Stallings graphs for (free-abelian)-by-free groups

Enric Ventura

Departament de Matemàtiques
Universitat Politècnica de Catalunya

and

University of Southampton, UK

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Outline

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Free group: the construction

Definition

- *Let $A = \{a_1, \ldots, a_r\}$ be a finite alphabet, and consider (formally) $\tilde{A} = \{a_1, \ldots, a_r, a_1^{-1}, \ldots, a_r^{-1}\}$.*
- *A word on A is a finite sequence of symbols $w = a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n}$, where $a_{i_j} \in A$ and $\epsilon_j = \pm 1$. The length of w is $\ell(w) = n$.*
- *The empty word is the only one with zero letters, denoted 1; $\ell(1) = 0$.*
- *The collection of all words on A is denoted $\tilde{A}^*$.*
- *Operation of concatenation in $\tilde{A}^*$: $u \cdot v = uv$; $\ell(uv) = \ell(u) + \ell(v)$.*

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- *Two consecutive letters in $w \in \tilde{A}^*$ of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$ are called a cancellation. A word w is called reduced if it has no cancellations. Denote $R(A) \subseteq \tilde{A}^*$ the set of reduced words.*
- *The reduction is the equivalence relation $\sim$ generated by*

$$ua_i^{\epsilon} a_i^{-\epsilon} v \sim uv.$$

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Definition

- *The free group on A is $F(A) = \tilde{A}^* / \sim$ with the operation of concatenation (i.e. concatenation + reduction).*
- *The neutral element is 1, and the inverse of $w = a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n}$ is $w^{-1} = (a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n})^{-1} = a_{i_n}^{-\epsilon_n} \cdots a_{i_1}^{-\epsilon_1}$.*
- *Of course, $(a_i^{-1})^{-1} = a_i$.*

Lemma

For every $w \in \tilde{A}^*$, there is a unique $\overline{w} \in R(A)$, s.t. $w = \overline{w}$ in $F(A)$.

This allows us to "forget" the $\sim$, and work in $F(A)$ by just manipulating words (and reducing every time it is possible).

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Membership problem

Definition

Let G be a group. The membership problem in G consists on finding an algorithm which, on

$$\textit{input: } g_0, g_1, \ldots, g_n \in G;$$

decides whether $g_0 \in \langle g_1, \ldots, g_n \rangle \leqslant G$, or not.

Proposition

(i) *Finite groups have solvable membership problem.*

(ii) *$\mathbb{Z}^n$ and $\mathbb{Q}^n$ have solvable membership problem.*

(iii) *There are groups G with UNSOLVABLE membership problem.*

(iv) *What about F_r ?*

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Intersection problem

Definition

A group G has the Howson property if the intersection of any two finitely generate subgroups is again finitely generated.

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Let G be a group. The intersection problem in G consists on finding an algorithm which, on

$$\text{input: } u_1, \ldots, u_n, v_1, \ldots, v_m \in G;$$

decides whether $\langle u_1, \ldots, u_n \rangle \cap \langle v_1, \ldots, v_m \rangle$ is finitely generated or not and, if yes, computes a set of generators $w_1, \ldots, w_p$ for it.

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A first example

Theorem (Howson, 50's)

Free groups are Howson.

Example

Consider F_2 and the subgroups $H = \langle a, b^2, bab^{-1} \rangle$ and $K = \langle b^2, ba^2 \rangle$. Can you find generators for $H \cap K$?

- *Clearly, $b^2 \in H \cap K$ ...*
- *Less obvious but still easy, $a^{-2}b^2a^2 \in H \cap K$ because*

$$a^{-2}b^2a^2 = (a)^{-2}(b^2)(a)^2 \in H,$$

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- *Something else? $H \cap K = \langle b^2, a^{-2}b^2a^2, \ldots (?) \ldots \rangle$*
- *How to be sure you found everything ?*

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Stallings automata

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A Stallings automaton over A is a finite A-graph (V, E, q_0), such that:

1- *it is connected,*

2- *it is trim, (no vertex of degree 1 except possibly q_0),*

3- *it is deterministic (no two edges with the same label go out of (or into) the same vertex).*

NO :

YES :

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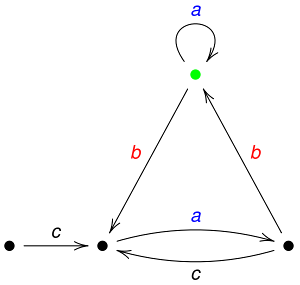

