Endofixedness and computation of fixed closures in free groups

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Outline

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1. Some history
2. Algorithmic results
3. Needed tools
4. The proof

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{Mono}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\phi\colon F_n \to F_n$ act on the right, $x \mapsto x\phi$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.
- If $S \subseteq \mathrm{End}\,(F_n)$ then
  $\mathrm{Fix}\,(S) = \{x \in F_n \mid x\phi = x\ \forall \phi \in S\} = \cap_{\phi \in S} \mathrm{Fix}\,(\phi) \leqslant F_n$.

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Fixed subgroups are complicated

$$\begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & ba \\ c & \mapsto & ca^2 \end{array}$$

$\mathrm{Fix}\,\phi = \langle a, bab^{-1}, cac^{-1} \rangle$

$$\begin{array}{rcl} \varphi\colon F_4 & \to & F_4 \\ a & \mapsto & dac \\ b & \mapsto & c^{-1}a^{-1}d^{-1}ac \\ c & \mapsto & c^{-1}a^{-1}b^{-1}ac \\ d & \mapsto & c^{-1}a^{-1}bc \end{array}$$

$\mathrm{Fix}\,\varphi = \langle w \rangle$, where...

$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1}$
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What is known about fixed subgroups ?

Theorem (Dyer-Scott, 75)

Let $G \leqslant Aut(F_n)$ be a finite group of automorphisms of F_n. Then, $Fix(G) \leqslant_{\rm ff} F_n$; in particular, $r(Fix(G)) \leqslant n$.

Conjecture (Scott)

For every $\phi \in Aut(F_n)$, $r(Fix(\phi)) \leqslant n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) < \infty$.

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Train-tracks

Main result in this story:

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Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

introducing the theory of train-tracks for graphs.

After Bestvina-Handel, live continues ...

Theorem (Imrich-Turner, 89)

Let $\phi \in End(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

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Let $\phi \in End(F_n)$. If ϕ is not bijective then $r(Fix(\phi)) \leqslant n-1$.

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Description of fixed subgroups

There are three easy ways of building fixed points:

(Construction-1)

Let $\phi\colon F_n \to F_n$ be an automorphism and $Fix(\phi)$ its fixed subgroup. Then, there are many ways of extending ϕ to $\phi'\colon F_n * F_m \to F_n * F_m$ such that $Fix(\phi') = Fix(\phi)$ (for example, invert all generators of F_m).

(Construction-2)

Let $\phi_1\colon F_n \to F_n$ and $\phi_2\colon F_m \to F_m$ be two automorphisms and $Fix(\phi_1)$ and $Fix(\phi_2)$ their fixed subgroups. Then, $\phi_1 * \phi_2\colon F_n * F_m \to F_n * F_m$ has $Fix(\phi_1 * \phi_2) = Fix(\phi_1) * Fix(\phi_2)$.

(Construction-3)

Let $\phi\colon F_n \to F_n$ be an automorphism and $Fix(\phi)$ its fixed subgroup. Let $h, h' \in F_n$ be such that $h\phi = h'hh'^{-1}$. Then, the extension $\phi'\colon F_n * \langle z\rangle \to F_n * \langle z\rangle$ defined by $z \mapsto h'h^r z$ satisfies $Fix(\phi') = Fix(\phi) * \langle z^{-1}hz\rangle$.

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Description of fixed subgroups

These are essentially the only possibilities:

Observation

A cyclic subgroup $\langle w \rangle \leqslant F_n$ is the fixed subgroup of some $\phi \in Aut(F_n)$ if and only if w is not a proper power.

Theorem (Martino-V., 04)

Every automorphism $\phi\colon F_n \to F_n$ and its fixed subgroup $Fix(\phi)$ can be built from finitely many automorphisms $\phi_i\colon F_{m_i} \to F_{m_i}$ $(m_i \leqslant n)$, $i = 1, \ldots, r$, with cyclic fixed subgroup, $r(Fix(\phi_i)) = 1$, by finitely many applications of Constructions 1, 2 and 3.

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Inertia

Definition

A subgroup $H \leqslant F_n$ is called inert if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F_n$.

