1. Historical context	<ol> <li>CP for F<sub>n</sub>-by-Z</li> </ol>	3. CP for Fn-by-Fm	4. Main result	

5. Applications

◆□▶ ◆□▶ ◆□▶ ◆□▶ ● ● ● ●

6. Negative results

# The conjugacy problem and other algorithmically related questions

# **Enric Ventura**

Departament de Matemàtica Aplicada III Universitat Politècnica de Catalunya

# **EMS-SCM** joint meeting

## Barcelona

May 28th, 2015.

1. Historical context	<ol> <li>CP for F<sub>n</sub>-by-Z</li> </ol>	3. CP for Fn-by-Fm	<ol><li>Main result</li></ol>	5. Applications	<ol><li>Negative results</li></ol>

**Mathematics** Algebra **Group Theory** 

◆□▶ ◆□▶ ◆豆▶ ◆豆▶ □ のへで

1. Historical context	2. CP for <i>F</i> <sub><i>n</i></sub> -by-ℤ 0000000000	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results

**Mathematics** Algebra

・ コット (雪) ( 小田) ( コット 日)

1. Historical context	2. CP for <i>F</i> <sub><i>n</i></sub> -by-ℤ 0000000000	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results

**Mathematics** Algebra **Group Theory** 

・ コット (雪) ( 小田) ( コット 日)

<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-Z</li> </ol>	3. CP for Fn-by-Fm	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>

**Mathematics** Algebra **Group Theory Discrete groups** 

・ コット (雪) ( 小田) ( コット 日)

1. Historical context	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	3. CP for Fn-by-Fm	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>

**Mathematics** Algebra **Group Theory Discrete groups** focus on algorithmic questions

◆□ ▶ ◆□ ▶ ◆ □ ▶ ◆ □ ▶ ● ● ● ● ●

- - O. Bogopolski, A. Martino, O. Maslakova, E. Ventura, Freeby-cyclic groups have solvable conjugacy problem, Bulletin of the London Mathematical Society, 38(5) (2006), 787–794.
  - O. Bogopolski, A. Martino, E. Ventura, Orbit decidability and the conjugacy problem for extensions of groups, **Transactions of the American Mathematical Society** 362 (2010), 2003–2036.
  - V. Romanko'v, E. Ventura, Twisted conjugacy problem for endomorphisms of metabelian groups, **Algebra and Logic** 48(2) (2009), 89–98. Translation from **Algebra i Logika** 48(2) (2009), 157–173.
  - J. González-Meneses, E. Ventura, Twisted conjugacy in the braid group, **Israel Journal of Mathematics** 201 (2014), 455–476.
  - J. Burillo, F. Matucci, E. Ventura, The conjugacy problem for extensions of Thompson's group, to appear at **Israel Journal of Mathematics**.
  - Z. Sŭnic, E. Ventura, The conjugacy problem in automaton groups is not solvable, **Journal of Algebra** 364 (2012), 148–154.
  - E. Ventura, Group theoretic orbit decidability, **Groups, Complexity, Cryptology** 6(2) (2014), 133–148.

• • • •

- 1. Historical context
   2. CP for F<sub>I</sub>-by-Z
   3. CP for F<sub>I</sub>-by-F<sub>I</sub>
   4. Main result
   5. Applications
   6. Negative result

   0000
   000000000
   000000
   00000000
   000000000
   000000000
   000000000
  - O. Bogopolski, A. Martino, O. Maslakova, E. Ventura, Freeby-cyclic groups have solvable conjugacy problem, Bulletin of the London Mathematical Society, 38(5) (2006), 787–794.
  - O. Bogopolski, A. Martino, E. Ventura, Orbit decidability and the conjugacy problem for extensions of groups, **Transactions of the American Mathematical Society** 362 (2010), 2003–2036.
  - V. Romanko'v, E. Ventura, Twisted conjugacy problem for endomorphisms of metabelian groups, **Algebra and Logic** 48(2) (2009), 89–98. Translation from **Algebra i Logika** 48(2) (2009), 157–173.
  - J. González-Meneses, E. Ventura, Twisted conjugacy in the braid group, **Israel Journal of Mathematics** 201 (2014), 455–476.
  - J. Burillo, F. Matucci, E. Ventura, The conjugacy problem for extensions of Thompson's group, to appear at **Israel Journal of Mathematics**.
  - Z. Sŭnic, E. Ventura, The conjugacy problem in automaton groups is not solvable, **Journal of Algebra** 364 (2012), 148–154.
  - E. Ventura, Group theoretic orbit decidability, **Groups, Complexity, Cryptology** 6(2) (2014), 133–148.

...

- - O. Bogopolski, A. Martino, O. Maslakova, E. Ventura, Freeby-cyclic groups have solvable conjugacy problem, **Bulletin of the London Mathematical Society**, 38(5) (2006), 787–794.
  - O. Bogopolski, A. Martino, E. Ventura, Orbit decidability and the conjugacy problem for extensions of groups, **Transactions of the American Mathematical Society** 362 (2010), 2003–2036.
  - V. Romanko'v, E. Ventura, Twisted conjugacy problem for endomorphisms of metabelian groups, **Algebra and Logic** 48(2) (2009), 89–98. Translation from **Algebra i Logika** 48(2) (2009), 157–173.
  - J. González-Meneses, E. Ventura, Twisted conjugacy in the braid group, Israel Journal of Mathematics 201 (2014), 455–476.
  - J. Burillo, F. Matucci, E. Ventura, The conjugacy problem for extensions of Thompson's group, to appear at **Israel Journal of Mathematics**.
  - Z. Sŭnic, E. Ventura, The conjugacy problem in automaton groups is not solvable, Journal of Algebra 364 (2012), 148–154.
  - E. Ventura, Group theoretic orbit decidability, **Groups, Complexity, Cryptology** 6(2) (2014), 133–148.

o ...

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○○○○○○○○○○	3. CP for <i>F<sub>n</sub>-by-F<sub>m</sub></i> 00000	4. Main result	5. Applications	6. Negative results
Outline					

- The historical context
- 2 The conjugacy problem for free-by-cyclic groups
- The conjugacy problem for free-by-free groups
- 4 The main result
- 5 Applications
- 6 Negative results

1. Historical context	2. CP for <i>F<sub>n</sub>-by-</i> ℤ ○○○○○○○○○○	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results
Outline					



- 2 The conjugacy problem for free-by-cyclic groups
- The conjugacy problem for free-by-free groups
- 4 The main result
- 5 Applications
- 6 Negative results

•000		00000	0000000	000000000	00000000000						
Presenta	Presentations of groups										

A finite presentation of a (discrete) group G is

$$G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle.$$

- $a_1, \ldots, a_n$  are the generators;
- $r_1, \ldots, r_m$  are the relators;
- elements of G are words (i.e., non-commutative! formal

•  $\mathbb{Z} = \langle a | - \rangle$ :

- $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle; \quad ba \cdot ba^{-2} = a^{-1}b^2$  $a^4 \cdot a^3 = a^2$
- $\mathbb{Z}/5\mathbb{Z} = \langle a \mid a^5 \rangle;$

<ol> <li>Historical context</li> <li>0000</li> </ol>	2. CP for <i>F<sub>n</sub>-by-</i> ℤ ○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results
Presenta	tions of gr	roups			

A finite presentation of a (discrete) group G is

$$G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle.$$

#### • $a_1, \ldots, a_n$ are the generators;

- $r_1, \ldots, r_m$  are the relators;
- elements of G are words (i.e., non-commutative! formal

• 
$$\mathbb{Z} = \langle a \mid - \rangle;$$
  
•  $\mathbb{Z}^2 = \langle a \mid b \mid aba^{-1}b^{-1} \rangle = \langle a \mid b \mid ab = ba \rangle;$   $ba \cdot ba^{-2} = a^{-1}b^2$ 

• 
$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle;$$
  $ba \cdot ba^{-2} =$   
•  $\mathbb{Z}/5\mathbb{Z} = \langle a \mid a^5 \rangle;$   $a^4 \cdot a^3 = a^2$ 

• 
$$\mathbb{Z}/5\mathbb{Z} = \langle a \mid a^5 \rangle;$$

<ol> <li>Historical context</li> <li>0000</li> </ol>	2. CP for <i>F<sub>n</sub>-by-</i> ℤ ○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results
Presenta	tions of gr	roups			

A finite presentation of a (discrete) group G is

$$G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle.$$

•  $a_1, \ldots, a_n$  are the generators;

#### • $r_1, \ldots, r_m$ are the relators;

• elements of G are words (i.e., non-commutative! formal

• 
$$\mathbb{Z} = \langle a \mid - \rangle;$$
  
•  $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle;$   
 $ba \cdot ba^{-2} = a^{-1}b^2$ 

• 
$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle;$$
  $ba \cdot ba^{-2} = \mathbb{Z} / 5\mathbb{Z} = \langle a \mid a^5 \rangle;$   $a^4 \cdot a^3 = a^2$ 

• 
$$\mathbb{Z}/5\mathbb{Z} = \langle a \mid a^{\mathsf{s}} \rangle;$$

<ol> <li>Historical context</li> <li>OOO</li> </ol>	2. CP for <i>F<sub>Π</sub>-</i> by-ℤ 0000000000	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results					
Presentations of groups										

A finite presentation of a (discrete) group G is

$$G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle.$$

- $a_1, \ldots, a_n$  are the generators;
- $r_1, \ldots, r_m$  are the relators;
- elements of G are words (i.e., non-commutative! formal products) of the  $a_i^{\pm 1}$ 's, subject to the rules  $r_i = 1$ .

• 
$$\mathbb{Z} = \langle a \mid - \rangle;$$
  
•  $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle;$   
 $ba \cdot ba^{-2} = a^{-1}b^2$ 

• 
$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle;$$
 ba  $\cdot$  ba

• 
$$\mathbb{Z}/5\mathbb{Z} = \langle a \mid a^5 \rangle;$$

•000		00000	0000000	000000000	00000000000
Presenta	ations of g	roups			

A finite presentation of a (discrete) group G is

$$G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle.$$

- $a_1, \ldots, a_n$  are the generators;
- $r_1, \ldots, r_m$  are the relators;
- elements of *G* are words (i.e., non-commutative! formal products) of the  $a_i^{\pm 1}$ 's, subject to the rules  $r_i = 1$ .

#### Example

• 
$$\mathbb{Z} = \langle a \mid - \rangle;$$

• 
$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle;$$

• 
$$\mathbb{Z}/5\mathbb{Z} = \langle a \mid a^{t} \rangle$$

$$a^{5} \cdot a^{-3} = a^{2}$$
  
ba \cdot ba^{-2} = a^{-1}b^{2}  
$$a^{4} \cdot a^{3} = a^{2}$$

▲□▶▲圖▶▲≣▶▲≣▶ ≣ のQ@

•000		00000	0000000	000000000	00000000000
Presenta	ations of g	roups			

A finite presentation of a (discrete) group G is

$$G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle.$$

- $a_1, \ldots, a_n$  are the generators;
- $r_1, \ldots, r_m$  are the relators;
- elements of *G* are words (i.e., non-commutative! formal products) of the  $a_i^{\pm 1}$ 's, subject to the rules  $r_i = 1$ .

#### Example

• 
$$\mathbb{Z} = \langle a | - \rangle;$$
  $a^5 \cdot a^-$ 

• 
$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle;$$

$$a^{5} \cdot a^{-3} = a^{2}$$
  
 $ba \cdot ba^{-2} = a^{-1}$ 

•000		00000	0000000	000000000	00000000000
Presenta	ations of g	roups			

A finite presentation of a (discrete) group G is

$$G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle.$$

- $a_1, \ldots, a_n$  are the generators;
- $r_1, \ldots, r_m$  are the relators;

•  $\mathbb{Z}/5\mathbb{Z} = \langle a \mid a^{\circ} \rangle;$ 

 elements of G are words (i.e., non-commutative! formal products) of the  $a_i^{\pm 1}$ 's, subject to the rules  $r_i = 1$ .

### Example

• 
$$\mathbb{Z} = \langle a \mid - \rangle;$$
  
•  $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle;$   
•  $ba \cdot ba^{-2} = a^{-1}b^2$ 

• 
$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle; \quad ba \cdot ba^{-2} =$$

•000		00000	0000000	000000000	00000000000
Presenta	ations of g	roups			

A finite presentation of a (discrete) group G is

$$G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle.$$

- $a_1, \ldots, a_n$  are the generators;
- $r_1, \ldots, r_m$  are the relators;
- elements of G are words (i.e., non-commutative! formal products) of the  $a_i^{\pm 1}$ 's, subject to the rules  $r_i = 1$ .

#### Example

• 
$$\mathbb{Z} = \langle a \mid - \rangle;$$
  
•  $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle;$   
•  $ba \cdot ba^{-2} = a^{-1}b^2$ 

• 
$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle; \quad ba \cdot ba^{-1}$$

• 
$$\mathbb{Z}/5\mathbb{Z} = \langle a \mid a^5 \rangle$$

•000		00000	0000000	000000000	00000000000
Presenta	ations of g	roups			

A finite presentation of a (discrete) group G is

$$G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle.$$

- $a_1, \ldots, a_n$  are the generators;
- $r_1, \ldots, r_m$  are the relators;
- elements of G are words (i.e., non-commutative! formal products) of the  $a_i^{\pm 1}$ 's, subject to the rules  $r_i = 1$ .

### Example

• 
$$\mathbb{Z} = \langle a \mid - \rangle;$$
  $a^5 \cdot a^{-3} = a^2$ 

$$\mathbb{Z} = \langle a \mid -\rangle, \qquad a \cdot a = a \\ \mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle; \qquad ba \cdot ba^{-2} = a^{-1}b^2 \\ \mathbb{Z}/5\mathbb{Z} = \langle a \mid a^5 \rangle; \qquad a^4 \cdot a^3 = a^2$$

• 
$$\mathbb{Z}/5\mathbb{Z} = \langle a \mid a^5 \rangle;$$

	tions of g		0000000	00000000	
0000	000000000	00000	0000000	000000000	00000000000
<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-F<sub>m</sub></li> </ol>	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \Rightarrow b^{-1}a^{-1}ba = b \Rightarrow (b^{-1}ab)^{-1}a = b$$
  
 $\Rightarrow a^{-2}a = b \Rightarrow a^{-1} = b.$ 

But then

$$a^{-1}ba = b^2 \quad \Rightarrow \ a^{-1} = b^2 = a^{-2} \quad \Rightarrow \ a = 1$$
$$b^{-1}ab = a^2 \quad \Rightarrow \ b^{-1} = a^2 = b^{-2} \quad \Rightarrow \ b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

▲□▶ ▲□▶ ▲□▶ ▲□▶ □ のQで

	tions of g		0000000	00000000	
0000	000000000	00000	0000000	000000000	00000000000
<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-F<sub>m</sub></li> </ol>	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \quad \Rightarrow \ b^{-1}a^{-1}ba = b \quad \Rightarrow \ (b^{-1}ab)^{-1}a = b$$
$$\Rightarrow \ a^{-2}a = b \qquad \Rightarrow \ a^{-1} = b.$$

But then

$$a^{-1}ba = b^2 \quad \Rightarrow \ a^{-1} = b^2 = a^{-2} \quad \Rightarrow \ a = 1$$
$$b^{-1}ab = a^2 \quad \Rightarrow \ b^{-1} = a^2 = b^{-2} \quad \Rightarrow \ b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

	tions of g		0000000	00000000	
0000	000000000	00000	0000000	000000000	00000000000
<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-F<sub>m</sub></li> </ol>	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \quad \Rightarrow \ b^{-1}a^{-1}ba = b \quad \Rightarrow \ (b^{-1}ab)^{-1}a = b$$
$$\Rightarrow \ a^{-2}a = b \qquad \Rightarrow \ a^{-1} = b.$$

But then

$$a^{-1}ba = b^2 \quad \Rightarrow \ a^{-1} = b^2 = a^{-2} \quad \Rightarrow \ a = 1$$
$$b^{-1}ab = a^2 \quad \Rightarrow \ b^{-1} = a^2 = b^{-2} \quad \Rightarrow \ b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

	tions of g		0000000	00000000	
0000	000000000	00000	0000000	000000000	00000000000
<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-F<sub>m</sub></li> </ol>	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \Rightarrow b^{-1}a^{-1}ba = b \Rightarrow (b^{-1}ab)^{-1}a = b$$
  
 $\Rightarrow a^{-2}a = b \Rightarrow a^{-1} = b.$ 

But then

$$a^{-1}ba = b^2 \Rightarrow a^{-1} = b^2 = a^{-2} \Rightarrow a = 1$$
  
$$b^{-1}ab = a^2 \Rightarrow b^{-1} = a^2 = b^{-2} \Rightarrow b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

	tions of g		0000000	00000000	
0000	000000000	00000	0000000	000000000	00000000000
<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-F<sub>m</sub></li> </ol>	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \quad \Rightarrow b^{-1}a^{-1}ba = b \quad \Rightarrow (b^{-1}ab)^{-1}a = b$$
  
 $\Rightarrow a^{-2}a = b \qquad \Rightarrow a^{-1} = b.$ 

But then

$$a^{-1}ba = b^2 \Rightarrow a^{-1} = b^2 = a^{-2} \Rightarrow a = 1$$
  
$$b^{-1}ab = a^2 \Rightarrow b^{-1} = a^2 = b^{-2} \Rightarrow b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

	tions of g		0000000	00000000	
0000	000000000	00000	0000000	000000000	00000000000
<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-F<sub>m</sub></li> </ol>	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \quad \Rightarrow \ b^{-1}a^{-1}ba = b \quad \Rightarrow \ (b^{-1}ab)^{-1}a = b$$
  
 $\Rightarrow \ a^{-2}a = b \qquad \Rightarrow \ a^{-1} = b.$ 

But then

$$a^{-1}ba = b^2 \Rightarrow a^{-1} = b^2 = a^{-2} \Rightarrow a = 1$$
  
$$b^{-1}ab = a^2 \Rightarrow b^{-1} = a^2 = b^{-2} \Rightarrow b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

	tions of g		0000000	00000000	
0000	000000000	00000	0000000	000000000	00000000000
<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-F<sub>m</sub></li> </ol>	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \quad \Rightarrow \ b^{-1}a^{-1}ba = b \quad \Rightarrow \ (b^{-1}ab)^{-1}a = b$$
  
 $\Rightarrow \ a^{-2}a = b \qquad \Rightarrow \ a^{-1} = b.$ 

But then

$$a^{-1}ba = b^2 \Rightarrow a^{-1} = b^2 = a^{-2} \Rightarrow a = 1$$
  
$$b^{-1}ab = a^2 \Rightarrow b^{-1} = a^2 = b^{-2} \Rightarrow b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

0000	0000000000	00000	0000000	000000000	000000000000000000000000000000000000000
<ol> <li>Historical context</li> <li>O●OO</li> </ol>	0000000000	00000	0000000	000000000	000000000000000000000000000000000000000