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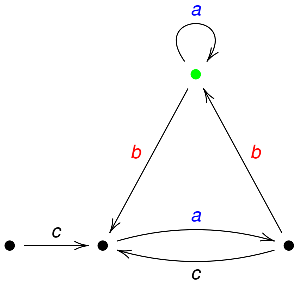

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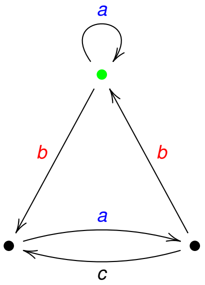

Stallings automata

In the influent paper

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Stallings (building on previous works) gave a bijection between finitely generated subgroups of $F(A)$ and Stallings automata:

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Reading the subgroup from the automata

Definition

To any given Stallings automaton $\mathcal{A} = (V, E, q_0)$, we associate its language:

$$L(\mathcal{A}) = \{ \text{ labels of closed paths at } q_0 \} \leqslant F(A).$$

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$L(\mathcal{A}) = \{1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \ldots\}$

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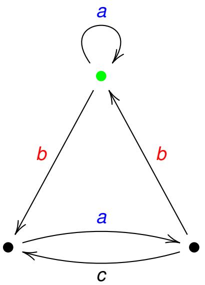
$\mathcal{A}=$ 

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Determinism is crucial

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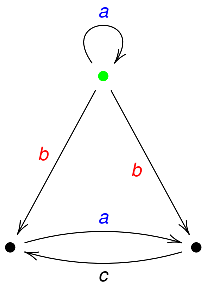

$a \in L(\mathcal{A})$

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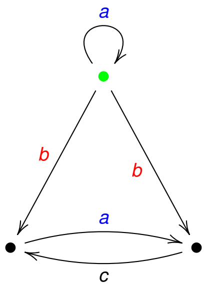

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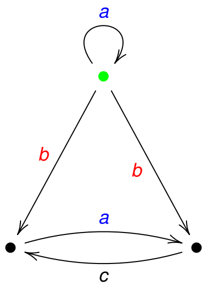

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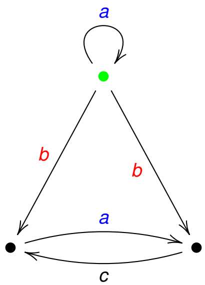

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A basis for $L(\mathcal{A})$

Proposition

For every Stallings automaton $\mathcal{A} = (V, E, q_0)$, and every maximal tree T, the group $L(\mathcal{A})$ is free with free basis

$$\{x_e = label(T[q_0, \iota e] \cdot e \cdot T[\tau e, q_0]) \in L(\mathcal{A}) \mid e \in EX - ET\},$$

where $T[p, q]$ denotes the geodesic in T from p to q. In particular, $rk(L(\mathcal{A})) = 1 - |V| + |E|$.

Constructing the automaton from the subgroup

Given $H = \langle w_1, \ldots, w_n \rangle \in F(A)$, construct the flower automaton, denoted $\mathcal{F}(H)$.

Clearly, $L(\mathcal{F}(H)) = H$.

... But $\mathcal{F}(H)$ is not in general deterministic...

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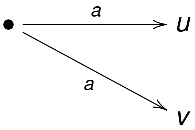

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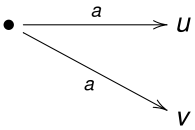

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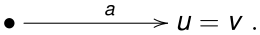

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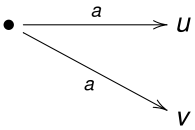

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$$\bullet \xrightarrow{\quad a \quad} u = v \ .$$

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Lemma (Stallings)

If $\mathcal{A} \rightsquigarrow \mathcal{A}'$ is a Stallings folding then $L(\mathcal{A}) = L(\mathcal{A}')$.

Given a f.g. subgroup $H = \langle w_1, \ldots, w_n \rangle \leqslant F_A$ (we assume w_i are reduced words), do the following:

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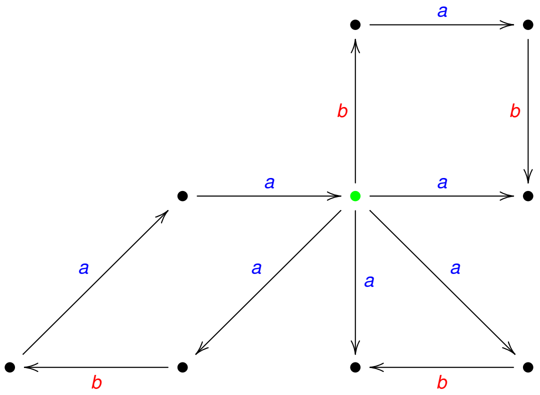

