On the difficulty of inverting automorphisms of free groups

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Outline

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Motivation

(Joint work with P. Silva and M. Ladra.)

Find a group G where $\cdot$ is “easy" but $(\,)^{-1}$ is “difficult".

Natural candidate: $\mathrm{Aut}\,(F_n)$, where $F_r = \langle a_1, \ldots, a_r \mid \rangle$.

$F_3 = \langle a, b, c \mid \rangle$.

$$\begin{array}{rcl} \phi\colon F_3 & \longrightarrow & F_3 \\ a & \mapsto & ab \\ b & \mapsto & ab^2c \\ c & \mapsto & bc^2 \end{array} \qquad \begin{array}{rcl} \psi\colon F_3 & \longrightarrow & F_3 \\ a & \mapsto & bc^{-1} \\ b & \mapsto & a^{-1}bc \\ c & \mapsto & c^{-1}. \end{array}$$

$$\begin{array}{rcl} \phi\psi\colon F_3 & \longrightarrow & F_3 \\ a & \mapsto & bc^{-1}a^{-1}bc \\ b & \mapsto & bc^{-1}a^{-1}bca^{-1}b \\ c & \mapsto & a^{-1}bc^{-1}. \end{array}$$

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$F_5 = \langle a, b, c, d, e \mid \rangle.$

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- We have formalized the situation.
- We have seen that inverting in $\mathrm{Aut}(F_r)$ is not that bad.
- We now want to look for worse groups G.

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Main definition

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Let $A = \{a_1, \ldots, a_r\}$ be a finite alphabet, and $G = \langle A \mid R \rangle$ be a finite presentation for a group G. We have the word metric:

$$\text{for } g \in G, \quad |g| = \min\{n \mid g = a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n}\}.$$

Definition

For $\theta \in Aut(G)$, note θ is determined by $a_1\theta, \ldots, a_r\theta$ and define

$$||\theta||_1 = |a_1\theta| + \cdots + |a_r\theta|,$$

$$||\theta||_\infty = \max\{|a_1\theta|, \ldots, |a_r\theta|\}.$$

Observation

For every $\theta \in Aut(F_r)$, $||\theta||_\infty \leqslant ||\theta||_1 \leqslant r||\theta||_\infty$

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$$\alpha_A(n) = \max\{||\theta^{-1}||_1 \mid \theta \in Aut(G), \ ||\theta||_1 \leqslant n\}.$$

Clearly, $\alpha_A(n) \leqslant \alpha_A(n+1)$.

The bigger is α_A, the more "difficult" will be to invert automorphisms of G (with respect to the given set of generators A).

Question

Determine the asymptotic growth of the function α_A.

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Free group case

For the rest of the talk, $G = F_r = \langle a_1, \ldots, a_r \mid \rangle$.

Definition

Every $w \in F_r$ has its length, $|w|$, and its cyclic length, $\cdot|w|\cdot$:
$|a_1 a_1^{-1} a_2| = |a_2| = \cdot|a_2|\cdot = 1,$
$|a_1 a_2 a_1^{-2}| = 4,$
$\cdot|a_1 a_2 a_1^{-2}|\cdot = \cdot|a_2 a_1^{-1}|\cdot = 2.$

Observation

i) $|w^n| \leqslant |n||w|$ and $\cdot|w^n|\cdot = |n|\cdot|w|\cdot$;
ii) $|vw| \leqslant |v| + |w|$, but $\cdot|vw|\cdot \leqslant \cdot|v|\cdot + \cdot|w|\cdot$ is not true in general.

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$|\cdot|\theta|\cdot|_1 \leqslant |||\theta|||_1 \leqslant ||\theta||_1$, *but not equal in general.*

Example

Consider $\theta\colon F_4 \to F_4$, $a \mapsto a$, $b \mapsto a^{-1}ba$, $c \mapsto a^{-1}ca$, $d \mapsto d$. *We have* $|\cdot|\theta|\cdot|_1 = 4$, $|||\theta|||_1 = 6$ *and* $||\theta||_1 = 8$.

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Are these functions equal up to multiplicative constants ?