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \quad \Rightarrow \ b^{-1}a^{-1}ba = b \quad \Rightarrow \ (b^{-1}ab)^{-1}a = b$$
$$\Rightarrow \ a^{-2}a = b \qquad \Rightarrow \ a^{-1} = b.$$

But then

$$a^{-1}ba = b^2 \quad \Rightarrow \ a^{-1} = b^2 = a^{-2} \quad \Rightarrow \ a = 1$$
$$b^{-1}ab = a^2 \quad \Rightarrow \ b^{-1} = a^2 = b^{-2} \quad \Rightarrow \ b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

0000	0000000000	00000	0000000	000000000	000000000000000000000000000000000000000
<ol> <li>Historical context</li> <li>O●OO</li> </ol>	0000000000	00000	0000000	000000000	000000000000000000000000000000000000000

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \quad \Rightarrow \ b^{-1}a^{-1}ba = b \quad \Rightarrow \ (b^{-1}ab)^{-1}a = b$$
$$\Rightarrow \ a^{-2}a = b \qquad \Rightarrow \ a^{-1} = b.$$

But then

$$a^{-1}ba = b^2 \quad \Rightarrow a^{-1} = b^2 = a^{-2} \quad \Rightarrow a = 1$$
  
$$b^{-1}ab = a^2 \quad \Rightarrow b^{-1} = a^2 = b^{-2} \quad \Rightarrow b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

0000	0000000000	00000	0000000	000000000	000000000000000000000000000000000000000
<ol> <li>Historical context</li> <li>O●OO</li> </ol>	0000000000	00000	0000000	000000000	000000000000000000000000000000000000000

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \quad \Rightarrow \ b^{-1}a^{-1}ba = b \quad \Rightarrow \ (b^{-1}ab)^{-1}a = b$$
  
 $\Rightarrow \ a^{-2}a = b \qquad \Rightarrow \ a^{-1} = b.$ 

But then

$$a^{-1}ba = b^2 \Rightarrow a^{-1} = b^2 = a^{-2} \Rightarrow a = 1$$
  
$$b^{-1}ab = a^2 \Rightarrow b^{-1} = a^2 = b^{-2} \Rightarrow b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

0000	0000000000	00000	0000000	000000000	00000000000
<ol> <li>Historical context</li> <li>O●OO</li> </ol>	0000000000	00000	0000000	000000000	000000000000000000000000000000000000000

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \quad \Rightarrow \ b^{-1}a^{-1}ba = b \quad \Rightarrow \ (b^{-1}ab)^{-1}a = b$$
  
 $\Rightarrow \ a^{-2}a = b \qquad \Rightarrow \ a^{-1} = b.$ 

But then

$$a^{-1}ba = b^2 \Rightarrow a^{-1} = b^2 = a^{-2} \Rightarrow a = 1$$
  
$$b^{-1}ab = a^2 \Rightarrow b^{-1} = a^2 = b^{-2} \Rightarrow b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

0000	0000000000	00000	0000000	000000000	00000000000
<ol> <li>Historical context</li> <li>O●OO</li> </ol>	0000000000	00000	0000000	000000000	000000000000000000000000000000000000000

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \quad \Rightarrow \ b^{-1}a^{-1}ba = b \quad \Rightarrow \ (b^{-1}ab)^{-1}a = b$$
  
 $\Rightarrow \ a^{-2}a = b \qquad \Rightarrow \ a^{-1} = b.$ 

But then

$$a^{-1}ba = b^2 \quad \Rightarrow \ a^{-1} = b^2 = a^{-2} \quad \Rightarrow \ a = 1$$
$$b^{-1}ab = a^2 \quad \Rightarrow \ b^{-1} = a^2 = b^{-2} \quad \Rightarrow \ b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

1. Historical context O●OO	2. CP for <i>Fn</i> -by-ℤ 0000000000	3. CP for F <sub>n</sub> -by-F <sub>m</sub> 00000	4. Main result	5. Applications	6. Negative results
Presenta	tions of gr	roups			

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \quad \Rightarrow \ b^{-1}a^{-1}ba = b \quad \Rightarrow \ (b^{-1}ab)^{-1}a = b$$
  
 $\Rightarrow \ a^{-2}a = b \qquad \Rightarrow \ a^{-1} = b.$ 

But then

$$a^{-1}ba = b^2 \quad \Rightarrow \ a^{-1} = b^2 = a^{-2} \quad \Rightarrow \ a = 1$$
$$b^{-1}ab = a^2 \quad \Rightarrow \ b^{-1} = a^2 = b^{-2} \quad \Rightarrow \ b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize G from a given presentation.

0000	0000000000	00000	0000000	000000000	00000000000
<ol> <li>Historical context</li> <li>O●OO</li> </ol>	0000000000	00000	0000000	000000000	000000000000000000000000000000000000000

Which group is  $G = \langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ ?

$$a^{-1}ba = b^2 \quad \Rightarrow \ b^{-1}a^{-1}ba = b \quad \Rightarrow \ (b^{-1}ab)^{-1}a = b$$
  
 $\Rightarrow \ a^{-2}a = b \qquad \Rightarrow \ a^{-1} = b.$ 

But then

$$a^{-1}ba = b^2 \quad \Rightarrow \ a^{-1} = b^2 = a^{-2} \quad \Rightarrow \ a = 1$$
$$b^{-1}ab = a^2 \quad \Rightarrow \ b^{-1} = a^2 = b^{-2} \quad \Rightarrow \ b = 1.$$

Hence, G = 1 is the trivial group.

It is not easy, in general, to recognize *G* from a given presentation.

<ol> <li>Historical context</li> <li>OO●O</li> </ol>	2. CP for <i>F<sub>n</sub></i> -by-Z 0000000000	3. CP for <i>F<sub>n</sub></i> -by- <i>F<sub>m</sub></i> 00000	4. Main result	5. Applications	6. Negative results
Dehn's p	roblems				

### Word Problem, WP(G)

For any given presentation  $G = \langle a_1, \dots, a_n | r_1, \dots, r_m \rangle$ , find an algorithm W with:

- Input: a word  $w(a_1, \ldots, a_n)$  on the  $a_i^{\pm 1}$ 's;
- **Output:** "yes" or "no" depending on whether  $w =_{G} 1$ .

#### Conjugacy Problem, *CP*(*G*)

For any given presentation  $G = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$ , find an algorithm C with:

- Input: two words  $u(a_1, \ldots, a_n)$  and  $v(a_1, \ldots, a_n)$ ;
- Output: "yes" or "no" depending on whether u and v are conjugate in G, u ∼<sub>G</sub> v (i.e., v =<sub>G</sub> g<sup>-1</sup>ug for some g ∈ G).

<ol> <li>Historical context</li> <li>OO●O</li> </ol>	2. CP for <i>F<sub>n</sub></i> -by-Z 0000000000	3. CP for <i>F<sub>n</sub></i> -by- <i>F<sub>m</sub></i> 00000	4. Main result	5. Applications	6. Negative results
Dehn's p	roblems				

### Word Problem, WP(G)

For any given presentation  $G = \langle a_1, \dots, a_n | r_1, \dots, r_m \rangle$ , find an algorithm W with:

- Input: a word  $w(a_1, \ldots, a_n)$  on the  $a_i^{\pm 1}$ 's;
- **Output:** "yes" or "no" depending on whether  $w =_{G} 1$ .

### Conjugacy Problem, CP(G)

For any given presentation  $G = \langle a_1, \dots, a_n | r_1, \dots, r_m \rangle$ , find an algorithm C with:

- Input: two words  $u(a_1, \ldots, a_n)$  and  $v(a_1, \ldots, a_n)$ ;
- Output: "yes" or "no" depending on whether u and v are conjugate in G, u ∼<sub>G</sub> v (i.e., v =<sub>G</sub> g<sup>-1</sup>ug for some g ∈ G).

Dehn's p	roblems				
<ol> <li>Historical context</li> <li>OOOO</li> </ol>	2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>Fn</i> -by- <i>Fm</i>	4. Main result	5. Applications	6. Negative results

Find an algorithm  $\mathcal{I}$  with:

- **Input:** two presentations  $G_i = \langle a_1, ..., a_{n_i} | r_1, ..., r_{m_i} \rangle$ , i = 1, 2;
- **Output:** "yes" or "no" depending on whether  $G_1 \simeq G_2$  as groups.

Theorem (Novikov '55; Boone '58)

There exist finitely presented groups with unsolvable word problem.

Theorem (Adyan '57; Rabin '58)

The Isomorphism Problem is unsolvable.

## Theorem (Miller '71)

Dehn's p	roblems				
<ol> <li>Historical context</li> <li>OOOO</li> </ol>	2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>Fn</i> -by- <i>Fm</i>	4. Main result	5. Applications	6. Negative results

Find an algorithm  $\mathcal{I}$  with:

- **Input:** two presentations  $G_i = \langle a_1, ..., a_{n_i} | r_1, ..., r_{m_i} \rangle$ , i = 1, 2;
- **Output:** "yes" or "no" depending on whether  $G_1 \simeq G_2$  as groups.

# Theorem (Novikov '55; Boone '58)

There exist finitely presented groups with unsolvable word problem.

Theorem (Adyan '57; Rabin '58)

The Isomorphism Problem is unsolvable.

## Theorem (Miller '71)

Dehn's p	roblems				
<ol> <li>Historical context</li> <li>OOOO</li> </ol>	2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>Fn</i> -by- <i>Fm</i>	4. Main result	5. Applications	6. Negative results

Find an algorithm  $\mathcal{I}$  with:

- **Input:** two presentations  $G_i = \langle a_1, ..., a_{n_i} | r_1, ..., r_{m_i} \rangle$ , i = 1, 2;
- **Output:** "yes" or "no" depending on whether  $G_1 \simeq G_2$  as groups.

Theorem (Novikov '55; Boone '58)

There exist finitely presented groups with unsolvable word problem.

Theorem (Adyan '57; Rabin '58)

The Isomorphism Problem is unsolvable.

## Theorem (Miller '71)

Dehn's p	roblems				
<ol> <li>Historical context</li> <li>OOOO</li> </ol>	2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>Fn</i> -by- <i>Fm</i>	4. Main result	5. Applications	6. Negative results

Find an algorithm  $\mathcal{I}$  with:

- **Input:** two presentations  $G_i = \langle a_1, ..., a_{n_i} | r_1, ..., r_{m_i} \rangle$ , i = 1, 2;
- **Output:** "yes" or "no" depending on whether  $G_1 \simeq G_2$  as groups.

Theorem (Novikov '55; Boone '58)

There exist finitely presented groups with unsolvable word problem.

Theorem (Adyan '57; Rabin '58)

The Isomorphism Problem is unsolvable.

# Theorem (Miller '71)

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results
Outline					

- The historical context
- 2 The conjugacy problem for free-by-cyclic groups
- The conjugacy problem for free-by-free groups
- 4 The main result
- 5 Applications
- 6 Negative results



1. Historical context	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	3. CP for Fn-by-Fm	4. Main result	5. Applications	6. Negative results
	•000000000				

# Step 1:

# Find a problem you like

# (2004)

Conjuga	cy problen	n for free-b	y-cyclic	groups	
1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○●○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results

## Definition

Let  $F_n = \langle a_1, \ldots, a_n | - \rangle$  be a free group on  $\{a_1, \ldots, a_n\}$   $(n \ge 2)$ , and let  $\varphi \in Aut(F_n)$ . The free-by-cyclic group  $F_n \rtimes_{\varphi} \mathbb{Z}$  is defined as

$$F_n \rtimes_{\varphi} \mathbb{Z} = \langle a_1, \ldots, a_n, t \mid t^{-1}a_i t = a_i \varphi \rangle.$$

◆□▶ ◆□▶ ▲□▶ ▲□▶ ■ ののの

#### Observation

The word problem in  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$  is solvable.

#### Open problem since 2004

Solve the conjugacy problem in  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ .

Conjuga	cy problen	n for free-b	y-cyclic	groups	
1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○●○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results

# Definition

Let  $F_n = \langle a_1, ..., a_n | - \rangle$  be a free group on  $\{a_1, ..., a_n\}$   $(n \ge 2)$ , and let  $\varphi \in Aut(F_n)$ . The free-by-cyclic group  $F_n \rtimes_{\varphi} \mathbb{Z}$  is defined as

$$F_n \rtimes_{\varphi} \mathbb{Z} = \langle a_1, \ldots, a_n, t \mid t^{-1}a_i t = a_i \varphi \rangle.$$

#### Observation

The word problem in  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$  is solvable.

#### Open problem since 2004

Solve the conjugacy problem in  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ .

くりょう 小田 マイビット 日 うくの

Conjuga	cy problen	n for free-b	y-cyclic	groups	
1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○●○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results

# Definition

Let  $F_n = \langle a_1, \dots, a_n | - \rangle$  be a free group on  $\{a_1, \dots, a_n\}$   $(n \ge 2)$ , and let  $\varphi \in Aut(F_n)$ . The free-by-cyclic group  $F_n \rtimes_{\varphi} \mathbb{Z}$  is defined as

$$F_n \rtimes_{\varphi} \mathbb{Z} = \langle a_1, \ldots, a_n, t \mid t^{-1}a_i t = a_i \varphi \rangle.$$

#### Observation

The word problem in  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$  is solvable.

## Open problem since 2004

Solve the conjugacy problem in  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ .

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ● ●

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 00000000
 00000000
 00000000
 00000000
 000000000
 000000000

 Conjugacy problem for free-by-cyclic groups
 00000
 00000000
 000000000
 000000000

Let's consider an example:  $M_{\varphi} = \langle a, b, t \mid t^{-1}at = a\varphi, t^{-1}bt = b\varphi \rangle$ 

 $\varphi: F_2 \rightarrow F_2 \qquad \varphi^{-1}: F_2 \rightarrow F_2$   $a \mapsto ab \qquad a \mapsto a^{-1}b$   $b \mapsto aba \qquad b \mapsto b^{-1}a^2$   $wt = t(w\varphi) \qquad wt^{-1} = t^{-1}(w\varphi^{-1})$   $tab^{-1}t^{-1}at^2a = tab^{-1}t^{-1}t(ab)ta = tab^{-1}abta$   $= tab^{-1}at(aba)a = tab^{-1}ataba^2$   $= tat(a^{-1}b^{-1}a^{-1})ababa^2 = tatba^2$   $= tt(ab)ba^2 = t^2ab^2a^2.$ 

#### \_emma

Every element from  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$  has a unique normal form:

 $t^rw$  for some  $r\in\mathbb{Z},\;w\in F_n.$ 

 1. Historical context
 2. OP for Fn-by-Z
 3. OP for Fn-by-Em
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 000000000
 000000000
 000000000
 000000000
 000000000

 Conjugacy problem for free-by-cyclic groups
 000000000
 000000000
 0000000000
 0000000000

Let's consider an example:  $M_{\varphi} = \langle a, b, t \mid t^{-1}at = a\varphi, t^{-1}bt = b\varphi \rangle$ 

 $\varphi: F_2 \rightarrow F_2 \qquad \varphi^{-1}: F_2 \rightarrow F_2$   $a \mapsto ab \qquad a \mapsto a^{-1}b$   $b \mapsto aba \qquad b \mapsto b^{-1}a^2$   $wt = t(w\varphi) \qquad wt^{-1} = t^{-1}(w\varphi^{-1})$   $tab^{-1}t^{-1}at^2a = tab^{-1}t^{-1}t(ab)ta = tab^{-1}abta$   $= tab^{-1}at(aba)a = tab^{-1}ataba^2$   $= tat(a^{-1}b^{-1}a^{-1})ababa^2 = tatba^2$   $= tt(ab)ba^2 = t^2ab^2a^2.$ 

Lemma

Every element from  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$  has a unique normal form:

 $f^rw$  for some  $r\in\mathbb{Z}, \,\,w\in F_n.$ 

 1. Historical context
 2. OP for Fn-by-Z
 3. OP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 000000000
 000000000
 000000000
 000000000
 000000000

 Conjugacy problem for free-by-cyclic groups
 000000000
 000000000
 0000000000
 0000000000

Let's consider an example:  $M_{\varphi} = \langle a, b, t \mid t^{-1}at = a\varphi, t^{-1}bt = b\varphi \rangle$ 

 $\varphi: F_2 \rightarrow F_2 \qquad \varphi^{-1}: F_2 \rightarrow F_2$   $a \mapsto ab \qquad a \mapsto a^{-1}b$   $b \mapsto aba \qquad b \mapsto b^{-1}a^2$   $wt = t(w\varphi) \qquad wt^{-1} = t^{-1}(w\varphi^{-1})$   $tab^{-1}t^{-1}at^2a = tab^{-1}t^{-1}t(ab)ta = tab^{-1}abta$   $= tab^{-1}at(aba)a = tab^{-1}ataba^2$   $= tat(a^{-1}b^{-1}a^{-1})ababa^2 = tatba^2$   $= tt(ab)ba^2 = t^2ab^2a^2.$ 

#### Lemma

Every element from  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$  has a unique normal form:

 $t^r w$  for some  $r \in \mathbb{Z}, w \in F_n$ .

<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-Z</li> </ol>	3. CP for Fn-by-Fm	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>
	000000000				

Step 2:

Push the problem into your

favorite territory

(2005)

▲□▶ ▲□▶ ▲□▶ ▲□▶ = 三 のへで

 1. Historical context
 2. CP for Fa-by-Z
 3. CP for Fa-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 0000
 00000
 000000
 00000000
 000000000
 000000000

 Converting it into a free group problem

Let  $t^r u$ ,  $t^s v$ ,  $t^k g$  be arbitrary elements in  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ . Then,

$$egin{array}{rcl} (g^{-1}t^{-k})(t^ru)(t^kg)&=&g^{-1}t^r(uarphi^k)g\ &=&t^r(garphi^r)^{-1}(uarphi^k)g \end{array}$$

$$t^r u \sim_{_{M_{\varphi}}} t^s v \quad \Longleftrightarrow \quad r = s \quad \& \quad v \sim_{\varphi^r} (u \varphi^k) \text{ for some } k \in \mathbb{Z}.$$

#### Definition

For  $\phi \in Aut(G)$ , two elements  $u, v \in G$  are said to be  $\phi$ -twisted conjugated, denoted  $u \sim_{\phi} v$ , if  $v = (g\phi)^{-1}ug$  for some  $g \in G$ .