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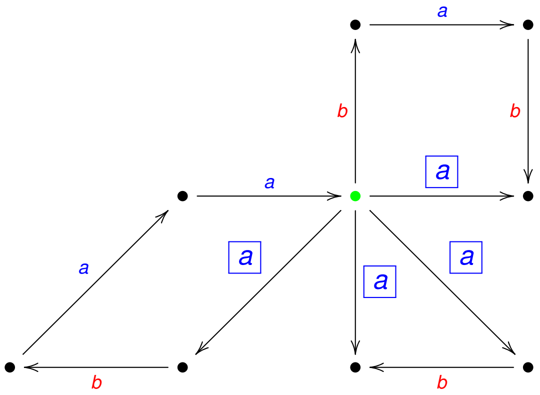

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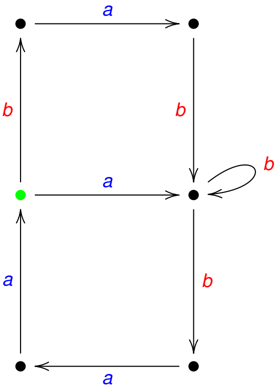

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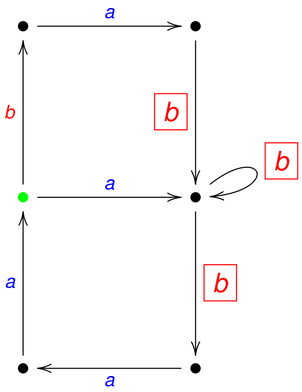

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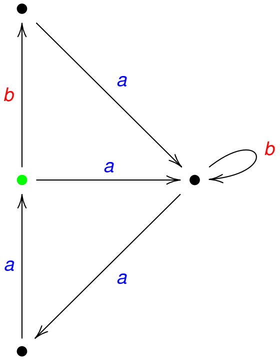

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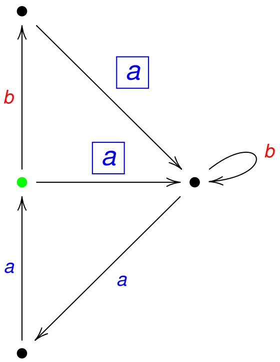

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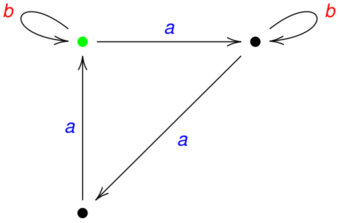

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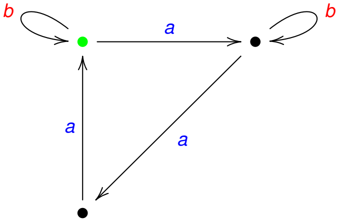

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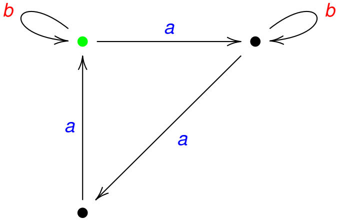

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Local confluence

It can be shown that

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The automaton $\Gamma(H)$ does not depend on the sequence of foldings.

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The automaton $\Gamma(H)$ does not depend on the generators of H.

Theorem

The following is a well defined bijection:

$$\begin{array}{ccc} \{\text{f.g. subgroups of } F_A\} & \longleftrightarrow & \{\text{Stallings automata}\} \\ H & \rightarrow & \Gamma(H) \\ L(\mathcal{A}) & \leftarrow & \mathcal{A} \end{array}$$

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Outline

1. Free groups
2. Stallings' graphs
3. Applications to free groups
4. Applications to (free-abelian)-by-free groups

Nielsen–Schreier Theorem

Corollary (Nielsen-Schreier)

Every subgroup of F_A is free.

- Finite automata work for the finitely generated case, but everything extends easily to the general case (using infinite graphs).
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Free groups have solvable membership problem.