α_r *and* γ_r *are not;*
β_r *is not clear.*

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Main results

Theorem

For rank $r = 2$ we have

(i) *for* $n \geqslant 4$, $\alpha_2(n) \leqslant \frac{(n-1)^2}{2}$,

(ii) *for* $n \geqslant n_0$, $\alpha_2(n) \geqslant \frac{n^2}{16}$,

(iii) *for* $n \geqslant 1$, $\beta_2(n) = n$,

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Theorem

For $r \geqslant 3$ *there exist* $K = K(r)$ *and* $M = M(r)$ *such that, for* $n \geqslant 1$,

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(ii) *for* $n \geqslant n_0$, $\alpha_2(n) \geqslant \frac{n^2}{16}$,

(iii) *for* $n \geqslant 1$, $\beta_2(n) = n$,

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For $r \geqslant 3$ *there exist* $K = K(r)$ *and* $M = M(r)$ *such that, for* $n \geqslant 1$,

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Main results

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Outline

A lower bound for γ_r

Theorem

For $r \geqslant 2$, and $n \geqslant n_0$, we have $\gamma_r(n) \geqslant \frac{1}{2r^{r-1}} n^{r-1}$.

Proof: For $r \geqslant 2$ and $n \geqslant 1$, consider

$$\begin{array}{rcl} \psi_{r,n}\colon F_r & \to & F_r \\ a_1 & \mapsto & a_1 \\ a_2 & \mapsto & a_1^n a_2 \\ a_3 & \mapsto & a_2^n a_3 \\ & \vdots & \\ a_r & \mapsto & a_{r-1}^n a_r \end{array} \qquad \begin{array}{rcl} \psi_{r,n}^{-1}\colon F_r & \to & F_r \\ a_1 & \mapsto & a_1 \\ a_2 & \mapsto & a_1^{-n} a_2 \\ & \vdots & \\ a_i & \mapsto & (a_{i-1}^{-n})\psi_{r,n}^{-1} \cdot a_i \\ & (2 \leqslant i \leqslant r) & \end{array}$$

A straightforward calculation shows that
$|\!|\psi_{r,n}|\!|_1 = ||\psi_{r,n}||_1 = (r-1)n + r$, and
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Hence, for $n \geqslant r$,

$$\gamma_r(rn) \geqslant \gamma_r((r-1)n + r) \geqslant n^{r-1}.$$

Now, for n big enough, take the closest multiple of r below,

$$n \geqslant rm > n - r,$$

and

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For $r \geqslant 2$, and $n \geqslant n_0$, we have $\alpha_r(n) \geqslant \frac{(r-1)^{r-1}}{2r^{2r-1}} n^r$.

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Outline

Outer space

To prove the upper bound

(ii) $\beta_r(n) \leqslant Kn^M$,

we'll need to use the recently discovered metric in the outer space $\mathcal{X}_r$.

Definition

- *By graf Γ we mean a finite, connected graph of rank r, with no vertices of degree 1 or 2.*
- *A metric on Γ is a map $\ell\colon E\Gamma \to [0,1]$ such that $\sum_{e \in E\Gamma} \ell(e) = 1$, and $\{e \in E\Gamma \mid \ell(e) = 0\}$ is a forest.*
- *For a graph Γ, $\Sigma_\Gamma = \{\text{metrics on } \Gamma\}$ = a simplex with missing faces.*
- *If $\Gamma' = \Gamma/\text{forest}$, then we identify points in $\Sigma_{\Gamma'}$ with the corresponding points in Σ_Γ by assigning length 0 to the collapsed edges.*
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The outer space $\mathcal{X}_r$ is

$$\mathcal{X}_r = \{ (\Gamma, f, \ell) \} / \sim$$

(where $\sim$ is an equivalence relation).

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There is a natural action of $Aut(F_r)$ on $\mathcal{X}_r$, given by

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Metric on $\mathcal{X}_r$

Definition

Let $x, x' \in \mathcal{X}_r$, $x = (\Gamma, f, \ell)$, $x' = (\Gamma', f', \ell')$. A difference of markings is a map $\alpha \colon \Gamma \to \Gamma'$, which is linear over edges and $f\alpha \simeq f'$.
For such an α, define $\sigma(\alpha)$ to be its maximum slope over edges.

Definition

$\mathcal{X}_r$ admits the following "metric":

$$d(x, x') = \min\{\log(\sigma(\alpha)) \mid \alpha \text{ diff. markings}\}.$$

This minimum is achieved by Arzela-Ascoli's theorem.

This is Bestvina-AlgomKfir version of Martino-Francaviglia's original metric.

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(i) $d(x, y) \geqslant 0$, *and* $= 0 \Leftrightarrow x = y$.