### Twisted Conjugacy Problem, *TCP*(*G*)

 1. Historical context
 2. CP for Fa-by-Z
 3. CP for Fa-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 0000
 00000
 000000
 0000000
 00000000
 000000000

 Converting it into a free group problem

Let  $t^r u$ ,  $t^s v$ ,  $t^k g$  be arbitrary elements in  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ . Then,

$$\begin{array}{rcl} (g^{-1}t^{-k})(t^r u)(t^k g) &=& g^{-1}t^r(u\varphi^k)g\\ &=& t^r(g\varphi^r)^{-1}(u\varphi^k)g \end{array}$$

$$t^r u \sim_{_{M_{\varphi}}} t^s v \quad \Longleftrightarrow \quad r = s \quad \& \quad v \sim_{\varphi^r} (u \varphi^k) \text{ for some } k \in \mathbb{Z}.$$

イロト 不良 とくほ とくほう 二日

#### Definition

For  $\phi \in Aut(G)$ , two elements  $u, v \in G$  are said to be  $\phi$ -twisted conjugated, denoted  $u \sim_{\phi} v$ , if  $v = (g\phi)^{-1}ug$  for some  $g \in G$ .

#### Twisted Conjugacy Problem, *TCP*(*G*)

 1. Historical context
 2. CP for Fa-by-Z
 3. CP for Fa-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 0000
 00000
 000000
 0000000
 00000000
 000000000

 Converting it into a free group problem

Let  $t^r u$ ,  $t^s v$ ,  $t^k g$  be arbitrary elements in  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ . Then,

$$\begin{array}{rcl} (g^{-1}t^{-k})(t^r u)(t^k g) &=& g^{-1}t^r(u\varphi^k)g\\ &=& t^r(g\varphi^r)^{-1}(u\varphi^k)g \end{array}$$

$$t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s \& v \sim_{\varphi^r} (u \varphi^k) \text{ for some } k \in \mathbb{Z}.$$

#### Definition

For  $\phi \in Aut(G)$ , two elements  $u, v \in G$  are said to be  $\phi$ -twisted conjugated, denoted  $u \sim_{\phi} v$ , if  $v = (g\phi)^{-1}ug$  for some  $g \in G$ .

## Twisted Conjugacy Problem, *TCP*(*G*)

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 0000
 0000
 00000
 000000
 00000000
 00000000

 Converting it into a free group problem

Let  $t^r u$ ,  $t^s v$ ,  $t^k g$  be arbitrary elements in  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ . Then,

$$(g^{-1}t^{-k})(t^r u)(t^k g) = g^{-1}t^r(u\varphi^k)g$$
  
=  $t^r(g\varphi^r)^{-1}(u\varphi^k)g$ 

$$t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s \& v \sim_{\varphi^r} (u \varphi^k) \text{ for some } k \in \mathbb{Z}.$$

## Definition

For  $\phi \in Aut(G)$ , two elements  $u, v \in G$  are said to be  $\phi$ -twisted conjugated, denoted  $u \sim_{\phi} v$ , if  $v = (g\phi)^{-1}ug$  for some  $g \in G$ .

## Twisted Conjugacy Problem, TCP(G)

 1. Historical context
 2. CP for Fa-by-Z
 3. CP for Fa-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 0000
 00000
 000000
 0000000
 00000000
 000000000

 Converting it into a free group problem

Let  $t^r u$ ,  $t^s v$ ,  $t^k g$  be arbitrary elements in  $M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ . Then,

$$(g^{-1}t^{-k})(t^r u)(t^k g) = g^{-1}t^r(u\varphi^k)g$$
  
=  $t^r(g\varphi^r)^{-1}(u\varphi^k)g$ 

$$t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s \& v \sim_{\varphi^r} (u \varphi^k) \text{ for some } k \in \mathbb{Z}.$$

#### Definition

For  $\phi \in Aut(G)$ , two elements  $u, v \in G$  are said to be  $\phi$ -twisted conjugated, denoted  $u \sim_{\phi} v$ , if  $v = (g\phi)^{-1}ug$  for some  $g \in G$ .

## Twisted Conjugacy Problem, TCP(G)

1. Historical context	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	3. CP for Fn-by-Fm	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>
	0000000000				

Step 3:

Solve it

(2005)

◆□▶ ◆□▶ ◆ □▶ ◆ □▶ ● □ ● ● ● ●

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^s v \in M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ .

•  $t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s$  &  $v \sim_{\varphi^r} (u\varphi^k)$  for some  $k \in \mathbb{Z}$ .

• To reduce to finitely many k's, note that  $u \sim_{\varphi} u \varphi$  because

 $u = (u\varphi)^{-1}(u\varphi)u$ 

• so  $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$  and hence,

 $t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s \& v \sim_{\varphi^r} (u\varphi^k) \text{ for } k = 0, \dots r - 1.$ 

Thus, CP(M<sub>φ</sub>) reduces to finitely many checks of TCP(F<sub>n</sub>).
BUT...

・ロト・日本・日本・日本・日本・日本

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^s v \in M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ .

•  $t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s$  &  $v \sim_{\varphi^r} (u\varphi^k)$  for some  $k \in \mathbb{Z}$ .

• To reduce to finitely many *k*'s, note that  $u \sim_{\varphi} u \varphi$  because

 $u = (u\varphi)^{-1}(u\varphi)u$ 

• so  $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$  and hence,

 $t^r u \sim_{M_{\alpha}} t^s v \iff r = s \& v \sim_{\varphi^r} (u\varphi^k) \text{ for } k = 0, \dots r - 1.$ 

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^s v \in M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ .

- $t^r u \sim_{M_{\varphi}} t^s v \iff r = s$  &  $v \sim_{\varphi^r} (u\varphi^k)$  for some  $k \in \mathbb{Z}$ . • To reduce to finitely many k's, note that  $u \sim_{\varphi} u\varphi$  because  $u = (u\varphi)^{-1}(u\varphi)u$
- so  $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$  and hence,

 $t^r u \sim_{_{M_{o}}} t^s v \quad \Longleftrightarrow \quad r = s \quad \& \quad v \sim_{\varphi^r} (u\varphi^k) \text{ for } k = 0, \dots r - 1.$ 

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^s v \in M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ .

•  $t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s$  &  $v \sim_{\varphi^r} (u\varphi^k)$  for some  $k \in \mathbb{Z}$ .

• To reduce to finitely many k's, note that  $u \sim_{\varphi} u\varphi$  because  $u = (u\varphi)^{-1}(u\varphi)u$ 

• so  $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$  and hence,

 $t^r u \sim_{_{M_{\omega}}} t^s v \iff r = s \& v \sim_{\varphi^r} (u\varphi^k) \text{ for } k = 0, \dots r - 1.$ 

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^s v \in M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ .

- $t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s$  &  $v \sim_{\varphi^r} (u\varphi^k)$  for some  $k \in \mathbb{Z}$ .
- To reduce to finitely many k's, note that  $u\sim_{\varphi} u\varphi$  because  $u=(u\varphi)^{-1}(u\varphi)u$
- so  $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$  and hence,

 $t^r u \sim_{M_{\alpha}} t^s v \iff r = s \& v \sim_{\varphi^r} (u\varphi^k) \text{ for } k = 0, \dots r - 1.$ 

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^s v \in M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ .

- $t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s$  &  $v \sim_{\varphi^r} (u\varphi^k)$  for some  $k \in \mathbb{Z}$ .
- To reduce to finitely many k's, note that  $u \sim_{\varphi} u\varphi$  because

$$u = (u\varphi)^{-1}(u\varphi)u$$

• so  $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$  and hence,

 $t^{r}u \sim_{M_{\varphi}} t^{s}v \iff r = s \& v \sim_{\varphi^{r}} (u\varphi^{k}) \text{ for } k = 0, \dots r - 1.$ • Thus,  $CP(M_{\varphi})$  reduces to finitely many checks of  $TCP(F_{n})$ . • BUT...

▲□▶▲□▶▲□▶▲□▶ □ のQ@

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^s v \in M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ .

- $t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s$  &  $v \sim_{\varphi^r} (u\varphi^k)$  for some  $k \in \mathbb{Z}$ .
- To reduce to finitely many k's, note that  $u \sim_{\varphi} u\varphi$  because

$$u=(u\varphi)^{-1}(u\varphi)u$$

• so  $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$  and hence,

 $t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s \& v \sim_{\varphi^r} (u\varphi^k) \text{ for } k = 0, \dots r - 1.$ 

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^s v \in M_{\varphi} = F_n \rtimes_{\varphi} \mathbb{Z}$ .

- $t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s$  &  $v \sim_{\varphi^r} (u\varphi^k)$  for some  $k \in \mathbb{Z}$ .
- To reduce to finitely many k's, note that  $u \sim_{\varphi} u\varphi$  because

$$u=(u\varphi)^{-1}(u\varphi)u$$

• so  $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$  and hence,

 $t^r u \sim_{_{M_{\varphi}}} t^s v \iff r = s \& v \sim_{\varphi^r} (u\varphi^k) \text{ for } k = 0, \dots r - 1.$ 

<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	3. CP for Fn-by-Fm	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>
	00000000000				

Step 4:

# Ups ... a technical problem!

(2005)

・ロト・日本・日本・日本・日本

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

# **Proof.** Given $t^r u$ , $t^r v \in F_n \rtimes_{\varphi} \mathbb{Z}$ ,

- ► Case 1: r ≠ 0
- $t^r u \sim_{_{M_{\varphi}}} t^r v \iff v \sim_{\varphi^r} (u\varphi^k)$  for  $k = 0, \ldots r 1$ .
- Thus,  $CP(M_{\varphi})$  reduces to finitely many checks of  $TCP(F_n)$ .
- ▶ <u>Case 2: *r* = 0</u>

• Still infinitely many k's to check:

 $u\sim_{_{M_{arphi}}}v\quad \Longleftrightarrow\quad v\sim uarphi^k$  for some  $k\in\mathbb{Z}_{+}$ 

(日) (日) (日) (日) (日) (日) (日)

• Fortunately, this is precisely Brinkmann's result:

# Theorem (Brinkmann, 2006)

Given an automorphism  $\phi: F_n \to F_n$  and  $u, v \in F_n$ , it is decidable whether  $v \sim u\phi^k$  for some  $k \in \mathbb{Z}$ .

1. Historical context2. CP for  $F_n$ -by-Z<br/>oocooocooo3. CP for  $F_n$ -by- $F_m$ 4. Main result5. Applications<br/>oocooocooocooo6. Negative results<br/>oocooocoocoo $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^r v \in F_n \rtimes_{\varphi} \mathbb{Z}$ ,

- ► Case 1: r ≠ 0
- $t^r u \sim_{_{M_{\varphi}}} t^r v \iff v \sim_{\varphi^r} (u\varphi^k)$  for  $k = 0, \ldots r 1$ .
- Thus,  $CP(M_{\varphi})$  reduces to finitely many checks of  $TCP(F_n)$ .

• Still infinitely many k's to check:

 $u\sim_{_{M_{arphi}}} v \quad \Longleftrightarrow \quad v\sim uarphi^k$  for some  $k\in\mathbb{Z}.$ 

• Fortunately, this is precisely Brinkmann's result:

## Theorem (Brinkmann, 2006)

Given an automorphism  $\phi: F_n \to F_n$  and  $u, v \in F_n$ , it is decidable whether  $v \sim u\phi^k$  for some  $k \in \mathbb{Z}$ .

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^r v \in F_n \rtimes_{\varphi} \mathbb{Z}$ ,

- ► Case 1: r ≠ 0
- $t^r u \sim_{_{M_{\varphi}}} t^r v \iff v \sim_{\varphi^r} (u\varphi^k)$  for  $k = 0, \ldots r 1$ .
- Thus,  $CP(M_{\varphi})$  reduces to finitely many checks of  $TCP(F_n)$ .

• Still infinitely many k's to check:

 $u\sim_{_{M_{arphi}}}v\quad\Longleftrightarrow\quad v\sim uarphi^k$  for some  $k\in\mathbb{Z}$ 

• Fortunately, this is precisely Brinkmann's result:

# Theorem (Brinkmann, 2006)

Given an automorphism  $\phi: F_n \to F_n$  and  $u, v \in F_n$ , it is decidable whether  $v \sim u\phi^k$  for some  $k \in \mathbb{Z}$ .

1. Historical context2. CP for  $F_n$ -by-Z<br/>oocooocooo3. CP for  $F_n$ -by- $F_m$ 4. Main result5. Applications<br/>oocooocooocooo6. Negative results<br/>oocooocoocoo $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^r v \in F_n \rtimes_{\varphi} \mathbb{Z}$ ,

- ► Case 1: r ≠ 0
- $t^r u \sim_{_{M_{\varphi}}} t^r v \iff v \sim_{\varphi^r} (u\varphi^k)$  for  $k = 0, \ldots r 1$ .
- Thus,  $CP(M_{\varphi})$  reduces to finitely many checks of  $TCP(F_n)$ .

• Still infinitely many k's to check:

 $u\sim_{_{M_{\varphi}}} v \quad \Longleftrightarrow \quad v\sim u\varphi^k ext{ for some } k\in\mathbb{Z}.$ 

• Fortunately, this is precisely Brinkmann's result:

Theorem (Brinkmann, 2006)

Given an automorphism  $\phi: F_n \to F_n$  and  $u, v \in F_n$ , it is decidable whether  $v \sim u\phi^k$  for some  $k \in \mathbb{Z}$ .

1. Historical context2. CP for  $F_n$ -by-Z<br/>oocooocooo3. CP for  $F_n$ -by- $F_m$ 4. Main result5. Applications<br/>oocooocooocooo6. Negative results<br/>oocooocoocoo $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^r v \in F_n \rtimes_{\varphi} \mathbb{Z}$ ,

- ► Case 1: r ≠ 0
- $t^r u \sim_{_{M_{\varphi}}} t^r v \iff v \sim_{\varphi^r} (u\varphi^k)$  for  $k = 0, \ldots r 1$ .
- Thus,  $CP(M_{\varphi})$  reduces to finitely many checks of  $TCP(F_n)$ .

• Still infinitely many k's to check:

 $u \sim_{_{M_{\omega}}} v \iff v \sim u \varphi^k$  for some  $k \in \mathbb{Z}$ .

(日) (日) (日) (日) (日) (日) (日)

• Fortunately, this is precisely Brinkmann's result:

# Theorem (Brinkmann, 2006)

Given an automorphism  $\phi \colon F_n \to F_n$  and  $u, v \in F_n$ , it is decidable whether  $v \sim u\phi^k$  for some  $k \in \mathbb{Z}$ .

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

**Proof.** Given  $t^r u$ ,  $t^r v \in F_n \rtimes_{\varphi} \mathbb{Z}$ ,

- ► Case 1: r ≠ 0
- $t^r u \sim_{_{M_{\varphi}}} t^r v \iff v \sim_{\varphi^r} (u\varphi^k)$  for  $k = 0, \ldots r 1$ .
- Thus,  $CP(M_{\varphi})$  reduces to finitely many checks of  $TCP(F_n)$ .

• Still infinitely many k's to check:

 $u \sim_{_{M_{\omega}}} v \iff v \sim u \varphi^k$  for some  $k \in \mathbb{Z}$ .

(日) (日) (日) (日) (日) (日) (日)

• Fortunately, this is precisely Brinkmann's result:

# Theorem (Brinkmann, 2006)

Given an automorphism  $\phi \colon F_n \to F_n$  and  $u, v \in F_n$ , it is decidable whether  $v \sim u\phi^k$  for some  $k \in \mathbb{Z}$ .



- Our solution to *TCP*(*F<sub>n</sub>*) uses a previous deep result by Bogopolski–Maslakova about computability of fixed subgroups of automorphisms of free groups.
- In 2008 a problem was found in the proof of Bogopolski–Maslakova; the authors claim to have fixed it, but no correction has been published yet.
- In 2014 Feighn–Handel give an alternative proof for Bogopolski–Maslakova's result.

• • •

• Alternative solution to  $CP(F_n \rtimes_{\phi} \mathbb{Z})$  combining deep results by Ol'shanskii–Sapir 2006, and Bridson–Groves 2010.

◆□▶ ◆□▶ ▲□▶ ▲□▶ ■ ののの



- Our solution to *TCP*(*F<sub>n</sub>*) uses a previous deep result by Bogopolski–Maslakova about computability of fixed subgroups of automorphisms of free groups.
- In 2008 a problem was found in the proof of Bogopolski–Maslakova; the authors claim to have fixed it, but no correction has been published yet.
- In 2014 Feighn–Handel give an alternative proof for Bogopolski–Maslakova's result.

• • •

• Alternative solution to  $CP(F_n \rtimes_{\phi} \mathbb{Z})$  combining deep results by Ol'shanskii–Sapir 2006, and Bridson–Groves 2010.

◆□▶ ◆□▶ ▲□▶ ▲□▶ ■ ののの



- Our solution to *TCP*(*F<sub>n</sub>*) uses a previous deep result by Bogopolski–Maslakova about computability of fixed subgroups of automorphisms of free groups.
- In 2008 a problem was found in the proof of Bogopolski–Maslakova; the authors claim to have fixed it, but no correction has been published yet.
- In 2014 Feighn–Handel give an alternative proof for Bogopolski–Maslakova's result.

• Alternative solution to  $CP(F_n \rtimes_{\phi} \mathbb{Z})$  combining deep results by Ol'shanskii–Sapir 2006, and Bridson–Groves 2010.

(日) (日) (日) (日) (日) (日) (日)



- Our solution to *TCP*(*F<sub>n</sub>*) uses a previous deep result by Bogopolski–Maslakova about computability of fixed subgroups of automorphisms of free groups.
- In 2008 a problem was found in the proof of Bogopolski–Maslakova; the authors claim to have fixed it, but no correction has been published yet.
- In 2014 Feighn–Handel give an alternative proof for Bogopolski–Maslakova's result.

• • •

• Alternative solution to  $CP(F_n \rtimes_{\phi} \mathbb{Z})$  combining deep results by Ol'shanskii–Sapir 2006, and Bridson–Groves 2010.

◆□▶ ◆□▶ ◆□▶ ◆□▶ ● ● ● ●



- Our solution to *TCP*(*F<sub>n</sub>*) uses a previous deep result by Bogopolski–Maslakova about computability of fixed subgroups of automorphisms of free groups.
- In 2008 a problem was found in the proof of Bogopolski–Maslakova; the authors claim to have fixed it, but no correction has been published yet.
- In 2014 Feighn–Handel give an alternative proof for Bogopolski–Maslakova's result.

• • •

• Alternative solution to  $CP(F_n \rtimes_{\phi} \mathbb{Z})$  combining deep results by Ol'shanskii–Sapir 2006, and Bridson–Groves 2010.