Proof:

- Given w_0 and $H = \langle w_1, \ldots, w_n \rangle$ in F_m,
- Construct the flower automaton $\mathcal{F}(H)$,
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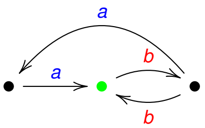

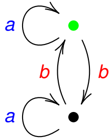

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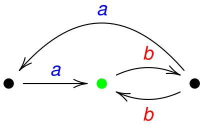

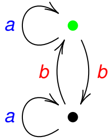

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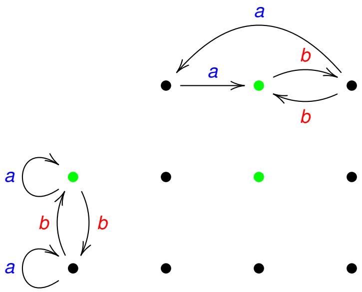

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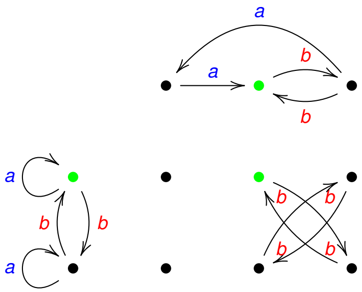

$H \cap K = \langle b^2, \quad \rangle$

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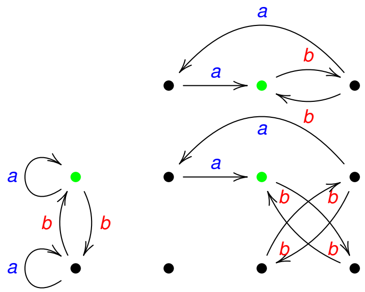

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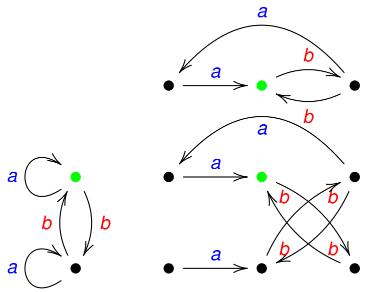

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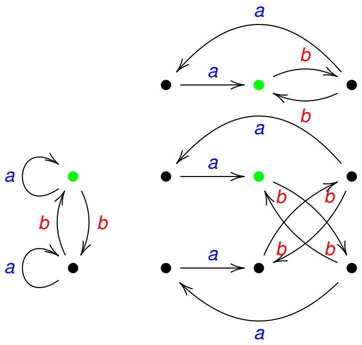

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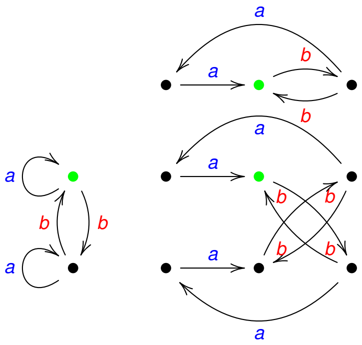

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Outline

1. Free groups
2. Stallings' graphs
3. Applications to free groups
4. Applications to (free-abelian)-by-free groups

(Free-abelian)-by-free groups

Definition

Consider $\{t^v \mid v \in \mathbb{Z}^m\}$ (i.e., $\mathbb{Z}^m$ in multiplicative notation), let $A_1, \ldots, A_n \in GL_m(\mathbb{Z})$ acting as $A_i \colon t^v \mapsto t^{vA_i}$, and consider the group

$$G = F_n \ltimes_{A_1,\ldots,A_n} \mathbb{Z}^m = \langle a_1, \ldots, a_n, t_1, \ldots, t_m \mid [t_i, t_j] = 1,\ a_i^{-1} t^v a_i = t^{vA_i} \rangle$$

Observation

We have the split short exact sequence

$$1 \to \mathbb{Z}^m \to G \to F_n \to 1,$$

and normal forms $w(\vec{a})t^v$ for the elements of G (where $v \in \mathbb{Z}^m$ and $w \in F(\{a_1, \ldots, a_n\})$), computable using $t^v a_i = a_i t^{vA_i}$. Furthermore,

$$t^v w(\vec{a}) = w(\vec{a}) t^{vW},$$

where $W = W(A_1, \ldots, A_n) \in GL_m(\mathbb{Z})$.

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(Free-abelian)-by-free groups

Proposition

For every subgroup $H \leqslant G = F_n \ltimes_{A_1,\ldots,A_n} \mathbb{Z}^m$, the sub- short exact sequence

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z}^m & \to & G & \xrightarrow{\pi} & F_n & \to & 1 \\ & & \vee & & \vee & & \vee & & \\ 1 & \to & L = H \cap \mathbb{Z}^m & \to & H & \xrightarrow{\pi} & H\pi & \to & 1 \end{array}$$

also splits and so, $H \simeq H\pi \ltimes_{\mathcal{A}} L$, where $\mathcal{A}$ is the restriction of the defining action $F_n \to Aut(\mathbb{Z}^m)$ to $\mathcal{A} \colon H\pi \to Aut(L)$.