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For $\epsilon > 0$, *the* ϵ*-thick part of* $\mathcal{X}_r$ *is*

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Bestvina-AlgomKfir theorem

Theorem (Bestvina-AlgomKfir)

For any $\epsilon > 0$ there is constant $M = M(r, \epsilon)$ such that for all $x, y \in \mathcal{X}_r(\epsilon)$,

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Corollary

For $r \geqslant 2$, there exists $M = M(r)$ such that

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Proof

Remind $\beta_r(n) = \max\{|||\theta^{-1}|||_1 \mid \theta \in Aut\, F_r,\ |||\theta|||_1 \leqslant n\}$.

Proof. Given $\phi \in \mathrm{Aut}\,(F_r)$, consider $x = (R_r, id, \ell_0) \in \mathcal{X}_r$, and $\phi \cdot x = (R_r, \phi, \ell_0) \in \mathcal{X}_r$, where ℓ_0 is the uniform metric.

$$\begin{aligned} d(x,\ \phi\cdot x) &= \min\{\log(\sigma(\alpha)) \mid \alpha \text{ diff. markings}\} \\ &= \log\left(\min\{\sigma(\phi\gamma_w\gamma_p) \mid w \in F_r,\ p = \text{“half petal"}\}\right) \\ &\sim \log\left(\min\{\sigma(\phi\gamma_w) \mid w \in F_r\}\right) \\ &= \log\left(\min\{||\phi\gamma_w||_\infty \mid w \in F_r\}\right) \\ &= \log(|||\phi|||_\infty) \\ &\sim \log(|||\phi|||_1). \end{aligned}$$

Now, using Bestvina-AlgomKfir theorem,

$$\log(|||\phi^{-1}|||_1) \sim d(x, \phi^{-1}\cdot x) = d(\phi\cdot x, x) \leqslant Md(x, \phi\cdot x) \sim M\log(|||\phi|||_1).$$

Hence, for every $\phi \in \mathrm{Aut}\,(F_r)$, $|||\phi^{-1}|||_1 \leqslant r\,|||\phi|||_1^M$. $\square$

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Outline

The rank 2 case

These functions for $\mathrm{Aut}(F_2)$ are much easier to understand due to the following technical lemmas.

Lemma

Let $\varphi \in Aut(F_2)$ be positive. Then φ^{-1} is cyclically reduced and $||\varphi^{-1}||_1 = ||\varphi||_1$.

Lemma

For every $\theta \in Aut(F_2)$, there exist two letter permuting autos $\psi_1, \psi_2 \in Aut(F_2)$, a positive one $\varphi \in Aut^+(F_2)$, and an element $g \in F_2$, such that $\theta = \psi_1 \varphi \psi_2 \lambda_g$ and $||\varphi||_1 + 2|g| \leqslant ||\theta||_1$.

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The rank 2 case: γ_2

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For every $\theta \in Aut(F_2)$, $|\!\cdot\!|\theta^{-1}|\!\cdot\!|_1 = |\!\cdot\!|\theta|\!\cdot\!|_1$. Hence, $\gamma_2(n) = n$.

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For $n \geqslant 4$ we have $\alpha_2(n) \leqslant \frac{(n-1)^2}{2}$.

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Theorem

For $n \geqslant n_0$ we have $\alpha_2(n) \geqslant \frac{n^2}{16}$.

So, the global known picture is

(i) $\frac{n^2}{16} \leqslant \alpha_2(n) \leqslant \frac{(n-1)^2}{2}$,

(ii) $\beta_2(n) = n$,

(iii) $\gamma_2(n) = n$,

(iv) $Kn^r \leqslant \alpha_r(n)$,

(v) $\beta_r(n) \leqslant Kn^M$,

(iii) $Kn^{r-1} \leqslant \gamma_r(n)$.

for some constants $K = K(r)$, $M = M(r)$, and for $n \geqslant n_0$.

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For $n \geqslant n_0$ we have $\alpha_2(n) \geqslant \frac{n^2}{16}$.

So, the global known picture is

(i) $\frac{n^2}{16} \leqslant \alpha_2(n) \leqslant \frac{(n-1)^2}{2}$,

(ii) $\beta_2(n) = n$,

(iii) $\gamma_2(n) = n$,

(iv) $Kn^r \leqslant \alpha_r(n)$,

(v) $\beta_r(n) \leqslant Kn^M$,

(iii) $Kn^{r-1} \leqslant \gamma_r(n)$.

for some constants $K = K(r)$, $M = M(r)$, and for $n \geqslant n_0$.