◆□▶ ◆□▶ ◆□▶ ◆□▶ ● ● ● ●

1. Historical context	2. CP for <i>Fn</i> -by-ℤ 0000000000	3. CP for <i>Fn</i> -by- <i>Fm</i>	4. Main result	5. Applications	6. Negative results
Outline					

- The historical context
- 2 The conjugacy problem for free-by-cyclic groups
- The conjugacy problem for free-by-free groups
- 4 The main result
- 5 Applications
- 6 Negative results



<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	<ol> <li>CP for Fn-by-Fm</li> </ol>	<ol><li>Main result</li></ol>	5. Applications	<ol><li>Negative results</li></ol>
		●0000			

# Step 5:

# Intuition always ahead

(2006)

・ロト・日本・日本・日本・日本

0000	0000000000	00000	0000000	000000000	00000000000
A crucial	comment				

Armando Martino: "The whole argument essentially works the same way in presence of more stable letters, i.e., for free-by-free groups"

#### Definition

Let  $F_n = \langle x_1, \ldots, x_n | \rangle$  be the free group on  $\{x_1, \ldots, x_n\}$   $(n \ge 2)$ , and let  $\varphi_1, \ldots, \varphi_m \in Aut(F_n)$ . The free-by-free group  $F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m$  is

 $M_{\varphi_1,\ldots,\varphi_m} = F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m = \langle x_1,\ldots,x_n, t_1,\ldots,t_m \mid t_j^{-1} x_i t_j = x_i \varphi_j \rangle.$ 

But this must be wrong ...

Theorem (Miller '71)

There exist free-by-free groups with unsolvable conjugacy problem.

Surprise was that ...

... Armando was "essentially" right !!

(日) (日) (日) (日) (日) (日) (日)

0000	0000000000	00000	0000000	000000000	00000000000
A crucial	comment				

Armando Martino: "The whole argument essentially works the same way in presence of more stable letters, i.e., for free-by-free groups"

#### Definition

Let  $F_n = \langle x_1, \ldots, x_n | \rangle$  be the free group on  $\{x_1, \ldots, x_n\}$   $(n \ge 2)$ , and let  $\varphi_1, \ldots, \varphi_m \in Aut(F_n)$ . The free-by-free group  $F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m$  is

 $M_{\varphi_1,\ldots,\varphi_m} = F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m = \langle x_1,\ldots,x_n, t_1,\ldots,t_m \mid t_j^{-1} x_i t_j = x_i \varphi_j \rangle.$ 

# But this must be wrong ...

Theorem (Miller '71)

There exist free-by-free groups with unsolvable conjugacy problem.

Surprise was that ...

... Armando was "essentially" right !!

(ロ) (同) (三) (三) (三) (○) (○)

0000	0000000000	00000	0000000	000000000	00000000000
A crucial	comment				

Armando Martino: "The whole argument essentially works the same way in presence of more stable letters, i.e., for free-by-free groups"

#### Definition

Let  $F_n = \langle x_1, \ldots, x_n | \rangle$  be the free group on  $\{x_1, \ldots, x_n\}$   $(n \ge 2)$ , and let  $\varphi_1, \ldots, \varphi_m \in Aut(F_n)$ . The free-by-free group  $F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m$  is

 $M_{\varphi_1,\ldots,\varphi_m} = F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m = \langle x_1,\ldots,x_n, t_1,\ldots,t_m \mid t_j^{-1} x_i t_j = x_i \varphi_j \rangle.$ 

But this must be wrong ...

Theorem (Miller '71)

There exist free-by-free groups with unsolvable conjugacy problem.

Surprise was that ...

... Armando was "essentially" right !!

(ロ) (同) (三) (三) (三) (○) (○)

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000
 000000
 0000000
 0000000
 00000000
 00000000

# The comment was right...

In Case 1, the whole argument essentially works the same way;

But in Case 2, a much stronger problem arises:

 $u\sim_{_{M_{\iota\sigma}}} v \quad \Longleftrightarrow \quad v\sim u\varphi^k \text{ for some } k\in\mathbb{Z}.$ 

 $u \sim_{_{M_{\varphi}}} v \iff v \sim u\phi$  for some  $\phi \in \langle \varphi \rangle \leqslant Aut(F_n)$ 

 $u \sim_{M_{\varphi_1,\ldots,\varphi_m}} v \iff v \sim u\phi \text{ for some } \phi \in \langle \varphi_1,\ldots,\varphi_m \rangle \leqslant Aut(F_n).$ 

Theorem (Bogopolski–Martino–V., 2010)

 $CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$  is solvable if and only if  $\langle \varphi_1,\ldots,\varphi_m \rangle \leq Aut(F_n)$  is orbit decidable.

### Definition

The comment was right...

In Case 1, the whole argument essentially works the same way;

But in Case 2, a much stronger problem arises:

 $u\sim_{_{M_{\omega}}} v \iff v\sim u\varphi^k$  for some  $k\in\mathbb{Z}.$ 

 $u \sim_{_{M_{\varphi}}} v \iff v \sim u\phi$  for some  $\phi \in \langle \varphi \rangle \leqslant Aut(F_n)$ 

 $u \sim_{_{M_{\varphi_1},\ldots,\varphi_m}} v \iff v \sim u\phi \text{ for some } \phi \in \langle \varphi_1,\ldots,\varphi_m \rangle \leqslant Aut(F_n).$ 

Theorem (Bogopolski–Martino–V., 2010)

 $CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$  is solvable if and only if  $\langle \varphi_1,\ldots,\varphi_m \rangle \leq Aut(F_n)$  is orbit decidable.

# Definition

The comment was right...

In Case 1, the whole argument essentially works the same way;

But in Case 2, a much stronger problem arises:

 $u\sim_{_{M_{\iota \sigma}}} v \quad \Longleftrightarrow \quad v\sim u \varphi^k \text{ for some } k\in \mathbb{Z}.$ 

 $u \sim_{_{M_{\varphi}}} v \iff v \sim u\phi$  for some  $\phi \in \langle \varphi \rangle \leqslant Aut(F_n)$ .

 $u \sim_{M_{\varphi_1},\ldots,\varphi_m} v \iff v \sim u\phi \text{ for some } \phi \in \langle \varphi_1,\ldots,\varphi_m \rangle \leqslant Aut(F_n).$ 

Theorem (Bogopolski–Martino–V., 2010)

 $CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$  is solvable if and only if  $\langle \varphi_1,\ldots,\varphi_m \rangle \leq Aut(F_n)$  is orbit decidable.

# Definition

The comment was right...

In Case 1, the whole argument essentially works the same way;

But in Case 2, a much stronger problem arises:

 $u\sim_{_{M_{\iota \sigma}}} v \quad \Longleftrightarrow \quad v\sim u \varphi^k \text{ for some } k\in \mathbb{Z}.$ 

 $u \sim_{_{M_{\varphi}}} v \iff v \sim u\phi$  for some  $\phi \in \langle \varphi \rangle \leqslant Aut(F_n)$ .

$$u \sim_{M_{\varphi_1,\ldots,\varphi_m}} v \iff v \sim u\phi \text{ for some } \phi \in \langle \varphi_1,\ldots,\varphi_m \rangle \leqslant Aut(F_n).$$

Theorem (Bogopolski–Martino–V., 2010)

 $CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$  is solvable if and only if  $\langle \varphi_1,\ldots,\varphi_m \rangle \leq Aut(F_n)$  is orbit decidable.

### Definition

The comment was right...

In Case 1, the whole argument essentially works the same way;

But in Case 2, a much stronger problem arises:

 $u\sim_{_{M_{\iota \sigma}}} v \quad \Longleftrightarrow \quad v\sim u \varphi^k \text{ for some } k\in \mathbb{Z}.$ 

$$u \sim_{_{M_{\alpha}}} v \iff v \sim u\phi$$
 for some  $\phi \in \langle \varphi \rangle \leqslant Aut(F_n)$ 

$$u \sim_{_{M_{\varphi_1},\ldots,\varphi_m}} v \iff v \sim u\phi \text{ for some } \phi \in \langle \varphi_1,\ldots,\varphi_m \rangle \leqslant Aut(F_n).$$

# Theorem (Bogopolski–Martino–V., 2010)

 $CP(F_n \rtimes_{\varphi_1,...,\varphi_m} F_m)$  is solvable if and only if  $\langle \varphi_1,...,\varphi_m \rangle \leq Aut(F_n)$  is orbit decidable.

### Definition

The comment was right...

In Case 1, the whole argument essentially works the same way;

But in Case 2, a much stronger problem arises:

 $u\sim_{_{M_{\iota \sigma}}} v \quad \Longleftrightarrow \quad v\sim u \varphi^k \text{ for some } k\in \mathbb{Z}.$ 

$$u \sim_{_{M_{\alpha}}} v \iff v \sim u\phi$$
 for some  $\phi \in \langle \varphi \rangle \leqslant Aut(F_n)$ 

$$u \sim_{M_{\varphi_1,\ldots,\varphi_m}} v \iff v \sim u\phi \text{ for some } \phi \in \langle \varphi_1,\ldots,\varphi_m \rangle \leqslant Aut(F_n).$$

# Theorem (Bogopolski–Martino–V., 2010)

 $CP(F_n \rtimes_{\varphi_1,...,\varphi_m} F_m)$  is solvable if and only if  $\langle \varphi_1,...,\varphi_m \rangle \leq Aut(F_n)$  is orbit decidable.

# Definition

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> ○○○●○	4. Main result	5. Applications	6. Negative results
Reformu	lating				

A subgroup  $A \leq Aut(F_n)$  is orbit decidable (O.D.) if  $\exists$  an algorithm A s.t., given  $u, v \in F_n$  decides whether  $v \sim u\alpha$  for some  $\alpha \in A$ .

#### Theorem (Brinkmann, 2006)

Cyclic subgroups of  $Aut(F_n)$  are orbit decidable.

Theorem (Bogopolski–Martino–V., 2010)

 $CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$  is solvable  $\iff \langle \varphi_1,\ldots,\varphi_m \rangle \leqslant Aut(F_n)$  is O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> ○○○●○	4. Main result	5. Applications	6. Negative results
Reformu	lating				

A subgroup  $A \leq Aut(F_n)$  is orbit decidable (O.D.) if  $\exists$  an algorithm A s.t., given  $u, v \in F_n$  decides whether  $v \sim u\alpha$  for some  $\alpha \in A$ .

# Theorem (Brinkmann, 2006)

Cyclic subgroups of  $Aut(F_n)$  are orbit decidable.

#### Theorem (Bogopolski–Martino–V., 2010)

 $CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$  is solvable  $\iff \langle \varphi_1,\ldots,\varphi_m \rangle \leqslant Aut(F_n)$  is O.D.

Corollary (Bogopolski–Martino–Maslakova–V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> ○○○●○	4. Main result	5. Applications	6. Negative results
Reformu	lating				

A subgroup  $A \leq Aut(F_n)$  is orbit decidable (O.D.) if  $\exists$  an algorithm A s.t., given  $u, v \in F_n$  decides whether  $v \sim u\alpha$  for some  $\alpha \in A$ .

#### Theorem (Brinkmann, 2006)

Cyclic subgroups of  $Aut(F_n)$  are orbit decidable.

# Theorem (Bogopolski–Martino–V., 2010)

 $CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$  is solvable  $\iff \langle \varphi_1,\ldots,\varphi_m \rangle \leqslant Aut(F_n)$  is O.D.

Corollary (Bogopolski–Martino–Maslakova–V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> ○○○●○	4. Main result	5. Applications	6. Negative results
Reformu	lating				

A subgroup  $A \leq Aut(F_n)$  is orbit decidable (O.D.) if  $\exists$  an algorithm A s.t., given  $u, v \in F_n$  decides whether  $v \sim u\alpha$  for some  $\alpha \in A$ .

#### Theorem (Brinkmann, 2006)

Cyclic subgroups of  $Aut(F_n)$  are orbit decidable.

# Theorem (Bogopolski–Martino–V., 2010)

 $CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$  is solvable  $\iff \langle \varphi_1,\ldots,\varphi_m \rangle \leqslant Aut(F_n)$  is O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> ○○○●○	4. Main result	5. Applications	6. Negative results
Reformu	lating				

A subgroup  $A \leq Aut(F_n)$  is orbit decidable (O.D.) if  $\exists$  an algorithm A s.t., given  $u, v \in F_n$  decides whether  $v \sim u\alpha$  for some  $\alpha \in A$ .

### Theorem (Brinkmann, 2006)

Cyclic subgroups of  $Aut(F_n)$  are orbit decidable.

# Theorem (Bogopolski–Martino–V., 2010)

 $CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$  is solvable  $\iff \langle \varphi_1,\ldots,\varphi_m \rangle \leqslant Aut(F_n)$  is O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

For every  $\varphi \in Aut(F_n)$ ,  $CP(F_n \rtimes_{\varphi} \mathbb{Z})$  is solvable.

 And Miller's examples must correspond to orbit undecidable subgroups ⟨φ<sub>1</sub>,...,φ<sub>m</sub>⟩ ≤ Aut (F<sub>n</sub>).

<ol> <li>Historical context</li> </ol>	<ol> <li>CP for <i>F<sub>n</sub></i>-by-ℤ</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-F<sub>m</sub></li> </ol>	<ol><li>Main result</li></ol>	5. Applications	<ol><li>Negative results</li></ol>
		00000			

# Step 6:

# Extend as much as possible

# (2007)

◆□▶ ◆□▶ ◆ □▶ ◆ □▶ ● □ ● ● ●

1. Historical context	2. CP for <i>F⊓</i> -by-ℤ 0000000000	3. CP for <i>F<sub>n</sub>-</i> by- <i>F<sub>m</sub></i>	4. Main result	5. Applications	6. Negative results
Outline					

- The historical context
- The conjugacy problem for free-by-cyclic groups
- The conjugacy problem for free-by-free groups
- 4 The main result
- 5 Applications
- 6 Negative results



1. Historical context	2. CP for <i>F<sub>∩</sub></i> -by-ℤ ○○○○○○○○○	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result ●000000	5. Applications	6. Negative results
Orbit dec	cidability				

Let X be a set. A collection of maps  $A \subseteq Map(X, X)$  is said to be orbit decidable (O.D.) if there is an algorithm A with:

- Input: two elements  $x, y \in X$ ;
- **Output:** "yes" or "no" depending on  $x\alpha = y$  for some  $\alpha \in A$ .

#### Definition

For X,  $A \subseteq Map(X, X)$ , the A-orbit of  $x \in X$  is  $\mathcal{O}(x) = \{x \alpha \mid \alpha \in A\}$ .

#### Observation

O.D. is membership in A-orbits.

#### Observation

The conjugacy problem for group G, CP(G), is just the O.D. for  $A = \text{lnn}(G) = \{\gamma_g \colon G \to G, x \mapsto g^{-1}xg \mid g \in G\} \trianglelefteq Aut(G).$ 

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>F<sub>n</sub>-</i> by- <i>F<sub>m</sub></i> 00000	4. Main result ●000000	5. Applications	6. Negative results
Orbit dec	dability				

Let X be a set. A collection of maps  $A \subseteq Map(X, X)$  is said to be orbit decidable (O.D.) if there is an algorithm A with:

- Input: two elements  $x, y \in X$ ;
- **Output:** "yes" or "no" depending on  $x\alpha = y$  for some  $\alpha \in A$ .

### Definition

For X,  $A \subseteq Map(X, X)$ , the A-orbit of  $x \in X$  is  $\mathcal{O}(x) = \{x\alpha \mid \alpha \in A\}$ .

#### Observation

O.D. is membership in A-orbits.

#### Observation

The conjugacy problem for group G, CP(G), is just the O.D. for  $A = \text{lnn}(G) = \{\gamma_g \colon G \to G, x \mapsto g^{-1}xg \mid g \in G\} \trianglelefteq Aut(G).$ 

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>F<sub>n</sub>-</i> by- <i>F<sub>m</sub></i> 00000	4. Main result ●000000	5. Applications	6. Negative results
Orbit dec	dability				

Let X be a set. A collection of maps  $A \subseteq Map(X, X)$  is said to be orbit decidable (O.D.) if there is an algorithm A with:

- Input: two elements  $x, y \in X$ ;
- **Output:** "yes" or "no" depending on  $x\alpha = y$  for some  $\alpha \in A$ .

# Definition

For X,  $A \subseteq Map(X, X)$ , the A-orbit of  $x \in X$  is  $\mathcal{O}(x) = \{x\alpha \mid \alpha \in A\}$ .

# Observation

O.D. is membership in A-orbits.

#### Observation

The conjugacy problem for group G, CP(G), is just the O.D. for  $A = \text{lnn}(G) = \{\gamma_g \colon G \to G, x \mapsto g^{-1}xg \mid g \in G\} \trianglelefteq Aut(G).$ 

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>F<sub>n</sub>-</i> by- <i>F<sub>m</sub></i> 00000	4. Main result ●000000	5. Applications	6. Negative results
Orbit dec	dability				

Let X be a set. A collection of maps  $A \subseteq Map(X, X)$  is said to be orbit decidable (O.D.) if there is an algorithm A with:

- Input: two elements  $x, y \in X$ ;
- **Output:** "yes" or "no" depending on  $x\alpha = y$  for some  $\alpha \in A$ .

### Definition

For X,  $A \subseteq Map(X, X)$ , the A-orbit of  $x \in X$  is  $\mathcal{O}(x) = \{x \alpha \mid \alpha \in A\}$ .

# Observation

O.D. is membership in A-orbits.

#### Observation

The conjugacy problem for group G, CP(G), is just the O.D. for  $A = Inn(G) = \{\gamma_g : G \to G, x \mapsto g^{-1}xg \mid g \in G\} \trianglelefteq Aut(G).$ 

0000	0000000000	00000	000000	000000000	00000000000			
Short exact sequences								

# Observation

(i) For  $\varphi \in Aut(F_n)$ , we have the natural short exact sequence:

(ii) For φ<sub>1</sub>,..., φ<sub>m</sub> ∈ Aut(F<sub>n</sub>), we have the natural short exact sequence:

(iii) And their action subgroups are, respectively,  $\langle \varphi \rangle \leq Out(F_n)$  and  $\langle \varphi_1, \ldots, \varphi_m \rangle \leq Out(F_n)$ .

	0000000000	00000	000000	000000000	00000000000				
Short exact sequences									

### Observation

(i) For  $\varphi \in Aut(F_n)$ , we have the natural short exact sequence:

(ii) For  $\varphi_1, \ldots, \varphi_m \in Aut(F_n)$ , we have the natural short exact sequence:

(iii) And their action subgroups are, respectively,  $\langle \varphi \rangle \leq Out(F_n)$  and  $\langle \varphi_1, \ldots, \varphi_m \rangle \leq Out(F_n)$ .

Chart avaat aaguanaaa							
			000000				
<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-Z</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-F<sub>m</sub></li> </ol>	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>		

# Short exact sequences

# Observation

(i) For  $\varphi \in Aut(F_n)$ , we have the natural short exact sequence:

(ii) For  $\varphi_1, \ldots, \varphi_m \in Aut(F_n)$ , we have the natural short exact sequence:

(iii) And their action subgroups are, respectively,  $\langle \varphi \rangle \leq Out(F_n)$  and  $\langle \varphi_1, \ldots, \varphi_m \rangle \leq Out(F_n)$ .

