The automorphism group of a free-by-cyclic group

(joint work with A. Martino, O. Bogopolski)

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- $\begin{array}{rcl} \gamma_w \colon F_n & \to & F_n \\ x & \mapsto & w^{-1} x w \end{array}$ $\quad \begin{array}{rcl} \Gamma_g \colon M_\phi & \to & M_\phi \\ x & \mapsto & g^{-1} x g \end{array}$ $\quad w \in F_n,\ g \in M_\phi$.

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- Are $Aut(M_\phi)$ and $Out(M_\phi)$ finitely presented ?
- Find an explicit presentation for $Aut(M_\phi)$ and $Out(M_\phi)$.
- Understand the structure of $Aut(M_\phi)$ and $Out(M_\phi)$.

By using relations $wt = t(w\phi)$ and $wt^{-1} = t^{-1}(w\phi^{-1})$, M_ϕ has a left normal form:

$$\forall g \in M_\phi \quad \exists! k \in \mathbb{Z} \quad \exists! w \in F_n, \quad g = t^k w.$$

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Lemma. *Let $n \geq 2$, F_n, $\phi \in Aut(F_n)$ and M_ϕ be as above. The group M_ϕ has non-trivial center if and only if $\phi^k = \gamma_w$ for some $k \neq 0$ and some $w \in F_n$ with $w\phi = w$.*

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Proposition. *Let $n \geq 2$, F_n, $\phi \in Aut(F_n)$ and M_ϕ be as above. Let $\Psi \in Aut(M_\phi)$ be such that $F_n \Psi \leqslant F_n$, and let $\psi \colon F_n \to F_n$ be its restriction to F_n. Then,*

i) ψ is an automorphism of F_n,

ii) writing $t\Psi = t^\epsilon w$, we have $\phi\psi = \psi\phi^\epsilon \gamma_w$.

Unfortunately, this is not the full story:

For every vector $(r_1, \dots, r_n) \in \mathbb{Z}^n$, the group $M = M_{Id} = F_n \times \mathbb{Z}$ admit the following automorphism:

$$\begin{array}{rcl} M & \longrightarrow & M \\ x_1 & \mapsto & t^{r_1} x_1 \\ & \cdots & \\ x_n & \mapsto & t^{r_n} x_n \\ t & \mapsto & t^{\pm 1} \end{array}$$

where $F_n = \langle x_1, \dots, x_n \rangle$ is far from invariant.

Theorem. *Let $n \geq 2$, F_n, $\phi \in Aut(F_n)$, M_ϕ, ϕ^{ab} and $[\phi]$ be as above. The following are equivalent:*

(a) M_ϕ^{ab} is the direct sum of $\mathbb{Z}$ and a finite abelian group,
(b) the matrix ϕ^{ab} does not have eigenvalue 1,
(c) $F_n \leqslant M_\phi$ is the unique normal subgroup of M_ϕ with quotient isomorphic to $\mathbb{Z}$.

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Furthermore, if these conditions hold then every automorphism of M_ϕ leaves F_n invariant,

$$Aut^+(M_\phi) = \{\Psi \in Aut(M_\phi) \mid \Psi \text{ is positive}\}$$

is a normal subgroup of $Aut(M_\phi)$ of index at most 2, and its image $Out^+(M_\phi)$ in $Out(M_\phi)$ is also normal, of index at most two, and isomorphic to $C([\phi])/\langle[\phi]\rangle$, where $C([\phi])$ denotes the centralizer of $[\phi]$ in $Out(F_n)$.

In the extreme opposite case,

Theorem. *Let $n \geqslant 2$ and let $M = M_{Id} = F_n \times \mathbb{Z}$. Then,*

$$Aut(M) \cong (\mathbb{Z}^n \rtimes C_2) \rtimes Aut(F_n),$$

$$Out(M) \cong (\mathbb{Z}^n \rtimes C_2) \rtimes Out(F_n),$$

where C_2 acts on $\mathbb{Z}^n$ by sending u to $-u$; and $Aut(F_n)$ (and also $Out(F_n)$) acts on $Z^n \rtimes C_2$ by the trivial action on C_2, and the natural action after abelianization on $\mathbb{Z}^n$.

We remark that $Aut^+(M) \cong Aut(F_n)$ (t goes always to t).