Outline

Fixed subgroups are complicated

$$\begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & ba \\ c & \mapsto & ca^2 \end{array}$$

$\operatorname{Fix}\phi = \langle a, bab^{-1}, cac^{-1} \rangle$

$$\begin{array}{rcl} \varphi\colon F_4 & \to & F_4 \\ a & \mapsto & dac \\ b & \mapsto & c^{-1}a^{-1}d^{-1}ac \\ c & \mapsto & c^{-1}a^{-1}b^{-1}ac \\ d & \mapsto & c^{-1}a^{-1}bc \end{array}$$

$\operatorname{Fix}\varphi = \langle w \rangle$, where...

$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1}$
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What is known about fixed subgroups ?

Theorem (Dyer-Scott, 75)

Let $G \leqslant Aut(F_n)$ be a finite group of automorphisms of F_n. Then, $Fix(G) \leqslant_{\mathrm{ff}} F_n$; in particular, $r(Fix(G)) \leqslant n$.

Conjecture (Scott)

For every $\phi \in Aut(F_n)$, $r(Fix(\phi)) \leqslant n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) < \infty$.

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Main result in this story:

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Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

introducing the theory of train-tracks for graphs.

After Bestvina-Handel, live continues ...

Theorem (Imrich-Turner, 89)

Let $\phi \in End(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

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Inertia

Definition

A subgroup $H \leqslant F_n$ is called inert if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F_n$.

Theorem (Dicks-V, 96)

Let $G \subseteq Mon(F_n)$ be an arbitrary set of monomorphisms of F_n. Then, $Fix(G)$ is inert; in particular, $r(Fix(G)) \leqslant n$.

Theorem (Bergman, 99)

Let $G \subseteq End(F_n)$ be an arbitrary set of endomorphisms of F_n. Then, $r(Fix(G)) \leqslant n$.

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The four families

Definition

A subgroup $H \leqslant F_n$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_n)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_n)$,*
- *auto-fixed if $H = Fix(S)$ for some $S \subseteq Aut(F_n)$,*
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Easy to see that 1-mono-fixed = 1-auto-fixed.

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Relations between them

$$\begin{array}{ccc}
\boxed{1-auto-fixed} & \subseteq & \boxed{1-endo-fixed} \\
\cap| & & \cap| \\
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Example (Martino-V., 03; Ciobanu-Dicks, 06)

Let $F_3 = \langle a, b, c\rangle$ and $H = \langle b, cacbab^{-1}c^{-1}\rangle \leqslant F_3$. Then, $H = Fix(a \mapsto 1,\ b \mapsto b,\ c \mapsto cacbab^{-1}c^{-1})$, but H is NOT the fixed subgroup of any set of automorphism of F_3.

Relations between them

$$\begin{array}{ccc} \boxed{1-auto-fixed} & \underset{\neq}{\subseteq} & \boxed{1-endo-fixed} \\ \cap| \quad \| ? & & \cap| \quad \| ? \\ \boxed{auto-fixed} & \underset{\neq}{\subseteq} & \boxed{endo-fixed} \end{array}$$

Theorem (Martino-V., 00)

Let $S \subseteq End(F_n)$. Then, $\exists \phi \in \langle S \rangle$ such that $Fix(S) \leqslant_{\rm ff} Fix(\phi)$.

But... free factors of 1-endo-fixed (1-auto-fixed) subgroups need not be even endo-fixed (auto-fixed).

Outline

Computing fixed subgroups

Proposition (Turner, 86)

There exists a pseudo-algorithm to compute fix of an endo.

Easy but is not an algorithm...

Theorem (Maslakova, 03)

Fixed subgroups of automorphisms of F_n are computable.

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Deciding fixedness

What about the dual problem ?

Theorem (V. 2010)

Given $H \leqslant_{\mathrm{fg}} F_n$, one can algorithmically decide whether

i) *H is auto-fixed or not,*

ii) *H is endo-fixed or not,*

and in the affirmative case, find a finite family, $S = \{\phi_1, \ldots, \phi_m\}$, of automorphisms (endomorphisms) of F_n such that $Fix(S) = H$.

Conjecture

Given $H \leqslant_{\mathrm{fg}} F_n$, one can algorithmically decide whether

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Fixed closures

Definition

Given $H \leqslant_{\mathrm{fg}} F_n$, we define the (auto- and endo-) stabilizer of H, respectively, as

$$Aut_H(F_n) = \{\phi \in Aut(F_n) \mid H \leqslant Fix(\phi)\} \leqslant Aut(F_n)$$

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$$End_H(F_n) = \{\phi \in End(F_n) \mid H \leqslant Fix(\phi)\} \leqslant End(F_n)$$

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Given $H \leqslant F_n$, we define the auto-closure and endo-closure of H as

$$a\text{-}Cl(H) = Fix(Aut_H(F_n)) \geqslant H$$

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Main result

Theorem

For every $H \leqslant_{\mathrm{fg}} F_n$, a-$Cl(H)$ and e-$Cl(H)$ are finitely generated and one can algorithmically compute bases for them.