# Short exact sequences

# Definition

Consider an arbitrary short exact sequence of groups,

 $1 \rightarrow F \rightarrow G \rightarrow H \rightarrow 1$ .

Given  $g \in G$ , consider  $\gamma_g \colon G \to G$ , which restricts to an automorphism  $\gamma_g|_F \colon F \to F$ . Then, the action subgroup of the short exact sequence is:

 $A = \{\gamma_g|_F \mid g \in G\} \leqslant Aut(F)$ 

▲□▶ ▲□▶ ▲□▶ ▲□▶ □ のQで

To solve  $CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$  we have needed:

- $TCP(F_n)$ ,
- orbit decidability of  $\langle \varphi_1, \ldots, \varphi_m \rangle \in \operatorname{Aut}(F_n)$ ,
- computability up and down the short exact sequence.

・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・

These conditions (plus two more) will suffice ...

Chart av		00000	0000000	000000000	00000000000				
Short exact sequences									

To solve  $CP(F_n \rtimes_{\varphi_1,...,\varphi_m} F_m)$  we have needed:

•  $TCP(F_n)$ ,

• orbit decidability of  $\langle \varphi_1, \ldots, \varphi_m \rangle \in \operatorname{Aut}(F_n)$ ,

• computability up and down the short exact sequence.

・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・

These conditions (plus two more) will suffice ...

Short ov		00000	0000000	000000000	00000000000				
Short exact sequences									

To solve  $CP(F_n \rtimes_{\varphi_1,...,\varphi_m} F_m)$  we have needed:

•  $TCP(F_n)$ ,

• orbit decidability of  $\langle \varphi_1, \ldots, \varphi_m \rangle \in Aut(F_n)$ ,

• computability up and down the short exact sequence.

< □ > < 同 > < 三 > < 三 > < 三 > < ○ < ○ </p>

These conditions (plus two more) will suffice ...

Short ov		00000	0000000	000000000	00000000000				
Short exact sequences									

To solve  $CP(F_n \rtimes_{\varphi_1,...,\varphi_m} F_m)$  we have needed:

- $TCP(F_n)$ ,
- orbit decidability of  $\langle \varphi_1, \ldots, \varphi_m \rangle \in Aut(F_n)$ ,
- computability up and down the short exact sequence.

・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・

These conditions (plus two more) will suffice ....

1. Historical context	2. CP for <i>F⊓</i> -by-ℤ 000000000	3. CP for <i>F<sub>n</sub>-</i> by- <i>F<sub>m</sub></i>	4. Main result 000€000	5. Applications	6. Negative results			
Short exact sequences								

To solve  $CP(F_n \rtimes_{\varphi_1,...,\varphi_m} F_m)$  we have needed:

- $TCP(F_n)$ ,
- orbit decidability of  $\langle \varphi_1, \ldots, \varphi_m \rangle \in Aut(F_n)$ ,
- computability up and down the short exact sequence.

< □ > < 同 > < 三 > < 三 > < 三 > < ○ < ○ </p>

These conditions (plus two more) will suffice ....

1. Historio		2. CP for <i>F⊓</i> -by-ℤ 0000000000	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result 0000●00	5. Applications	6. Negative results					
The main result											
6	Theorem	(Boaopolski-	Martino-V., 200	)8)							
	Let										
	$1 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$ be an algorithmic short exact sequence of groups such that										
		P(F) is solvab									
		H) is solvable e is an algorit	e, hm which, give	n an innut 1	$\neq$ h $\in$ H con	nnutes					
			nents $z_{h,1}, \ldots,$			ipates					
			$C_H(h) = \langle h \rangle z_{h,1}$	$\Box \cdots \sqcup \langle h \rangle z$							
	Then,										
				$a : F \rightarrow I$		)					

CP(G) is solvable  $\iff$ 

 $A_G = \left\{ egin{array}{ccc} \gamma_g \colon F & o & F \ x & \mapsto & g^{-1}xg \end{array} \middle| g \in G 
ight\} lpha$ 

 $\leq$  Aut(F) is orbit decidable.

1. Histori 0000		2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result 0000●00	5. Applications	6. Negative results					
The main result											
	Thoorom	(Bogopolski-l	Martino-V., 200	18)		_					
			Martino-v., 200	00)							
	Let	1	$\longrightarrow F \stackrel{lpha}{\longrightarrow} G$ -	$\xrightarrow{\beta} H \longrightarrow 1$							
	be an algorithmic short exact sequence of groups such that										
	(i) <i>TCF</i>	P(F) is solvabl	<i>e</i> ,								
		H) is solvable									
			hm which, give hents z <sub>h,1</sub> ,,			nputes					
			$C_H(h) = \langle h \rangle Z_{h,1}$	$\sqcup \cdots \sqcup \langle h \rangle z$							
	Then,										
			()	$a: F \rightarrow h$							

CP(G) is solvable  $\Leftarrow$ 

 $\mathbf{h}_{G} = \left\{ egin{array}{ccc} \gamma_{g} \colon \mathbf{r} & 
ightarrow \mathbf{r} \ \mathbf{x} & \mapsto & g^{-1} \mathbf{x} g \end{array} \middle| egin{array}{ccc} g \in G \ \end{array} 
ight\}$ 

 $\leq$  Aut(F) is orbit decidable.

1. Histor 0000		2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○	3. CP for F <sub>n</sub> -by-F <sub>m</sub>	4. Main result 0000●00	5. Applications	6. Negative results
Th	e mai	n result				
	Theore	m (Bogopolski-I	Martino-V., 200	)8)		
	Let					
		1	$\longrightarrow F \xrightarrow{\alpha} G$ -	$\xrightarrow{r} H \longrightarrow 1$		
		algorithmic shor		ce of groups	such that	
		CP(F) is solvabl				
		P(H) is solvable				
		ere is an algorith inite set of elem				putes
			$C_H(h) = \langle h \rangle Z_{h,1}$	$\Box \cdots \Box \langle h \rangle z$		
	Then,					

$$A_G = \left\{ egin{array}{ccc} \gamma_g \colon F & o & F \ x & \mapsto & g^{-1}xg \end{array} \middle| g \in G 
ight\} \leqslant$$

CP(G) is solvable  $\iff$ 

 $\leq$  Aut(F) is orbit decidable.

1. Historical context	2. CP for <i>Fn</i> -by-ℤ 0000000000	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result 0000●00	5. Applications	6. Negative results
The main	result				

Theorem (Bogopolski-Martino-V., 2008)

Let

$$1 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$$

be an algorithmic short exact sequence of groups such that

- (i) TCP(F) is solvable,
- (ii) CP(H) is solvable,
- (iii) there is an algorithm which, given an input  $1 \neq h \in H$ , computes a finite set of elements  $z_{h,1}, \ldots, z_{h,t_h} \in H$  such that

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

Then,

$$A_G = \left\{ egin{array}{ccc} \gamma_g \colon F & o & F \ x & \mapsto & g^{-1}xg \end{array} \middle| g \in G 
ight\} \leqslant$$

CP(G) is solvable  $\iff$ 

 $\leq$  Aut(F) is orbit decidable.

1. Historical context	2. CP for <i>F<sub>n</sub>-by-</i> Z	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result 0000●00	5. Applications	6. Negative results
The mair	n result				
Theorem	n (Rogonolski-	Martino-V 200	18)		

Let

$$1 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$$

be an algorithmic short exact sequence of groups such that

- (i) TCP(F) is solvable,
- (ii) CP(H) is solvable,
- (iii) there is an algorithm which, given an input  $1 \neq h \in H$ , computes a finite set of elements  $z_{h,1}, \ldots, z_{h,t_h} \in H$  such that

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

Then.

$$A_{G} = \left\{ \begin{array}{ccc} \gamma_{g} \colon F & \to & F \\ x & \mapsto & g^{-1}xg \end{array} \middle| g \in G \right\} \leqslant$$

CP(G) is solvable  $\iff$ 

 $\leq$  Aut(F) is orbit decidable.

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>F<sub>n</sub>-by-F<sub>m</sub></i> 00000	4. Main result 00000●0	5. Applications	6. Negative results
The mair	n result				

# Proposition (Bogopolski-Martino-V., 2008)

*Torsion-free hyperbolic groups (in particular, free groups) satisfy hypothesis (ii) and (iii).* 

◆□▶ ◆□▶ ▲□▶ ▲□▶ □ のQ@

So, they all fit well as *H*.

- 1. Historical context
   2. CP for Fn-by-Z
   3. CP for Fn-by-Fm
   4. Main result
   5. Applications
   6. Negative results

   0000
   00000000
   0000000
   00000000
   00000000
   00000000
   000000000
  - O. Bogopolski, A. Martino, O. Maslakova, E. Ventura, Freeby-cyclic groups have solvable conjugacy problem, **Bulletin of the London Mathematical Society**, 38(5) (2006), 787–794.
  - O. Bogopolski, A. Martino, E. Ventura, Orbit decidability and the conjugacy problem for extensions of groups, **Transactions of the American Mathematical Society** 362 (2010), 2003–2036.
  - V. Romanko'v, E. Ventura, Twisted conjugacy problem for endomorphisms of metabelian groups, **Algebra and Logic** 48(2) (2009), 89–98. Translation from **Algebra i Logika** 48(2) (2009), 157–173.
  - J. González-Meneses, E. Ventura, Twisted conjugacy in the braid group, Israel Journal of Mathematics 201 (2014), 455–476.
  - J. Burillo, F. Matucci, E. Ventura, The conjugacy problem for extensions of Thompson's group, to appear at **Israel Journal of Mathematics**.
  - Z. Sŭnic, E. Ventura, The conjugacy problem in automaton groups is not solvable, Journal of Algebra 364 (2012), 148–154.
  - E. Ventura, Group theoretic orbit decidability, **Groups, Complexity, Cryptology** 6(2) (2014), 133–148.

o ...

1. Historical context	2. CP for <i>F⊓</i> -by-ℤ 0000000000	3. CP for <i>F<sub>n</sub>-</i> by- <i>F<sub>m</sub></i>	4. Main result	5. Applications	6. Negative results
Outline					

- The historical context
- The conjugacy problem for free-by-cyclic groups
- The conjugacy problem for free-by-free groups
- 4 The main result
- 5 Applications
- 6 Negative results

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 00000000
 00000000
 000000000
 000000000

 Free-by-free groups

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.

Theorem (Brinkmann, 2006)

Cyclic subgroups of  $Aut(F_n)$  are O.D.

Corollary (Bogopolski–Martino–Maslakova–V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead '36)

The full  $Aut(F_n)$  is O.D.

Corollary (Bogopolski–Martino–V., 2008)

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 00000000
 00000000
 00000000
 00000000
 000000000

 Free-by-free groups

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.

### • • •

Theorem (Brinkmann, 2006)

Cyclic subgroups of  $Aut(F_n)$  are O.D.

Corollary (Bogopolski–Martino–Maslakova–V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead '36)

The full  $Aut(F_n)$  is O.D.

Corollary (Bogopolski–Martino–V., 2008)

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 00000000
 00000000
 000000000
 000000000

 Free-by-free groups

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.



Theorem (Brinkmann, 2006)

Cyclic subgroups of  $Aut(F_n)$  are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead '36)

The full  $Aut(F_n)$  is O.D.

Corollary (Bogopolski–Martino–V., 2008)

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 00000000
 00000000
 00000000
 000000000

 Free-by-free groups

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.



Theorem (Brinkmann, 2006)

Cyclic subgroups of  $Aut(F_n)$  are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead '36)

The full  $Aut(F_n)$  is O.D.

Corollary (Bogopolski–Martino–V., 2008)

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 00000000
 00000000
 00000000
 000000000

 Free-by-free groups

Theorem (Bogopolski–Martino–Maslakova–V., 2005)

 $TCP(F_n)$  is solvable.



Theorem (Brinkmann, 2006)

Cyclic subgroups of  $Aut(F_n)$  are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead '36)

The full  $Aut(F_n)$  is O.D.

Corollary (Bogopolski-Martino-V., 2008)

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications ○●○○○○○○○	6. Negative results
Free-by-t	free group	S			

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of  $Aut(F_n)$  are O.D.

Corollary (Bogopolski–Martino–V., 2008)

If  $\langle \varphi_1, \ldots, \varphi_m \rangle$  is of finite index in  $Aut(F_n)$  then  $CP(F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m)$  is solvable.

Theorem (Bogopolski–Martino–V., 2008)

Every finitely generated subgroup of  $Aut(F_2)$  is O.D.

Corollary (Bogopolski–Martino–V., 2008) Every E. by free group has solvable conjugacy n

◆□▶ ◆□▶ ◆三▶ ◆三▶ ◆□ ◆ ○ ◆ ○

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○○○○○○○○○○	3. CP for <i>F<sub>n</sub>-by-F<sub>m</sub></i> 00000	4. Main result	5. Applications	6. Negative results
Free-by-	free group	S			

Theorem (Bogopolski–Martino–V., 2008)

Finite index subgroups of  $Aut(F_n)$  are O.D.

# Corollary (Bogopolski–Martino–V., 2008)

If  $\langle \varphi_1, \ldots, \varphi_m \rangle$  is of finite index in  $Aut(F_n)$  then  $CP(F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m)$  is solvable.

・ コット (雪) ( 小田) ( コット 日)

Theorem (Bogopolski–Martino–V., 2008)

Every finitely generated subgroup of  $Aut(F_2)$  is O.D.

Corollary (Bogopolski–Martino–V., 2008)

Every F<sub>2</sub>-by-free group has solvable conjugacy problem.

1. Historical context	2. CP for <i>F<sub>n</sub>-by-</i> ℤ ○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications ○●○○○○○○○	6. Negative results
Free-by-1	free group	S			

Theorem (Bogopolski–Martino–V., 2008)

Finite index subgroups of  $Aut(F_n)$  are O.D.

Corollary (Bogopolski–Martino–V., 2008)

If  $\langle \varphi_1, \ldots, \varphi_m \rangle$  is of finite index in  $Aut(F_n)$  then  $CP(F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m)$  is solvable.

Theorem (Bogopolski–Martino–V., 2008)

Every finitely generated subgroup of  $Aut(F_2)$  is O.D.

Corollary (Bogopolski–Martino–V., 2008) *Every F<sub>2</sub>-by-free group has solvable conjugacy proble* 

◆□ ▶ ◆□ ▶ ◆ □ ▶ ◆ □ ▶ ● ○ ● ● ● ●

1. Historical context	2. CP for <i>F∩</i> -by-ℤ ○○○○○○○○○	3. CP for <i>F<sub>n</sub>-by-F<sub>m</sub></i> 00000	4. Main result	5. Applications	6. Negative results
Free-by-	free group	S			

Theorem (Bogopolski–Martino–V., 2008)

Finite index subgroups of  $Aut(F_n)$  are O.D.

Corollary (Bogopolski–Martino–V., 2008)

If  $\langle \varphi_1, \ldots, \varphi_m \rangle$  is of finite index in  $Aut(F_n)$  then  $CP(F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m)$  is solvable.

Theorem (Bogopolski–Martino–V., 2008)

Every finitely generated subgroup of  $Aut(F_2)$  is O.D.

Corollary (Bogopolski–Martino–V., 2008)

Every  $F_2$ -by-free group has solvable conjugacy problem.

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \rtimes_{M_1, \dots, M_m} F_m \longrightarrow F_m \longrightarrow 1$$

Observation (linear algebra)  $TCP(\mathbb{Z}^n)$  is solvable.

So,

 $CP(\mathbb{Z}^n \rtimes_{M_1,...,M_m} F_m)$  is solvable  $\Leftrightarrow \langle M_1,...,M_m \rangle \leqslant GL_n(\mathbb{Z})$  is O.D.

Observe that now  $M_i \in Aut(\mathbb{Z}^n) = GL_n(\mathbb{Z})$  are just  $n \times n$  invertible matrices.

• • •

▲□▶ ▲□▶ ▲□▶ ▲□▶ □ のQで

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \rtimes_{M_1, \dots, M_m} F_m \longrightarrow F_m \longrightarrow 1$$



So

 $CP(\mathbb{Z}^n \rtimes_{M_1,...,M_m} F_m)$  is solvable  $\Leftrightarrow \langle M_1,...,M_m \rangle \leqslant GL_n(\mathbb{Z})$  is O.D.

Observe that now  $M_i \in Aut(\mathbb{Z}^n) = GL_n(\mathbb{Z})$  are just  $n \times n$  invertible matrices.

• • •

▲□▶ ▲□▶ ▲□▶ ▲□▶ □ のQで

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \rtimes_{M_1, \dots, M_m} F_m \longrightarrow F_m \longrightarrow 1$$

Observation (linear algebra)  $TCP(\mathbb{Z}^n)$  is solvable.

# So,

 $CP(\mathbb{Z}^n \rtimes_{M_1,...,M_m} F_m)$  is solvable  $\Leftrightarrow \langle M_1,...,M_m \rangle \leqslant GL_n(\mathbb{Z})$  is O.D.

Observe that now  $M_i \in Aut(\mathbb{Z}^n) = GL_n(\mathbb{Z})$  are just  $n \times n$  invertible matrices.

• • •

▲□▶ ▲□▶ ▲□▶ ▲□▶ □ のQで

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \rtimes_{M_1, \dots, M_m} F_m \longrightarrow F_m \longrightarrow 1$$

Observation (linear algebra)  $TCP(\mathbb{Z}^n)$  is solvable.

# So,

 $CP(\mathbb{Z}^n \rtimes_{M_1,...,M_m} F_m)$  is solvable  $\Leftrightarrow \langle M_1,...,M_m \rangle \leqslant GL_n(\mathbb{Z})$  is O.D.

Observe that now  $M_i \in Aut(\mathbb{Z}^n) = GL_n(\mathbb{Z})$  are just  $n \times n$  invertible matrices.

• • •

・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \rtimes_{M_1, \dots, M_m} F_m \longrightarrow F_m \longrightarrow 1$$

Observation (linear algebra)  $TCP(\mathbb{Z}^n)$  is solvable.

# So,

 $CP(\mathbb{Z}^n \rtimes_{M_1,...,M_m} F_m)$  is solvable  $\Leftrightarrow \langle M_1,...,M_m \rangle \leqslant GL_n(\mathbb{Z})$  is O.D.

Observe that now  $M_i \in Aut(\mathbb{Z}^n) = GL_n(\mathbb{Z})$  are just  $n \times n$  invertible matrices.

• • •

・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>Fn</i> -by- <i>Fm</i>	4. Main result	5. Applications	6. Negative results

# (Free abelian)-by-free groups

# Theorem (Kannan–Lipton '86)

*Cyclic subgroups of*  $GL_n(\mathbb{Z})$  *are O.D.* 

Corollary (Remeslennikov '69

 $\mathbb{Z}^n$ -by- $\mathbb{Z}$  groups have solvable conjugacy problem.