Theorem (Dicks-V, 96)

Let $G \subseteq Mon(F_n)$ be an arbitrary set of monomorphisms of F_n. Then, $Fix(G)$ is inert; in particular, $r(Fix(G)) \leqslant n$.

Theorem (Bergman, 99)

Let $G \subseteq End(F_n)$ be an arbitrary set of endomorphisms of F_n. Then, $r(Fix(G)) \leqslant n$.

Conjecture (V.)

Let $\phi \in End(F_n)$. Then $Fix(\phi)$ is inert.

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A subgroup $H \leqslant F_n$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_n)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_n)$,*
- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
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Easy to see that 1-mono-fixed = 1-auto-fixed.

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- *endo-fixed if $H = Fix(S)$ for some $S \subseteq End(F_n)$,*

Easy to see that 1-mono-fixed = 1-auto-fixed.

Relations between them

$$
\begin{array}{ccc}
\boxed{1-auto-fixed} & \subseteq & \boxed{1-endo-fixed} \\
\cap| & & \cap| \\
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Example (Martino-V., 03; Ciobanu-Dicks, 06)

Let $F_3 = \langle a, b, c\rangle$ and $H = \langle b, cacbab^{-1}c^{-1}\rangle \leqslant F_3$. Then, $H = Fix(a \mapsto 1,\ b \mapsto b,\ c \mapsto cacbab^{-1}c^{-1})$, but H is NOT the fixed subgroup of any set of automorphism of F_3.

Relations between them

$$\begin{array}{ccc} \boxed{1-auto-fixed} & \overset{\subseteq}{\neq} & \boxed{1-endo-fixed} \\ \cap| \quad \|\,? & & \cap| \quad \|\,? \\ \boxed{auto-fixed} & \overset{\subseteq}{\neq} & \boxed{endo-fixed} \end{array}$$

Conjecture

For every $S \subseteq End(F_n)$ ($S \subseteq Aut(F_n)$) there exists $\phi \in End(F_n)$ ($\phi \in End(F_n)$) such that $Fix(S) = Fix(\phi)$.

Theorem (Martino-V., 00)

Let $S \subseteq End(F_n)$. Then, $\exists \phi \in \langle S \rangle$ such that $Fix(S) \leqslant_{\mathrm{ff}} Fix(\phi)$.

But... free factors of 1-endo-fixed (1-auto-fixed) subgroups need not be even endo-fixed (auto-fixed).

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Computing fixed subgroups

Proposition (Turner, 86)

There exists a pseudo-algorithm to compute fix of an endo.

Easy but is not an algorithm...

Theorem (Maslakova, 03)

Fixed subgroups of automorphisms of F_n are computable.

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Deciding fixedness

In this talk, I'll solve the two dual problems:

Theorem

Given $H \leqslant_{\rm fg} F_n$, one can algorithmically decide whether

i) *H is auto-fixed or not,*

ii) *H is endo-fixed or not,*

and in the affirmative case, find a finite family, $S = \{\phi_1, \ldots, \phi_m\}$, of automorphisms (endomorphisms) of F_n such that $Fix(S) = H$.

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Fixed closures

Definition

Given $H \leqslant_{\mathrm{fg}} F_n$, we define the (auto- and endo-) stabilizer of H, respectively, as

$$Aut_H(F_n) = \{\phi \in Aut(F_n) \mid H \leqslant Fix(\phi)\} \leqslant Aut(F_n)$$

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Given $H \leqslant F_n$, we define the auto-closure and endo-closure of H as

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For every $H \leqslant_{\mathrm{fg}} F_n$, a-$Cl(H)$ and e-$Cl(H)$ are finitely generated and one can algorithmically compute bases for them.

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Auto-fixedness and endo-fixedness are decidable.

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Retracts

Definition

A subgroup $H \leqslant F_n$ is a retract if there exists a retraction, i.e. a morphism $\rho\colon F_n \to H$ which restricts to the identity of H.

Free factors are retracts, but there are more.

Observation

If $H \leqslant F_n$ is a retract then $r(H) \leqslant n$ (and, $r(H) = n \Leftrightarrow H = F_n$).