Corollary

Auto-fixedness and endo-fixedness are decidable.

Observe that $e\text{-}Cl(H) \leqslant a\text{-}Cl(H)$ but, in general, they are not equal.

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The automorphism case

Theorem (McCool, 70's)

Let $H \leqslant_{\text{fg}} F_n$. Then $Aut_H(F_n)$ is finitely generated (in fact, finitely presented) and a finite set of generators (and relations) is algorithmically computable from H.

Theorem

For every $H \leqslant_{\text{fg}} F_n$, a-$Cl(H)$ is finitely generated and algorithmically computable.

Proof.

$$\begin{aligned} a\text{-}Cl(H) &= \mathrm{Fix}\,(Aut_H(F_n)) \\ &= \mathrm{Fix}\,(\langle \phi_1, \ldots, \phi_m \rangle) \\ &= \mathrm{Fix}\,(\phi_1) \cap \cdots \cap \mathrm{Fix}\,(\phi_m). \quad \square \end{aligned}$$

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A similar approach does not work because:

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Example

Consider $F_3 = \langle a, b, c\rangle$, the element $d = ba[c^2, b]a^{-1}$, and the subgroup $H = \langle a, d\rangle \leqslant F_3$. Clearly, the morphisms

$$\begin{array}{rcl} \psi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & d \\ c & \mapsto & 1 \end{array} \qquad \begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & b \\ c & \mapsto & cb \end{array} \qquad \begin{array}{rcl} \phi^n\psi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & d \\ c & \mapsto & d^n \end{array}$$

satisfy $H \leqslant Fix(\phi^n\psi)$ for every $n \in \mathbb{Z}$.
With some computations, Ciobanu-Dicks-06 show that

$$End_H(F_3) = \{Id, \phi^n\psi \mid n \in \mathbb{Z}\}.$$

But, $\phi^m\psi \cdot \phi^n\psi = \phi^m\psi$. Hence, $End_H(F_3)$ is not finitely generated.

Furthermore, $a\text{-}Cl(H) = \mathrm{Fix}\,(Id) = F_3$ and $e\text{-}Cl(H) = \mathrm{Fix}\,(\psi) = H$.

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Consider $F_3 = \langle a, b, c\rangle$, the element $d = ba[c^2, b]a^{-1}$, and the subgroup $H = \langle a, d\rangle \leqslant F_3$. Clearly, the morphisms

$$\begin{array}{rcl} \psi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & d \\ c & \mapsto & 1 \end{array} \qquad \begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & b \\ c & \mapsto & cb \end{array} \qquad \begin{array}{rcl} \phi^n\psi\colon F_3 & \to & F_3 \\ a & \mapsto & a \\ b & \mapsto & d \\ c & \mapsto & d^n \end{array}$$

satisfy $H \leqslant Fix(\phi^n\psi)$ for every $n \in \mathbb{Z}$.
With some computations, Ciobanu-Dicks-06 show that

$$End_H(F_3) = \{Id,\ \phi^n\psi \mid n \in \mathbb{Z}\}.$$

But, $\phi^m\psi \cdot \phi^n\psi = \phi^m\psi$. Hence, $End_H(F_3)$ is not finitely generated.

Furthermore, $a\text{-}Cl(H) = \mathrm{Fix}\,(Id) = F_3$ and $e\text{-}Cl(H) = \mathrm{Fix}\,(\psi) = H$.

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Theorem

For every $H \leqslant_{\text{fg}} F_n$, e-$Cl(H)$ is finitely generated and algorithmically computable.

Proof. Given H (in generators),

- Compute $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$.
- Select those which are retracts, $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$ $(1 \leqslant r \leqslant q)$.
- Write the generators of H as words on the generators of each one of these H_i's, $i = 1, \ldots, r$.
- Compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$.
- Compute a basis for $a\text{-}Cl_{H_1}(H) \cap \cdots \cap a\text{-}Cl_{H_r}(H)$.

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- Take $\beta \in \mathrm{End}(F_n)$ with $H \leqslant \mathrm{Fix}(\beta)$.
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- Now, β restricts to an automorphism $\alpha \colon H_i \to H_i$.
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