**Observation** (elementary)

The full  $GL_n(\mathbb{Z})$  is O.D.

Corollary (Bogopolski-Martino-V., 2008)

If  $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$  then  $\mathbb{Z}^n \rtimes_{M_1, \ldots, M_m} F_m$  has solvable conjugacy problem.

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ● ●

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results
(Free ab	elian)-by-f	ree groups	;		

Theorem (Kannan–Lipton '86)

*Cyclic subgroups of*  $GL_n(\mathbb{Z})$  *are O.D.* 

Corollary (Remeslennikov '69)

 $\mathbb{Z}^n$ -by- $\mathbb{Z}$  groups have solvable conjugacy problem.

**Observation** (elementary)

The full  $GL_n(\mathbb{Z})$  is O.D.

Corollary (Bogopolski–Martino–V., 2008)

If  $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$  then  $\mathbb{Z}^n \rtimes_{M_1, \ldots, M_m} F_m$  has solvable conjugacy problem.

・ロト・日本・日本・日本・日本・日本

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>Fn</i> -by- <i>Fm</i>	4. Main result	5. Applications	6. Negative results

# (Free abelian)-by-free groups

Theorem (Kannan–Lipton '86)

*Cyclic subgroups of*  $GL_n(\mathbb{Z})$  *are O.D.* 

Corollary (Remeslennikov '69)

 $\mathbb{Z}^n$ -by- $\mathbb{Z}$  groups have solvable conjugacy problem.

**Observation** (elementary)

The full  $GL_n(\mathbb{Z})$  is O.D.

Corollary (Bogopolski–Martino–V., 2008)

If  $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$  then  $\mathbb{Z}^n \rtimes_{M_1, \ldots, M_m} F_m$  has solvable conjugacy problem.

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ● ●

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○○○○○○○○○○	3. CP for <i>F<sub>n</sub>-</i> by- <i>F<sub>m</sub></i> 00000	4. Main result	5. Applications 000●00000	6. Negative results
(Free ab	elian)-by-f	ree groups	;		

Theorem (Kannan–Lipton '86)

*Cyclic subgroups of*  $GL_n(\mathbb{Z})$  *are O.D.* 

Corollary (Remeslennikov '69)

 $\mathbb{Z}^n$ -by- $\mathbb{Z}$  groups have solvable conjugacy problem.

Observation (elementary)

The full  $GL_n(\mathbb{Z})$  is O.D.

Corollary (Bogopolski-Martino-V., 2008)

If  $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$  then  $\mathbb{Z}^n \rtimes_{M_1, \ldots, M_m} F_m$  has solvable conjugacy problem.

(Free abelian)-by-free groups

Theorem (Bogopolski–Martino–V., 2008)

Finite index subgroups of  $GL_n(\mathbb{Z})$  are O.D.

Corollary (Bogopolski–Martino–V., 2008)

If  $\langle M_1, \ldots, M_m \rangle$  is of finite index in  $GL_n(\mathbb{Z})$  then  $\mathbb{Z}^n \rtimes_{M_1, \ldots, M_m} F_m$  has solvable conjugacy problem.

・ ロ ト ・ 雪 ト ・ 雪 ト ・ 日 ト

Theorem (Bogopolski–Martino–V., 2008)

Every finitely generated subgroup of  $GL_2(\mathbb{Z})$  is O.D.

Corollary (Bogopolski–Martino–V., 2008)

(Free abelian)-by-free groups

Theorem (Bogopolski–Martino–V., 2008)

Finite index subgroups of  $GL_n(\mathbb{Z})$  are O.D.

Corollary (Bogopolski–Martino–V., 2008)

If  $\langle M_1, \ldots, M_m \rangle$  is of finite index in  $GL_n(\mathbb{Z})$  then  $\mathbb{Z}^n \rtimes_{M_1, \ldots, M_m} F_m$  has solvable conjugacy problem.

・ コット (雪) ( 小田) ( コット 日)

Theorem (Bogopolski–Martino–V., 2008)

Every finitely generated subgroup of  $GL_2(\mathbb{Z})$  is O.D.

Corollary (Bogopolski–Martino–V., 2008)

(Free abelian)-by-free groups

Theorem (Bogopolski–Martino–V., 2008)

Finite index subgroups of  $GL_n(\mathbb{Z})$  are O.D.

Corollary (Bogopolski–Martino–V., 2008)

If  $\langle M_1, \ldots, M_m \rangle$  is of finite index in  $GL_n(\mathbb{Z})$  then  $\mathbb{Z}^n \rtimes_{M_1, \ldots, M_m} F_m$  has solvable conjugacy problem.

◆□▶ ◆□▶ ▲□▶ ▲□▶ ■ ののの

Theorem (Bogopolski–Martino–V., 2008)

Every finitely generated subgroup of  $GL_2(\mathbb{Z})$  is O.D.

Corollary (Bogopolski–Martino–V., 2008)

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 00000
 00000
 00000
 00000
 0000000000

 (Free abelian)-by-free groups
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0

Theorem (Bogopolski–Martino–V., 2008)

Finite index subgroups of  $GL_n(\mathbb{Z})$  are O.D.

Corollary (Bogopolski–Martino–V., 2008)

If  $\langle M_1, \ldots, M_m \rangle$  is of finite index in  $GL_n(\mathbb{Z})$  then  $\mathbb{Z}^n \rtimes_{M_1, \ldots, M_m} F_m$  has solvable conjugacy problem.

Theorem (Bogopolski–Martino–V., 2008)

Every finitely generated subgroup of  $GL_2(\mathbb{Z})$  is O.D.

Corollary (Bogopolski-Martino-V., 2008)

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 00000
 000000000
 0000000000
 0000000000

 Braid-by-free groups
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0

Consider the braid group on *n* strands, given by the classical presentation:

$$B_n = \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \middle| \begin{array}{c} \sigma_i \sigma_j = \sigma_j \sigma_i & (|i-j| \ge 2) \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & (1 \le i \le n-2) \end{array} \right\rangle.$$

- $CP(B_n)$  is solvable.
- And the automorphism group is easy:

### Theorem (Dyer–Grossman '81)

 $|Out(B_n)| = 2$ ; more precisely,  $Aut(B_n) = Inn(B_n) \sqcup Inn(B_n) \cdot \varepsilon$ , where  $\varepsilon \colon B_n \to B_n$  is the automorphism which inverts all generators,  $\sigma_i \mapsto \sigma_i^{-1}$ .

◆□▶ ◆□▶ ▲□▶ ▲□▶ ■ ののの

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 00000
 000000000
 0000000000
 0000000000

 Braid-by-free groups
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0

Consider the braid group on *n* strands, given by the classical presentation:

$$B_n = \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \middle| \begin{array}{c} \sigma_i \sigma_j = \sigma_j \sigma_i & (|i-j| \ge 2) \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & (1 \le i \le n-2) \end{array} \right\rangle.$$

•  $CP(B_n)$  is solvable.

• And the automorphism group is easy:

### Theorem (Dyer–Grossman '81)

 $|Out(B_n)| = 2$ ; more precisely,  $Aut(B_n) = Inn(B_n) \sqcup Inn(B_n) \cdot \varepsilon$ , where  $\varepsilon \colon B_n \to B_n$  is the automorphism which inverts all generators,  $\sigma_i \mapsto \sigma_i^{-1}$ .

◆□▶ ◆□▶ ▲□▶ ▲□▶ ■ ののの

 1. Historical context
 2. CP for F<sub>n</sub>-by-Z
 3. CP for F<sub>n</sub>-by-F<sub>m</sub>
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 00000
 000000
 000000
 000000000

 Braid-by-free groups
 6. Negative results
 000000
 0000000000
 0000000000

Consider the braid group on *n* strands, given by the classical presentation:

$$B_n = \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \middle| \begin{array}{c} \sigma_i \sigma_j = \sigma_j \sigma_i & (|i-j| \ge 2) \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & (1 \le i \le n-2) \end{array} \right\rangle.$$

- $CP(B_n)$  is solvable.
- And the automorphism group is easy:

### Theorem (Dyer–Grossman '81)

 $|Out(B_n)| = 2$ ; more precisely,  $Aut(B_n) = Inn(B_n) \sqcup Inn(B_n) \cdot \varepsilon$ , where  $\varepsilon \colon B_n \to B_n$  is the automorphism which inverts all generators,  $\sigma_i \mapsto \sigma_i^{-1}$ .

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results		
Braid-by-free groups							

Theorem (González-Meneses–V. 2009)

 $TCP(B_n)$  is solvable.

#### • • •

Observation

Every subgroup  $A \leq Aut(B_n)$  is orbit decidable.

Corollary (González-Meneses–V. 2009)

Every extension of  $B_n$  by a torsion-free hyperbolic group has solvable conjugacy problem.

(日) (日) (日) (日) (日) (日) (日)

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results		
Braid-by-free groups							

Theorem (González-Meneses–V. 2009)

 $TCP(B_n)$  is solvable.

#### • • •

Observation

Every subgroup  $A \leq Aut(B_n)$  is orbit decidable.

Corollary (González-Meneses–V. 2009)

Every extension of  $B_n$  by a torsion-free hyperbolic group has solvable conjugacy problem.

(日) (日) (日) (日) (日) (日) (日)

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results		
Braid-by-free groups							

Theorem (González-Meneses–V. 2009)

 $TCP(B_n)$  is solvable.

#### • • •

Observation

Every subgroup  $A \leq Aut(B_n)$  is orbit decidable.

Corollary (González-Meneses-V. 2009)

Every extension of  $B_n$  by a torsion-free hyperbolic group has solvable conjugacy problem.

<b>T</b> L	he has free				
<ol> <li>Historical context</li> <li>0000</li> </ol>	2. CP for <i>F<sub>Π</sub>-</i> by-ℤ 0000000000	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result	<ol> <li>Applications</li> <li>0000000●0</li> </ol>	6. Negative results 00000000000

# Thompson-by-free groups

Consider Thompson's group F:

$$F = \begin{cases} -\text{increasing and piecewise linear,} \\ f: [0,1] \to [0,1] \mid f \text{ -with finitely many dyadic breakpoints,} \\ -\text{slopes being powers of 2.} \end{cases}$$

# • CP(F) is solvable.

• And the automorphism group is big, but easy:

### Theorem (Brin '97)

For every  $\varphi \in Aut(F)$ , there exists  $\tau \in EP_2$  such that  $\varphi(g) = \tau^{-1}g\tau$ , for every  $g \in F$ .

◆□▶ ◆□▶ ▲□▶ ▲□▶ ■ ののの

$$F \trianglelefteq EP_2 = igg\{ f \colon \mathbb{R} o \mathbb{R} \mid igg| egin{array}{c} f ext{ is p.l., dyadic bkp., slopes } 2^n \ eventually periodic \end{array} igg]$$

1. Historical context	2. CP for <i>F<sub>n</sub>-by-</i> Z	3. CP for F <sub>n</sub> -by-F <sub>m</sub>	4. Main result	5. Applications 0000000000	6. Negative results	
Thompson by free groups						

# I hompson-by-free groups

Consider Thompson's group F:

$$F = \begin{cases} -\text{increasing and piecewise linear,} \\ f: [0,1] \to [0,1] \mid f \text{ -with finitely many dyadic breakpoints,} \\ -\text{slopes being powers of 2.} \end{cases}$$

- CP(F) is solvable.
- And the automorphism group is big, but easy:

# Theorem (Brin '97)

For every  $\varphi \in Aut(F)$ , there exists  $\tau \in EP_2$  such that  $\varphi(g) = \tau^{-1}g\tau$ , for every  $g \in F$ .

 $F \trianglelefteq EP_2 = \left\{ f \colon \mathbb{R} \to \mathbb{R} \mid \begin{array}{c} f \text{ is p.l., dyadic bkp., slopes } 2^n \\ eventually periodic \end{array} \right\}.$ 

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results
Thompso	on-by-free	groups			

#### Theorem (Burillo-Matucci-V. 2010)

TCP(F) is solvable.

#### • • •

#### Conjecture

k - CP(F) (i.e., conjugacy problem for k-tuples) is solvable.

#### Proposition (Burillo–Matucci–V. 2010)

If conjecture is true then Aut(F) and  $Aut^+(F)$  are orbit decidable.

#### Corollary (Burillo–Matucci–V. 2010)

If conjecture is true and  $\varphi_1, \ldots, \varphi_m \in Aut(F)$  generate either Aut(F) or  $Aut^+(F)$ , then  $CP(F \rtimes_{\varphi_1, \ldots, \varphi_m} F_m)$  is solvable.

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications 00000000●	6. Negative results
Thompso	on-by-free	groups			

### Theorem (Burillo-Matucci-V. 2010)

TCP(F) is solvable.



### Conjecture

k - CP(F) (i.e., conjugacy problem for k-tuples) is solvable.

#### Proposition (Burillo–Matucci–V. 2010)

If conjecture is true then Aut(F) and  $Aut^+(F)$  are orbit decidable.

#### Corollary (Burillo-Matucci-V. 2010)

If conjecture is true and  $\varphi_1, \ldots, \varphi_m \in Aut(F)$  generate either Aut(F) or  $Aut^+(F)$ , then  $CP(F \rtimes_{\varphi_1, \ldots, \varphi_m} F_m)$  is solvable.

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result	5. Applications 00000000●	6. Negative results
Thompso	on-by-free	groups			

### Theorem (Burillo-Matucci-V. 2010)

TCP(F) is solvable.



#### Conjecture

k - CP(F) (i.e., conjugacy problem for k-tuples) is solvable.

#### Proposition (Burillo–Matucci–V. 2010)

If conjecture is true then Aut(F) and  $Aut^+(F)$  are orbit decidable.

#### Corollary (Burillo-Matucci-V. 2010)

If conjecture is true and  $\varphi_1, \ldots, \varphi_m \in Aut(F)$  generate either Aut(F) or  $Aut^+(F)$ , then  $CP(F \rtimes_{\varphi_1, \ldots, \varphi_m} F_m)$  is solvable.

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>F<sub>n</sub>-by-F<sub>m</sub></i> 00000	4. Main result	5. Applications	6. Negative results
Outline					

- The historical context
- The conjugacy problem for free-by-cyclic groups
- The conjugacy problem for free-by-free groups
- 4 The main result
- 5 Applications
- 6 Negative results



1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results ●0000000000
Free-by-1	free negat	ive results			

#### Theorem (Miller '71)

There exist free-by-free groups with unsolvable conjugacy problem.

#### Corollary

There exist 14 automorphisms  $\varphi_1, \ldots, \varphi_{14} \in Aut(F_3)$  such that  $\langle \varphi_1, \ldots, \varphi_{14} \rangle \leq Aut(F_3)$  is orbit undecidable.

Moreover, we were able to find the reason and generalize it to Aut(F) for many more grups F.

◆□▶ ◆□▶ ◆□▶ ◆□▶ ● ● ● ●

 1. Historical context
 2. CP for F<sub>n</sub>-by-Z
 3. CP for F<sub>n</sub>-by-F<sub>m</sub>
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 00000000
 00000000
 000000000
 000000000

 Ereco
 by/
 froe
 pococitive
 recoults
 00000000
 000000000

## Free-by-free negative results

Theorem (Miller '71)

There exist free-by-free groups with unsolvable conjugacy problem.

### Corollary

There exist 14 automorphisms  $\varphi_1, \ldots, \varphi_{14} \in Aut(F_3)$  such that  $\langle \varphi_1, \ldots, \varphi_{14} \rangle \leq Aut(F_3)$  is orbit undecidable.

Moreover, we were able to find the reason and generalize it to Aut (*F*) for many more grups *F*.

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 00000
 000000000
 000000000
 000000000

 Free-by-free negative results

Theorem (Miller '71)

There exist free-by-free groups with unsolvable conjugacy problem.

#### Corollary

There exist 14 automorphisms  $\varphi_1, \ldots, \varphi_{14} \in Aut(F_3)$  such that  $\langle \varphi_1, \ldots, \varphi_{14} \rangle \leq Aut(F_3)$  is orbit undecidable.

Moreover, we were able to find the reason and generalize it to Aut(F) for many more grups F.

◆□▶ ◆□▶ ◆□▶ ◆□▶ ● ● ● ●

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-Z	3. CP for <i>F<sub>n</sub></i> -by- <i>F<sub>m</sub></i> 00000	4. Main result	5. Applications	6. Negative results
Finding of	orbit undeo	cidable sub	ogroups		

Let *F* be a group, and let  $A \leq B \leq Aut(F)$  and  $u \in F$  be such that  $B \cap Stab^*(u) = 1$ . Then, *A* is *O*.*D*.  $\Rightarrow$  *MP*(*A*, *B*) solvable.

**Proof.** Given  $\varphi \in B \leq Aut(F)$ , let  $w = u\varphi$  and

 $\{\phi \in B \mid u\phi \sim w\} = (B \cap Stab^*(u)) \cdot \varphi = \{\varphi\}.$ 

So, u can be mapped to a conjugate of w by some automorphism in A

 $\Leftrightarrow \quad \varphi \in A. \quad \Box$ 

MP(A, B) is unsolvable  $\Rightarrow A \leq Aut(F)$  is orbit undecidable.

▲□▶▲圖▶▲≣▶▲≣▶ ■ のへで

		cidable sub		00000000	
1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>F<sub>n</sub>-</i> by- <i>F<sub>m</sub></i>	4. Main result	5. Applications	<ol> <li>Negative results</li> <li>000000000000000000000000000000000000</li></ol>

Let *F* be a group, and let  $A \leq B \leq Aut(F)$  and  $u \in F$  be such that  $B \cap Stab^*(u) = 1$ . Then, *A* is *O*.*D*.  $\Rightarrow$  *MP*(*A*, *B*) solvable.

**Proof.** Given  $\varphi \in B \leq Aut(F)$ , let  $w = u\varphi$  and

 $\{\phi \in B \mid u\phi \sim w\} = (B \cap Stab^*(u)) \cdot \varphi = \{\varphi\}.$ 

So, u can be mapped to a conjugate of w by some automorphism in A

MP(A, B) is unsolvable  $\Rightarrow A \leq Aut(F)$  is orbit undecidable.

		cidable sub		00000000	
1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>F<sub>n</sub>-</i> by- <i>F<sub>m</sub></i>	4. Main result	5. Applications	<ol> <li>Negative results</li> <li>000000000000000000000000000000000000</li></ol>

Let *F* be a group, and let  $A \leq B \leq Aut(F)$  and  $u \in F$  be such that  $B \cap Stab^*(u) = 1$ . Then, *A* is *O*.*D*.  $\Rightarrow$  *MP*(*A*, *B*) solvable.

