Aut}(H)$ and $v \in H$ be such that $B \cap Stab^*(v) = 1$. Then,

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Taking the copy B of $F_2 \times F_2$ in $\mathrm{Aut}(F_3)$ *via the embedding*

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Proposition (B-M-V, 08)

For $d \geqslant 4$, there exist f.g., orbit undecidable, subgroups $\Gamma \leqslant GL_d(\mathbb{Z})$.

Proof. *Consider* $F_2 \simeq \langle P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \rangle \leq_{24} GL_2(\mathbb{Z})$.

- $Stab(1,0) = \{M \mid (1,0)M = (1,0)\} = \{\begin{pmatrix} 1 & 0 \\ n & \pm 1 \end{pmatrix} \mid n \in \mathbb{Z}\}$.
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- *Write* $v = (1, 0, 1, 0)$. *By construction,* $B \cap Stab(v) = \{I\}$.
- *Take* $A \leq B \simeq F_2 \times F_2$ *with unsolvable membership problem.*
- *By previous Proposition,* $A \leqslant \mathrm{GL}_4(\mathbb{Z})$ *is orbit undecidable.*
- *Similarly for* $A \leqslant \mathrm{GL}_d(\mathbb{Z})$, $d \geqslant 4$. □

Corollary

For $d \geqslant 4$, *there exist* $\mathbb{Z}^d$*-by-free groups with unsolvable conjugacy problem.*

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Does there exist an orbit undecidable subgroup of $GL_3(\mathbb{Z})$ *?* i.e. *Does there exist* $\mathbb{Z}^3$*-by-free groups with unsolvable conjugacy problem ?*

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These orbit undecidable examples $\Gamma \leqslant GL_4(\mathbb{Z})$ come from Mihailova's construction, so they are not finitely presented...

Proposition (Sunic-V.)

For $d \geqslant 6$, $GL_d(\mathbb{Z})$ contains f.g., orbit undecidable, free, subgroups.

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- *Let $F_m = \langle f_1, \ldots, f_m \rangle$, and choose matrices $s_1, \ldots, s_m \in \mathrm{GL}_2(\mathbb{Z})$ such that $\langle s_1, \ldots, s_m \rangle \simeq F_m$.*
- *Consider the homomorphism given by*

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Playing with 2 extra dimensions...

These orbit undecidable examples $\Gamma \leqslant GL_4(\mathbb{Z})$ come from Mihailova's construction, so they are not finitely presented...

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- *Since* $\langle s_1, \ldots, s_m \rangle \leqslant GL_2(\mathbb{Z})$ *is free with basis* $\{s_1, \ldots, s_m\}$*, then* ϕ *must be one-to-one, and its image* F *is a free subgroup of* $\mathrm{GL}_d(\mathbb{Z})$ *or rank* m*.*
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Outline

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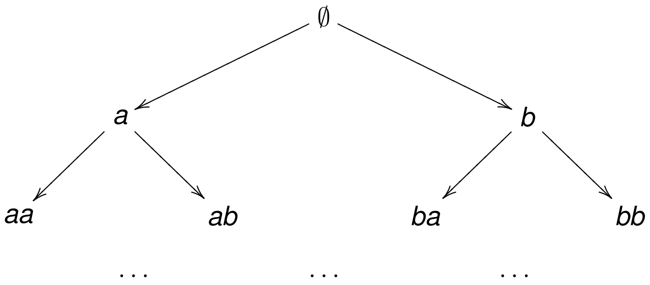

Automaton groups

Definition

- *A set of tree automorphisms is self-similar if it contains all sections of all of its elements.*
- *A finite automaton is a finite self-similar set (elements are called states).*
- *The group $G(\mathcal{A})$ of tree automorphisms generated by an automaton $\mathcal{A}$ is called an automaton group.*

Theorem (Sunic-V.)

Let $\Gamma \leqslant GL_d(\mathbb{Z})$ be f.g. Then, $\mathbb{Z}^d \rtimes \Gamma$ is an automaton group.

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Conclusion

So, we have proved that

Theorem

For $d \geqslant 6$, there exists $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$ free and orbit undecidable. Hence, the group $\mathbb{Z}^d \rtimes \Gamma$

- *is an automaton group,*
- *is free abelian-by-free,*
- *has unsolvable conjugacy problem.*

THANKS