Observation (Turner)

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The stable image

Definition

Let $\phi \in End(F_n)$. The stable image of ϕ is $F_n\phi^\infty = \cap_{i=1}^\infty F_n\phi^i$.

Theorem (Imrich-Turner, 89)

For every endomorphism $\phi \colon F_n \to F_n$,

i) *$F_n\phi^\infty$ is ϕ-invariant,*
ii) *the restriction $\phi \colon F_n\phi^\infty \to F_n\phi^\infty$ is an isomorphism,*
iii) *$F_n\phi^\infty$ is a retract.*
iv) *$Fix(\phi) \leqslant F_n\phi^\infty$.*

Example: For $\phi \colon F_2 \to F_2$, $a \mapsto a$, $b \mapsto b^2$, we have $F_2\phi = \langle a, b^2 \rangle$, $F_2\phi^2 = \langle a, b^4 \rangle$, $F_2\phi^3 = \langle a, b^8 \rangle$, ... So, $F_2\phi^\infty = \langle a \rangle \leqslant_{ff} F_2$.

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Stallings' graphs and intersections

Theorem (Stallings, 83)

For any free group $F_n = F(A)$, there is an effectively computable bijection

$$\{\textit{f.g. subgroups of } F_n\} \longleftrightarrow \{\textit{finite } A\textit{-labeled core graphs}\}$$

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Algebraic extensions

Definition

An extension of subgroups $H \leqslant K \leqslant F_n$ is called algebraic, denoted $H \leqslant_{\mathrm{alg}} K$, if H is not contained in any proper free factor of K. Write

$$\mathcal{AE}(H) = \{K \leqslant F_n \mid H \leqslant_{\mathrm{alg}} K\}.$$

Theorem (Takahasi, 51)

If $H \leqslant_{\mathrm{fg}} F_n$ then $\mathcal{AE}(H)$ is finite and computable (i.e. H has finitely many algebraic extensions, all of them are finitely generated, and bases are computable from H).

Theorem

Every extension of subgroups $H \leqslant K \leqslant F_n$ factors in a unique way as $H \leqslant_{\mathrm{alg}} L \leqslant_{\mathrm{ff}} K \leqslant F_n$.

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The automorphism case

Theorem (McCool)

Let $H \leqslant_{\text{fg}} F_n$. Then $Aut_H(F_n)$ is finitely generated (in fact, finitely presented) and a finite set of generators (and relations) is algorithmically computable from H.

Theorem

For every $H \leqslant_{\text{fg}} F_n$, a-$Cl(H)$ is finitely generated and algorithmically computable.

Proof.

$$\begin{aligned} a\text{-}Cl(H) &= \operatorname{Fix}(\operatorname{Aut}_H(F_n)) \\ &= \operatorname{Fix}(\langle \phi_1, \ldots, \phi_m \rangle) \\ &= \operatorname{Fix}(\phi_1) \cap \cdots \cap \operatorname{Fix}(\phi_m). \quad \square \end{aligned}$$

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For the endomorphism case, a similar approach does not work because:

- we don't know how to compute fix subgroups of endomorphisms
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Example (Ciobanu-Dicks, 06)

Consider $F_3 = \langle a, b, c\rangle$, the element $d = ba[c^2, b]a^{-1}$, and the subgroup $H = \langle a, d\rangle \leqslant F_3$. Clearly, the morphisms

$$\begin{array}{rcl} \psi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & d \\ c & \mapsto & 1 \end{array} \qquad \begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & b \\ c & \mapsto & cb \end{array} \qquad \begin{array}{rcl} \phi^n\psi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & d \\ c & \mapsto & d^n \end{array}$$

satisfy $H \leqslant \operatorname{Fix}(\phi^n\psi)$ for every $n \in \mathbb{Z}$.
With some computations, it can be shown that

$$End_H(F_3) = \{Id, \phi^n\psi \mid n \in \mathbb{Z}\}.$$

But, $\phi^m\psi \cdot \phi^n\psi = \phi^m\psi$. Hence, $End_H(F_3)$ is not finitely generated.

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For every $H \leqslant_{\mathrm{fg}} F_n$, e-$Cl(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
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- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

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THANKS