**Proof.** Given  $\varphi \in B \leq Aut(F)$ , let  $w = u\varphi$  and

 $\{\phi \in \boldsymbol{B} \mid \boldsymbol{u}\phi \sim \boldsymbol{w}\} = (\boldsymbol{B} \cap Stab^*(\boldsymbol{u})) \cdot \varphi = \{\varphi\}.$ 

So, u can be mapped to a conjugate of w by some automorphism in A

MP(A, B) is unsolvable  $\Rightarrow A \leq Aut(F)$  is orbit undecidable.

◆□▶ ◆□▶ ◆臣▶ ◆臣▶ ─臣 ─のへで

		cidable sub		00000000	
1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ 0000000000	3. CP for <i>F<sub>n</sub>-</i> by- <i>F<sub>m</sub></i>	4. Main result	5. Applications	<ol> <li>Negative results</li> <li>000000000000000000000000000000000000</li></ol>

Let F be a group, and let  $A \leq B \leq Aut(F)$  and  $u \in F$  be such that  $B \cap Stab^*(u) = 1$ . Then, A is O.D.  $\Rightarrow$  MP(A, B) solvable.

**Proof.** Given  $\varphi \in B \leq Aut(F)$ , let  $w = u\varphi$  and

 $\{\phi \in \boldsymbol{B} \mid \boldsymbol{u}\phi \sim \boldsymbol{w}\} = (\boldsymbol{B} \cap \boldsymbol{Stab}^*(\boldsymbol{u})) \cdot \varphi = \{\varphi\}.$ 

So, u can be mapped to a conjugate of w by some automorphism in A  $\Leftrightarrow \varphi \in$ 

MP(A, B) is unsolvable  $\Rightarrow A \leq Aut(F)$  is orbit undecidable.

◆□▶ ◆□▶ ◆□▶ ◆□▶ ● ● ● ●

Finding	orbit under	cidable sub	aroune		
1. Historical context	2. CP for <i>Fn</i> -by-ℤ 0000000000	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results ○●○○○○○○○○○

Let F be a group, and let  $A \leq B \leq Aut(F)$  and  $u \in F$  be such that  $B \cap Stab^*(u) = 1$ . Then, A is O.D.  $\Rightarrow$  MP(A, B) solvable.

**Proof.** Given  $\varphi \in B \leq Aut(F)$ , let  $w = u\varphi$  and

 $\{\phi \in B \mid u\phi \sim w\} = (B \cap Stab^*(u)) \cdot \varphi = \{\varphi\}.$ 

So, u can be mapped to a conjugate of w by some automorphism in A  $\Leftrightarrow \varphi \in A$ .  $\Box$ 

MP(A, B) is unsolvable  $\Rightarrow A \leq Aut(F)$  is orbit undecidable.

◆□ → ◆□ → ◆三 → ◆三 → ● ● ● ●

Finding	orbit under	cidable sub	aroune		
1. Historical context	2. CP for <i>Fn</i> -by-ℤ 0000000000	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results ○●○○○○○○○○○

Let F be a group, and let  $A \leq B \leq Aut(F)$  and  $u \in F$  be such that  $B \cap Stab^*(u) = 1$ . Then, A is O.D.  $\Rightarrow$  MP(A, B) solvable.

**Proof.** Given  $\varphi \in B \leq Aut(F)$ , let  $w = u\varphi$  and

 $\{\phi \in B \mid u\phi \sim w\} = (B \cap Stab^*(u)) \cdot \varphi = \{\varphi\}.$ 

So, u can be mapped to a conjugate of w by some automorphism in A  $\Leftrightarrow \varphi \in A$ .  $\Box$ 

MP(A, B) is unsolvable  $\Rightarrow A \leq Aut(F)$  is orbit undecidable.

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-Z 0000000000	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results 000000000000000000000000000000000000
Finding of	orbit unde	cidable sub	ogroups		

Let F be a group, and let  $F_2 \times F_2 \simeq B \leq Aut(F)$  and  $u \in F$  be such that  $B \cap Stab^*(u) = 1$ . Then, there exists f.g.  $A \leq Aut(F)$  which is orbit undecidable.

- Take a group U = (a<sub>1</sub>, a<sub>2</sub> | r<sub>1</sub>,..., r<sub>m</sub>) with unsolvable word problem;
- Consider  $A = \{(v, w) \mid v =_U w\} \leqslant F_2 \times F_2;$
- Easy to see that A = ⟨(a<sub>1</sub>, a<sub>1</sub>), (a<sub>2</sub>, a<sub>2</sub>), (r<sub>1</sub>, 1), ..., (r<sub>m</sub>, 1)⟩ so, A is finitely generated;
- $MP(A, F_2 \times F_2)$  is unsolvable;
- Hence,  $A \leq Aut(F)$  is orbit undecidable.

<b>Einding</b>	arbit undo	cidable sub	aroupe		
<ol> <li>Historical context</li> <li>0000</li> </ol>	2. CP for <i>Fn</i> -by-ℤ 0000000000	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result	5. Applications	<ol> <li>6. Negative results</li> <li>○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○</li></ol>

Let F be a group, and let  $F_2 \times F_2 \simeq B \leq Aut(F)$  and  $u \in F$  be such that  $B \cap Stab^*(u) = 1$ . Then, there exists f.g.  $A \leq Aut(F)$  which is orbit undecidable.

- Take a group U = (a<sub>1</sub>, a<sub>2</sub> | r<sub>1</sub>,..., r<sub>m</sub>) with unsolvable word problem;
- Consider  $A = \{(v, w) \mid v =_U w\} \leq F_2 \times F_2;$
- Easy to see that A = ⟨(a<sub>1</sub>, a<sub>1</sub>), (a<sub>2</sub>, a<sub>2</sub>), (r<sub>1</sub>, 1), ..., (r<sub>m</sub>, 1)⟩ so, A is finitely generated;
- $MP(A, F_2 \times F_2)$  is unsolvable;
- Hence,  $A \leq Aut(F)$  is orbit undecidable.

0000	000000000	cidable sub	0000000	000000000	0000000000
1. Historical context	2. CP for <i>F</i> <sub>Π</sub> -by-ℤ	3. CP for Fn-by-Fm	4. Main result	5. Applications	6. Negative results

Let F be a group, and let  $F_2 \times F_2 \simeq B \leq Aut(F)$  and  $u \in F$  be such that  $B \cap Stab^*(u) = 1$ . Then, there exists f.g.  $A \leq Aut(F)$  which is orbit undecidable.

- Take a group U = (a<sub>1</sub>, a<sub>2</sub> | r<sub>1</sub>,..., r<sub>m</sub>) with unsolvable word problem;
- Consider  $A = \{(v, w) \mid v =_U w\} \leq F_2 \times F_2;$
- Easy to see that A = ⟨(a<sub>1</sub>, a<sub>1</sub>), (a<sub>2</sub>, a<sub>2</sub>), (r<sub>1</sub>, 1), ..., (r<sub>m</sub>, 1)⟩ so, A is finitely generated;
- $MP(A, F_2 \times F_2)$  is unsolvable;
- Hence,  $A \leq Aut(F)$  is orbit undecidable.

Finding	orbit under	cidable sub	aroune		
1. Historical context	2. CP for <i>Fn</i> -by-ℤ 0000000000	3. CP for <i>F<sub>n</sub>-</i> by- <i>F<sub>m</sub></i>	4. Main result	5. Applications	6. Negative results 00●000000000

Let F be a group, and let  $F_2 \times F_2 \simeq B \leq Aut(F)$  and  $u \in F$  be such that  $B \cap Stab^*(u) = 1$ . Then, there exists f.g.  $A \leq Aut(F)$  which is orbit undecidable.

- Take a group U = (a<sub>1</sub>, a<sub>2</sub> | r<sub>1</sub>,..., r<sub>m</sub>) with unsolvable word problem;
- Consider  $A = \{(v, w) \mid v =_U w\} \leq F_2 \times F_2;$
- Easy to see that A = ⟨(a<sub>1</sub>, a<sub>1</sub>), (a<sub>2</sub>, a<sub>2</sub>), (r<sub>1</sub>, 1), ..., (r<sub>m</sub>, 1)⟩ so, A is finitely generated;
- $MP(A, F_2 \times F_2)$  is unsolvable;
- Hence,  $A \leq Aut(F)$  is orbit undecidable.

Finding	orbit under	cidable sub	aroune		
1. Historical context	2. CP for <i>Fn</i> -by-ℤ 0000000000	3. CP for <i>F<sub>n</sub>-</i> by- <i>F<sub>m</sub></i>	4. Main result	5. Applications	6. Negative results 00●000000000

Let F be a group, and let  $F_2 \times F_2 \simeq B \leq Aut(F)$  and  $u \in F$  be such that  $B \cap Stab^*(u) = 1$ . Then, there exists f.g.  $A \leq Aut(F)$  which is orbit undecidable.

- Take a group U = (a<sub>1</sub>, a<sub>2</sub> | r<sub>1</sub>,..., r<sub>m</sub>) with unsolvable word problem;
- Consider  $A = \{(v, w) \mid v =_U w\} \leq F_2 \times F_2;$
- Easy to see that A = ⟨(a<sub>1</sub>, a<sub>1</sub>), (a<sub>2</sub>, a<sub>2</sub>), (r<sub>1</sub>, 1), ..., (r<sub>m</sub>, 1)⟩ so, A is finitely generated;
- $MP(A, F_2 \times F_2)$  is unsolvable;
- Hence,  $A \leq Aut(F)$  is orbit undecidable.

0000	000000000	cidable sub	0000000	000000000	0000000000
1. Historical context	2. CP for <i>F</i> <sub>Π</sub> -by-ℤ	3. CP for Fn-by-Fm	4. Main result	5. Applications	6. Negative results

Let F be a group, and let  $F_2 \times F_2 \simeq B \leq Aut(F)$  and  $u \in F$  be such that  $B \cap Stab^*(u) = 1$ . Then, there exists f.g.  $A \leq Aut(F)$  which is orbit undecidable.

- Take a group U = (a<sub>1</sub>, a<sub>2</sub> | r<sub>1</sub>,..., r<sub>m</sub>) with unsolvable word problem;
- Consider  $A = \{(v, w) \mid v =_U w\} \leq F_2 \times F_2;$
- Easy to see that A = ⟨(a<sub>1</sub>, a<sub>1</sub>), (a<sub>2</sub>, a<sub>2</sub>), (r<sub>1</sub>, 1), ..., (r<sub>m</sub>, 1)⟩ so, A is finitely generated;
- $MP(A, F_2 \times F_2)$  is unsolvable;
- Hence,  $A \leq Aut(F)$  is orbit undecidable.  $\Box$

1. Historical context	2. CP for <i>F⊓</i> -by-ℤ 0000000000	3. CP for <i>F<sub>n</sub>-by-F<sub>m</sub></i> 00000	4. Main result	5. Applications	6. Negative results 000●0000000
Finding c	orbit undeo	cidable sub	ogroups		

Corollary (Bogopolski–Martino–V., 2008)

Aut( $F_r$ ) contains f.g. orbit undecidable subgroups, for  $r \ge 3$ .

**Proof.** Take the copy B of  $F_2 \times F_2$  in Aut( $F_3$ ) via the embedding

 $egin{array}{rccccc} F_2 imes F_2 & \leftrightarrow & Aut(F_3), \ (u,v) & \mapsto & _u heta_v \colon F_3 & \rightarrow & F_3 \ & a & \mapsto & a \ & b & \mapsto & b \ & q & \mapsto & u^{-1}qv; \end{array}$ 

(u = qaqbq satisfies  $B \cap Stab^*(u) = 1$ ). Now, take any Mihailova subgroup in there,  $A \leq B \leq Aut(F_3)$ , and A will be orbit undecidable.

Proposition (Bogopolski–Martino–V., 2008)

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-Z	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results 000●0000000
Finding o	orbit undeo	cidable sub	ogroups		

Corollary (Bogopolski-Martino-V., 2008)

Aut( $F_r$ ) contains f.g. orbit undecidable subgroups, for  $r \ge 3$ .

**Proof.** Take the copy B of  $F_2 \times F_2$  in Aut $(F_3)$  via the embedding

$$egin{array}{rcccccc} F_2 imes F_2 & \hookrightarrow & Aut(F_3), \ (u,v) & \mapsto & _u heta_v \colon F_3 & o & F_3 \ & a & \mapsto & a \ & b & \mapsto & b \ & q & \mapsto & u^{-1}qv; \end{array}$$

 $(u = qaqbq \text{ satisfies } B \cap Stab^*(u) = 1)$ . Now, take any Mihailova subgroup in there,  $A \leq B \leq Aut(F_3)$ , and A will be orbit undecidable.

Proposition (Bogopolski–Martino–V., 2008)

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-Z	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results 000●0000000
Finding o	orbit undeo	cidable sub	ogroups		

Corollary (Bogopolski-Martino-V., 2008)

Aut( $F_r$ ) contains f.g. orbit undecidable subgroups, for  $r \ge 3$ .

**Proof.** Take the copy B of  $F_2 \times F_2$  in Aut $(F_3)$  via the embedding

$$egin{array}{rcccccc} F_2 imes F_2 & \hookrightarrow & Aut(F_3), \ (u,v) & \mapsto & _u heta_v \colon F_3 & \to & F_3 \ & a & \mapsto & a \ & b & \mapsto & b \ & q & \mapsto & u^{-1}qv; \end{array}$$

( $u = qaqbq \text{ satisfies } B \cap Stab^*(u) = 1$ ). Now, take any Mihailova subgroup in there,  $A \leq B \leq Aut(F_3)$ , and A will be orbit undecidable.

Proposition (Bogopolski–Martino–V., 2008)

1. Historical context	2. CP for <i>F<sub>n</sub>-by-</i> ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results 000●0000000
Finding c	orbit unde	cidable sub	ogroups		

Corollary (Bogopolski-Martino-V., 2008)

Aut( $F_r$ ) contains f.g. orbit undecidable subgroups, for  $r \ge 3$ .

**Proof.** Take the copy B of  $F_2 \times F_2$  in Aut $(F_3)$  via the embedding

 $(u = qaqbq \text{ satisfies } B \cap Stab^*(u) = 1)$ . Now, take any Mihailova subgroup in there,  $A \leq B \leq Aut(F_3)$ , and A will be orbit undecidable.

Proposition (Bogopolski–Martino–V., 2008)

1. Historical context	2. CP for <i>F<sub>n</sub>-by-</i> ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results 000●0000000
Finding c	orbit unde	cidable sub	ogroups		

Corollary (Bogopolski–Martino–V., 2008)

Aut( $F_r$ ) contains f.g. orbit undecidable subgroups, for  $r \ge 3$ .

**Proof.** Take the copy B of  $F_2 \times F_2$  in Aut $(F_3)$  via the embedding

 $(u = qaqbq \text{ satisfies } B \cap Stab^*(u) = 1)$ . Now, take any Mihailova subgroup in there,  $A \leq B \leq Aut(F_3)$ , and A will be orbit undecidable.

#### Proposition (Bogopolski–Martino–V., 2008)

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results
Thompso	on-by-free	negative r	esults		

### For the braid group

- Aut  $(B_n)$  does not contain  $F_2 \times F_2$ ;
- we proved that every extension of *B<sub>n</sub>* (by torsion-free hyperbolic) has solvable conjugacy problem.

### For Thompson's group

Proposition (Burillo–Matucci–V. 2010)

 $F_2 \times F_2$  embeds in Aut(F).

#### Corollary (Burillo–Matucci–V. 2010)

There exist Thompson-by-free groups,  $F \rtimes F_m$ , with unsolvable conjugacy problem.

◆□▶ ◆□▶ ◆□▶ ◆□▶ ● ● ● ●

		no a still in a	ملابيهم		
					00000000000
<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-Z</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-F<sub>m</sub></li> </ol>	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>

### I nompson-by-free negative results

### For the braid group

- Aut  $(B_n)$  does not contain  $F_2 \times F_2$ ;
- we proved that every extension of B<sub>n</sub> (by torsion-free hyperbolic) has solvable conjugacy problem.

### For Thompson's group

Proposition (Burillo–Matucci–V. 2010)

 $F_2 \times F_2$  embeds in Aut(F).

#### Corollary (Burillo–Matucci–V. 2010)

There exist Thompson-by-free groups,  $F \rtimes F_m$ , with unsolvable conjugacy problem.

◆□▶ ◆□▶ ◆□▶ ◆□▶ ● ● ● ●

	a la fue a	no a a tiu a r	ملليهم		
<ol> <li>Historical context 0000</li> </ol>	2. CP for <i>F<sub>n</sub>-by-</i> ℤ ○○○○○○○○○	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result	5. Applications	<ol> <li>Negative results</li> <li>0000●0000000</li> </ol>

### i nompson-by-free negative results

### For the braid group

- Aut  $(B_n)$  does not contain  $F_2 \times F_2$ ;
- we proved that every extension of *B<sub>n</sub>* (by torsion-free hyperbolic) has solvable conjugacy problem.

### For Thompson's group

Proposition (Burillo–Matucci–V. 2010)

 $F_2 \times F_2$  embeds in Aut(F).

#### Corollary (Burillo–Matucci–V. 2010)

There exist Thompson-by-free groups,  $F \rtimes F_m$ , with unsolvable conjugacy problem.

(日) (日) (日) (日) (日) (日) (日)

	a la fue a	no a a tiu a r	ملليهم		
<ol> <li>Historical context 0000</li> </ol>	2. CP for <i>F<sub>n</sub>-by-</i> ℤ ○○○○○○○○○	3. CP for <i>Fn-</i> by- <i>Fm</i> 00000	4. Main result	5. Applications	<ol> <li>Negative results</li> <li>0000●0000000</li> </ol>

### i nompson-by-free negative results

### For the braid group

- Aut  $(B_n)$  does not contain  $F_2 \times F_2$ ;
- we proved that every extension of B<sub>n</sub> (by torsion-free hyperbolic) has solvable conjugacy problem.

### For Thompson's group

Proposition (Burillo–Matucci–V. 2010)

 $F_2 \times F_2$  embeds in Aut(F).

### Corollary (Burillo-Matucci-V. 2010)

There exist Thompson-by-free groups,  $F \rtimes F_m$ , with unsolvable conjugacy problem.

◆□▶ ◆□▶ ▲□▶ ▲□▶ ■ ののの

1. Historical context	2. CP for <i>F<sub>n</sub>-</i> by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results 00000●00000
(Free ab	elian)-bv-f	ree negativ	ve results	S	

### Corollary (Bogopolski-Martino-V., 2008)

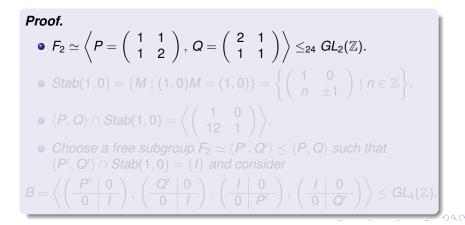
 $\operatorname{GL}_d(\mathbb{Z})$  contains f.g. orbit undecidable subgroups, for  $d \ge 4$ .