In particular, every $H \leqslant F_n \ltimes_{A_1,\ldots,A_n} \mathbb{Z}^m$, $n \geqslant 2$, is of the form $H \simeq F_{n'} \ltimes_{A'_1,\ldots,A'_{n'}} \mathbb{Z}^{m'}$ for some $n' \in \mathbb{N} \cup \{\infty\}$ and $m' \leqslant m$.

Stallings graphs with vectors

Definition

Let us consider now, vectored A-automata, i.e., A-graphs with vectors assigned at the heads and tails of the edges,

$$\bullet \xrightarrow{u_1 \quad a \quad u_2} \bullet \;,$$

reading $t^{-u_1} a t^{u_2} = a t^{u_2 - u_1 A}$ *(and the inverse if traversed backwards).*

... plus a subspace $L \leqslant \mathbb{Z}^m$ *attached to the basepoint (corresponding to the purely abelian elements).*

Example

For a f. g. subgroup $H = \langle w_1 t^{u_1}, \ldots, w_r t^{u_r}, \; t^{v_1}, \ldots, t^{v_s} \rangle$ *of* $G = F_n \ltimes_{A_1,\ldots,A_n} \mathbb{Z}^m$, *we can also construct the flower automaton.*

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Abelian moves

Definition

We need now some extra operations to allow moving abelian mass arround:

- *edge moves,*
- *vertex moves,*
- *vertex moves at the basepoint,*
- *open foldings,*
- *closed foldings,*
- *increase L to its closure by the labels of all closed paths at •.*

Definition

A vectored Stallings A-automata is a connected and trim vectored A-automata satisfying:
(i) $\mathcal{A}'$ is deterministic,
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(iii) vectors are zero everywhere except, maybe, at the heads of edges outside a maximal tree T.

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Main results

Proposition

With repeated use of the above operations, any vectored A-automata $\mathcal{A}$ can be converted into a Stallings vectored A-automata $\mathcal{A}'$.

Theorem (Delgado–V., 2016)

(i) $\mathcal{A}'$ with the above conditions is uniquely determined by the subgroup H (modulo the choice of the maximal tree, and with all vectors around being viewed 'modulo' L).
(ii) The membership problem is solvable in (free-abelian)-by-free groups.
(iii) The intersection problem is solvable in (free-abelian)-by-free groups.

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Example of intersection

Observation

$F_2 \times \mathbb{Z}$ is NOT Howson.

Example

In $F_2 \times \mathbb{Z}^2$, consider the subgroups

$$H = \langle at^{(1,0)}, b^2, bab^{-1}t^{(1,2)}, t^{(1,1)} \rangle,$$

$$K = \langle b^2, ba^2t^{(2,1)}, t^{(1,5)}, t^{(2,6)} \rangle.$$

After some computations...

$(4,4)-(4,2) = (0,2) \notin L_H + L_K = \langle (1,1), (1,5), (2,6) \rangle = \langle (1,1), (0,4) \rangle,$

But... $H \cap K$ is f.g. because

$$2((4,4)-(4,2)) = 2(0,2) = (0,4) = -(1,1) + (1,5) \in L_H + L_K,$$

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In fact, $b^2 \in H \cap K$,

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$$L_H \cap L_K = \langle (1,1) \rangle \cap \langle (1,5), (2,6) \rangle = \langle (1,1) \rangle.$$

Example of intersection

Example

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Example of intersection

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Note that $(ba^2)^2 \in H\pi \cap K\pi$:

$$(bab^{-1}t^{(1,2)})^2 b^2 (at^{(1,0)})^2 = (ba^2)^2 t^{(4,4)} \in H$$

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But ... $(ba^2)^2 \notin (H \cap K)\pi$ *because*

$$((4,4) + L_H) \cap ((4,2) + L_K) = \emptyset$$

However, $(ba^2)^4 \in (H \cap K)\pi$ *since*

$$(ba^2)^4 t^{(9,9)} \in H \cap K$$

corresponding to the fact that

$$(9,9) \in (2(4,4) + L_H) \cap (2(4,2) + L_K) \neq \emptyset.$$

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THANKS