In the case of rank $n = 2$, we give a complete description:

Theorem. *Let $F_2 = \langle a, b \rangle$, $\phi \in Aut(F_2)$, M_ϕ and $\phi^{\text{ab}} \in GL_2(\mathbb{Z})$ be as above.*

i) If $\phi^{\text{ab}} = I_2$, then $Out(M_\phi) \cong (\mathbb{Z}^2 \rtimes C_2) \rtimes GL_2(\mathbb{Z})$.

ii) If $\phi^{\text{ab}} = -I_2$, then $Out(M_\phi) \cong PGL_2(\mathbb{Z}) \times C_2$.

iii) If $\phi^{\text{ab}} \neq -I_2$ and does not have 1 as an eigenvalue, then $Out(M_\phi)$ is finite.

iv) If ϕ^{ab} is conjugate to $\left(\begin{smallmatrix} 1 & k \\ 0 & -1 \end{smallmatrix}\right)$, then $Out(M_\phi)$ is virtually-$\mathbb{Z}$.

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Furthermore, for every $\phi \in Aut(F_2)$, ϕ^{ab} fits into exactly one of the above cases.

Corollary. *Let* $F_2 = \langle a, b \rangle$ *be a free group of rank 2 and let* $\phi, \psi \in Aut(F_2)$*. The groups* M_ϕ *and* M_ψ *are isomorphic if and only if* $[\phi]$ *and* $[\psi]^{\pm 1}$ *are conjugate in* $Out(F_2)$*.*

Corollary. *Let* $F_2 = \langle a, b \rangle$ *be a free group of rank 2 and let* $\phi, \psi \in Aut(F_2)$. *The groups* M_ϕ *and* M_ψ *are isomorphic if and only if* $[\phi]$ *and* $[\psi]^{\pm 1}$ *are conjugate in* $Out(F_2)$.

"If" : This is a straightforward exercise (for arbitrary n).

Corollary. *Let* $F_2 = \langle a, b \rangle$ *be a free group of rank 2 and let* $\phi, \psi \in Aut(F_2)$*. The groups* M_ϕ *and* M_ψ *are isomorphic if and only if* $[\phi]$ *and* $[\psi]^{\pm 1}$ *are conjugate in* $Out(F_2)$*.*

"If": This is a straightforward exercise (for arbitrary n).

"Only if": $M_\phi \cong M_\psi$ implies $Out(M_\phi) \cong Out(M_\psi)$, so ϕ and ψ fit simultaneously into one of cases *(i)*-*(v)*. Then, $[\phi]$ and $[\psi]^{\pm 1}$ are conjugate in $Out(F_2) = GL_2(\mathbb{Z})$.

The “Only if” part is not true for higher rank. We thank J. Porti and W. Dicks for help in finding this example:

$$G = \langle\, s, t \mid t^{-3}st^{2}st^{-1}s^{-1}ts^{-2}ts \,\rangle$$

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On one hand we have

$$G \;\cong \langle\, s_0, s_1, s_2, s_3, t \mid s_1 = t^{-1}s_0t,\; s_2 = t^{-1}s_1t,\; s_3 = t^{-1}s_2t \\ s_3s_1s_2^{-1}s_1^{-2}s_0 = 1 \,\rangle$$

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where $\begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ s_0 & \mapsto & s_1 \\ s_1 & \mapsto & s_2 \\ s_2 & \mapsto & s_0^{-1}s_1^2s_2s_1^{-1} \end{array}$ (note that $|\phi^{\mathrm{ab}}| = \begin{vmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1$).

On the other hand,

$$G \cong \langle t_{-2}, t_{-1}, t_0, t_1, s \mid t_{-1} = s^{-1} t_{-2} s,\ t_0 = s^{-1} t_{-1} s,\ t_1 = s^{-1} t_0 s,\ t_0^{-3} t_{-1}^2 t_{-2}^{-1} t_{-1} t_1 = 1 \rangle$$

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&\cong \langle\, t_{-2}, t_{-1}, t_0, s \mid s^{-1} t_{-2} s = t_{-1},\ s^{-1} t_{-1} s = t_0 \\
&\qquad\qquad s^{-1} t_0 s = t_{-1}^{-1} t_{-2} t_{-1}^{-2} t_0^{3} \,\rangle
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where $\begin{array}{rcl} \psi\colon F_3 & \longrightarrow & F_3 \\ t_{-2} & \mapsto & t_{-1} \\ t_{-1} & \mapsto & t_0 \\ t_0 & \mapsto & t_{-1}^{-1}t_{-2}t_{-1}^{-2}t_0^{3} \end{array}$ (note that $|\phi^{\mathrm{ab}}| = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{vmatrix} = 1$).

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Thus, $M_\phi \cong M_\psi \cong G$, while $[\phi]$, $[\psi]^{\pm 1} \in Out(F_3)$ are not conjugate to each other.

THANKS