• 
$$F_2 \simeq \left\langle P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \leq_{24} GL_2(\mathbb{Z}).$$
  
•  $Stab(1,0) = \{M \mid (1,0)M = (1,0)\} = \left\{ \begin{pmatrix} 1 & 0 \\ n & \pm 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.$   
•  $\langle P, Q \rangle \cap Stab(1,0) = \left\langle \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix} \right\rangle.$   
• Choose a free subgroup  $F_2 \simeq \langle P', Q' \rangle \leq \langle P, Q \rangle$  such that  $\langle P', Q' \rangle \cap Stab(1,0) = \{I\}$  and consider  
 $B = \left\langle \left( \frac{P' \mid 0}{0 \mid I} \right), \left( \frac{Q' \mid 0}{0 \mid I} \right), \left( \frac{I \mid 0}{0 \mid P'} \right), \left( \frac{I \mid 0}{0 \mid Q'} \right) \right\rangle \leq GL_4(\mathbb{Z})$ 

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○○○○○○○○○○	3. CP for <i>F<sub>n</sub>-by-F<sub>m</sub></i> 00000	4. Main result	5. Applications	6. Negative results 00000€00000		
(Free abelian)-by-free negative results							

Corollary (Bogopolski-Martino-V., 2008)

 $\operatorname{GL}_d(\mathbb{Z})$  contains f.g. orbit undecidable subgroups, for  $d \ge 4$ .



1. Historical context	2. CP for <i>F<sub>Π</sub>-by-</i> ℤ ○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results 00000€00000			
(Free abelian)-by-free negative results								

### Corollary (Bogopolski-Martino-V., 2008)

 $\operatorname{GL}_d(\mathbb{Z})$  contains f.g. orbit undecidable subgroups, for  $d \ge 4$ .

#### Proof.

• 
$$F_2 \simeq \left\langle P = \left( \begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right), \ Q = \left( \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right) \right\rangle \leq_{24} GL_2(\mathbb{Z}).$$

• 
$$Stab(1,0) = \{M \mid (1,0)M = (1,0)\} = \left\{ \begin{pmatrix} 1 & 0 \\ n & \pm 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$$

• 
$$\langle P, Q \rangle \cap Stab(1,0) = \left\langle \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix} \right\rangle$$

• Choose a free subgroup  $F_2 \simeq \langle P', Q' \rangle \leq \langle P, Q \rangle$  such that  $\langle P', Q' \rangle \cap Stab(1,0) = \{I\}$  and consider

$$B = \left\langle \left( \begin{array}{c|c} P' & 0 \\ \hline 0 & I \end{array} \right), \left( \begin{array}{c|c} Q' & 0 \\ \hline 0 & I \end{array} \right), \left( \begin{array}{c|c} I & 0 \\ \hline 0 & P' \end{array} \right), \left( \begin{array}{c|c} I & 0 \\ \hline 0 & Q' \end{array} \right) \right\rangle \le GL_4(\mathbb{Z}).$$

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 0000
 000000000
 00000
 00000000
 00000000
 00000000
 000000000

 (Free abelian)\_by\_free pegative results
 00000000
 results
 00000000
 000000000

# (Free abelian)-by-free negative results

### For free abelian groups

### Corollary (Bogopolski-Martino-V., 2008)

 $\operatorname{GL}_d(\mathbb{Z})$  contains f.g. orbit undecidable subgroups, for  $d \ge 4$ .

### Proof.

• 
$$F_2 \simeq \left\langle P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \leq_{24} GL_2(\mathbb{Z}).$$

• 
$$Stab(1,0) = \{M \mid (1,0)M = (1,0)\} = \left\{ \begin{pmatrix} 1 & 0 \\ n & \pm 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.$$

• 
$$\langle P, Q \rangle \cap Stab(1,0) = \left\langle \left( \begin{array}{cc} 1 & 0 \\ 12 & 1 \end{array} \right) \right\rangle$$

• Choose a free subgroup  $F_2 \simeq \langle P', Q' \rangle \leq \langle P, Q \rangle$  such that  $\langle P', Q' \rangle \cap Stab(1,0) = \{I\}$  and consider

$$B = \left\langle \left( \begin{array}{c|c} P' & 0 \\ \hline 0 & I \end{array} \right), \left( \begin{array}{c|c} Q' & 0 \\ \hline 0 & I \end{array} \right), \left( \begin{array}{c|c} I & 0 \\ \hline 0 & P' \end{array} \right), \left( \begin{array}{c|c} I & 0 \\ \hline 0 & Q' \end{array} \right) \right\rangle \leq GL_4(\mathbb{Z}).$$

1

1. Historical context	2. CP for <i>Fn</i> -by-ℤ 0000000000	3. CP for <i>F<sub>n</sub></i> -by- <i>F<sub>m</sub></i> 00000	4. Main result	5. Applications	6. Negative results 00000€00000			
(Free abelian)-by-free negative results								

### Corollary (Bogopolski-Martino-V., 2008)

 $\operatorname{GL}_d(\mathbb{Z})$  contains f.g. orbit undecidable subgroups, for  $d \ge 4$ .

#### Proof.

• 
$$F_2 \simeq \left\langle P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \leq_{24} GL_2(\mathbb{Z}).$$

• 
$$Stab(1,0) = \{M \mid (1,0)M = (1,0)\} = \left\{ \begin{pmatrix} 1 & 0 \\ n & \pm 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$$

• 
$$\langle P, Q \rangle \cap Stab(1,0) = \left\langle \left( \begin{array}{cc} 1 & 0 \\ 12 & 1 \end{array} \right) \right\rangle$$

• Choose a free subgroup  $F_2 \simeq \langle P', Q' \rangle \leq \langle P, Q \rangle$  such that  $\langle P', Q' \rangle \cap Stab(1,0) = \{I\}$  and consider

 1. Historical context
 2. CP for Fn-by-Z
 3. CP for Fn-by-Fm
 4. Main result
 5. Applications
 6. Negative results

 (Free abelian)-by-free negative results

### For free abelian groups

### Corollary (Bogopolski-Martino-V., 2008)

 $\operatorname{GL}_d(\mathbb{Z})$  contains f.g. orbit undecidable subgroups, for  $d \ge 4$ .

#### Proof.

• 
$$F_2 \simeq \left\langle P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \leq_{24} GL_2(\mathbb{Z}).$$
  
•  $Stab(1,0) = \{M \mid (1,0)M = (1,0)\} = \left\{ \begin{pmatrix} 1 & 0 \\ n & \pm 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.$   
•  $\langle P, Q \rangle \cap Stab(1,0) = \left\langle \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix} \right\rangle.$   
• Choose a free subgroup  $F_2 \simeq \langle P', Q' \rangle \leq \langle P, Q \rangle$  such that

$$\langle P', Q' \rangle \cap Stab(1, 0) = \{I\} \text{ and consider} \\ \mathsf{B} = \left\langle \left( \frac{P' \mid 0}{0 \mid I} \right), \left( \frac{Q' \mid 0}{0 \mid I} \right), \left( \frac{I \mid 0}{0 \mid P'} \right), \left( \frac{I \mid 0}{0 \mid Q'} \right) \right\rangle \leq GL_4(\mathbb{Z}).$$



## • Note that $B \simeq F_2 \times F_2$ .

- Write *u* = (1,0,1,0). By construction, *B* ∩ Stab<sup>\*</sup>(*u*) = {*Id*}.
- Take  $A \le B \simeq F_2 \times F_2$  with unsolvable membership problem.
- By previous result, A ≤ GL<sub>4</sub>(ℤ) is orbit undecidable.
- Similarly for  $A \leq GL_d(\mathbb{Z})$ , with  $4 \leq d$ .  $\Box$

#### Proposition (Bogopolski–Martino–V., 2008)

Every finitely generated subgroup of  $GL_2(\mathbb{Z})$  is O.D.

#### Definition

A f.g. subgroup  $A \leq GL_d(\mathbb{Z})$  is orbit decidable is there exists an algorithm  $\mathcal{A}$  which, given two vectors  $u, v \in \mathbb{Z}^n$  decides whether v = uM by some matrix  $M \in A$ .



• Note that  $B \simeq F_2 \times F_2$ .

Write u = (1,0,1,0). By construction, B ∩ Stab\*(u) = {Id}.

- Take A ≤ B ≃ F<sub>2</sub> × F<sub>2</sub> with unsolvable membership problem.
- By previous result,  $\mathsf{A} \leqslant \mathsf{GL}_4(\mathbb{Z})$  is orbit undecidable
- Similarly for  $A \leq GL_d(\mathbb{Z})$ , with  $4 \leq d$ .  $\Box$

#### Proposition (Bogopolski–Martino–V., 2008)

Every finitely generated subgroup of  $GL_2(\mathbb{Z})$  is O.D.

#### Definition

A f.g. subgroup  $A \leq GL_d(\mathbb{Z})$  is orbit decidable is there exists an algorithm  $\mathcal{A}$  which, given two vectors  $u, v \in \mathbb{Z}^n$  decides whether v = uM by some matrix  $M \in A$ .



- Note that  $B \simeq F_2 \times F_2$ .
- Write u = (1, 0, 1, 0). By construction,  $B \cap Stab^*(u) = \{Id\}$ .
- Take  $A \leq B \simeq F_2 \times F_2$  with unsolvable membership problem.
- By previous result, A ≤ GL<sub>4</sub>(ℤ) is orbit undecidable.
- Similarly for  $A \leq GL_d(\mathbb{Z})$ , with  $4 \leq d$ .  $\Box$

Every finitely generated subgroup of  $GL_2(\mathbb{Z})$  is O.D.

#### Definition



- Note that  $B \simeq F_2 \times F_2$ .
- Write u = (1, 0, 1, 0). By construction,  $B \cap Stab^*(u) = \{Id\}$ .
- Take  $A \leq B \simeq F_2 \times F_2$  with unsolvable membership problem.
- By previous result,  $A \leq GL_4(\mathbb{Z})$  is orbit undecidable.
- Similarly for  $A \leq GL_d(\mathbb{Z})$ , with  $4 \leq d$ .  $\Box$

Every finitely generated subgroup of  $GL_2(\mathbb{Z})$  is O.D.

#### Definition



- Note that  $B \simeq F_2 \times F_2$ .
- Write u = (1, 0, 1, 0). By construction,  $B \cap Stab^*(u) = \{Id\}$ .
- Take  $A \leq B \simeq F_2 \times F_2$  with unsolvable membership problem.
- By previous result,  $A \leq GL_4(\mathbb{Z})$  is orbit undecidable.
- Similarly for  $A \leq GL_d(\mathbb{Z})$ , with  $4 \leq d$ .  $\Box$

Every finitely generated subgroup of  $GL_2(\mathbb{Z})$  is O.D.

#### Definition



- Note that  $B \simeq F_2 \times F_2$ .
- Write u = (1, 0, 1, 0). By construction,  $B \cap Stab^*(u) = \{Id\}$ .
- Take  $A \leq B \simeq F_2 \times F_2$  with unsolvable membership problem.
- By previous result,  $A \leq GL_4(\mathbb{Z})$  is orbit undecidable.
- Similarly for  $A \leq GL_d(\mathbb{Z})$ , with  $4 \leq d$ .  $\Box$

Every finitely generated subgroup of  $GL_2(\mathbb{Z})$  is O.D.

#### Definition



- Note that  $B \simeq F_2 \times F_2$ .
- Write u = (1, 0, 1, 0). By construction,  $B \cap Stab^*(u) = \{Id\}$ .
- Take  $A \leq B \simeq F_2 \times F_2$  with unsolvable membership problem.
- By previous result,  $A \leq GL_4(\mathbb{Z})$  is orbit undecidable.
- Similarly for  $A \leq GL_d(\mathbb{Z})$ , with  $4 \leq d$ .  $\Box$

Every finitely generated subgroup of  $GL_2(\mathbb{Z})$  is O.D.

# Definition

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results 000000000000000000000000000000000000
(Free ab	elian)-by-f	ree negativ	ve results	S	

There exist 14 matrices  $M_1, \ldots, M_{14} \in GL_d(\mathbb{Z})$ , for  $d \ge 4$ , such that  $\langle M_1, \ldots, M_{14} \rangle \leq GL_d(\mathbb{Z})$  is orbit undecidable.

#### Corollary (Bogopolski–Martino–V., 2008)

There exists a  $\mathbb{Z}^4$ -by- $F_{14}$  group with unsolvable conjugacy problem.

#### Question

Does  $GL_3(\mathbb{Z})$  contain orbit undecidable subgroups ?

#### Question

Does there exist  $\mathbb{Z}^3$ -by-free groups with unsolvable conjugacy problem ?

・ コット (雪) ( 小田) ( コット 日)

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results 000000000000000000000000000000000000
(Free ab	elian)-by-f	ree negativ	ve results	S	

There exist 14 matrices  $M_1, \ldots, M_{14} \in GL_d(\mathbb{Z})$ , for  $d \ge 4$ , such that  $\langle M_1, \ldots, M_{14} \rangle \leq GL_d(\mathbb{Z})$  is orbit undecidable.

#### Corollary (Bogopolski-Martino-V., 2008)

There exists a  $\mathbb{Z}^4$ -by- $F_{14}$  group with unsolvable conjugacy problem.

#### Question

Does  $GL_3(\mathbb{Z})$  contain orbit undecidable subgroups ?

#### Question

Does there exist  $\mathbb{Z}^3$ -by-free groups with unsolvable conjugacy problem ?

・ コット (雪) ( 小田) ( コット 日)

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results 000000000000000000000000000000000000
(Free ab	elian)-by-f	ree negativ	ve results	S	

There exist 14 matrices  $M_1, \ldots, M_{14} \in GL_d(\mathbb{Z})$ , for  $d \ge 4$ , such that  $\langle M_1, \ldots, M_{14} \rangle \leq GL_d(\mathbb{Z})$  is orbit undecidable.

#### Corollary (Bogopolski-Martino-V., 2008)

There exists a  $\mathbb{Z}^4$ -by- $F_{14}$  group with unsolvable conjugacy problem.

#### Question

Does  $GL_3(\mathbb{Z})$  contain orbit undecidable subgroups ?

#### Question

Does there exist  $\mathbb{Z}^3$ -by-free groups with unsolvable conjugacy problem ?

・ロット (雪) (日) (日) (日)

1. Historical context	2. CP for <i>Fn</i> -by-ℤ ○○○○○○○○○○	3. CP for <i>Fn</i> -by- <i>Fm</i> 00000	4. Main result	5. Applications	6. Negative results 000000000000000000000000000000000000
(Free ab	elian)-by-f	ree negativ	ve results	S	

There exist 14 matrices  $M_1, \ldots, M_{14} \in GL_d(\mathbb{Z})$ , for  $d \ge 4$ , such that  $\langle M_1, \ldots, M_{14} \rangle \leq GL_d(\mathbb{Z})$  is orbit undecidable.

#### Corollary (Bogopolski-Martino-V., 2008)

There exists a  $\mathbb{Z}^4$ -by- $F_{14}$  group with unsolvable conjugacy problem.

# Question

Does  $GL_3(\mathbb{Z})$  contain orbit undecidable subgroups ?

#### Question

Does there exist  $\mathbb{Z}^3$ -by-free groups with unsolvable conjugacy problem ?

Automata		00000	0000000	000000000	000000000000000000000000000000000000000
Automata	adrouns				

For  $d \ge 6$ , the group  $GL_d(\mathbb{Z})$  contains orbit undecidable, free subgroups.

So, for  $d \ge 6$ , there exists a group of the form

$$\Gamma = \mathbb{Z}^d \rtimes F_m \leqslant \mathbb{Z}^d \rtimes GL_d(\mathbb{Z})$$

with unsolvable conjugacy problem.

Theorem (Šunić–V., 2010)

All such groups  $\Gamma = \mathbb{Z}^d \rtimes F_m$  can be realized as automaton groups.

#### Corollary (Šunić–V., 2010)

There exists automaton groups with unsolvable conjugacy problem.

ヘロマ ヘロマ ヘロマ

Automata		00000	0000000	000000000	000000000000000000000000000000000000000
Automata	adrouns				

For  $d \ge 6$ , the group  $GL_d(\mathbb{Z})$  contains orbit undecidable, free subgroups.

So, for  $d \ge 6$ , there exists a group of the form

$$\Gamma = \mathbb{Z}^d \rtimes F_m \leqslant \mathbb{Z}^d \rtimes GL_d(\mathbb{Z})$$

with unsolvable conjugacy problem.

Theorem (Šunić–V., 2010)

All such groups  $\Gamma = \mathbb{Z}^d \rtimes F_m$  can be realized as automaton groups.

#### Corollary (Šunić–V., 2010)

There exists automaton groups with unsolvable conjugacy problem.

Automata		00000	0000000	000000000	000000000000000000000000000000000000000
Automata	adrouns				

For  $d \ge 6$ , the group  $GL_d(\mathbb{Z})$  contains orbit undecidable, free subgroups.

So, for  $d \ge 6$ , there exists a group of the form

$$\Gamma = \mathbb{Z}^d \rtimes F_m \leqslant \mathbb{Z}^d \rtimes GL_d(\mathbb{Z})$$

with unsolvable conjugacy problem.

# Theorem (Šunić–V., 2010)

All such groups  $\Gamma = \mathbb{Z}^d \rtimes F_m$  can be realized as automaton groups.

#### Corollary (Šunić–V., 2010)

There exists automaton groups with unsolvable conjugacy problem.

(日) (型) (モ) (モ) =

1. Historical context	2. CP for <i>F<sub>n</sub></i> -by-ℤ ○○○○○○○○○○	3. CP for <i>F<sub>n</sub>-by-F<sub>m</sub></i> 00000	4. Main result	5. Applications	6. Negative results 000000000000000000000000000000000000
Automata	a groups				

For  $d \ge 6$ , the group  $GL_d(\mathbb{Z})$  contains orbit undecidable, free subgroups.

So, for  $d \ge 6$ , there exists a group of the form

$$\Gamma = \mathbb{Z}^d \rtimes F_m \leqslant \mathbb{Z}^d \rtimes GL_d(\mathbb{Z})$$

with unsolvable conjugacy problem.

#### Theorem (Šunić–V., 2010)

All such groups  $\Gamma = \mathbb{Z}^d \rtimes F_m$  can be realized as automaton groups.

# Corollary (Šunić–V., 2010)

There exists automaton groups with unsolvable conjugacy problem.

<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	3. CP for Fn-by-Fm	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>
					0000000000000

Next step:

# What about TCP in

your favorite group ?

<ol> <li>Historical context</li> </ol>	<ol> <li>CP for F<sub>n</sub>-by-ℤ</li> </ol>	3. CP for Fn-by-Fm	<ol><li>Main result</li></ol>	<ol><li>Applications</li></ol>	<ol><li>Negative results</li></ol>
					0000000000

# THANKS