

# Deciding endofixedness in free groups

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# Outline

- 1 Some history
- 2 Algorithmic results
- 3 Needed tools
- 4 The proof

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# Notation

- $A = \{a_1, \dots, a_n\}$  is a finite alphabet ( $n$  letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \dots, a_n, a_n^{-1}\}$ .
- $F_n$  is the free group on  $A$ .
- $\text{Aut}(F_n) \subseteq \text{Mono}(F_n) \subseteq \text{End}(F_n)$ .
- I let endomorphisms  $\phi: F_n \rightarrow F_n$  act on the right,  $x \mapsto x\phi$ .
- $\text{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leq F_n$ .
- If  $S \subseteq \text{End}(F_n)$  then
 
$$\text{Fix}(S) = \{x \in F_n \mid x\phi = x \ \forall \phi \in S\} = \bigcap_{\phi \in S} \text{Fix}(\phi) \leq F_n.$$

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# Fixed subgroups are complicated

$$\begin{aligned}\phi: F_3 &\rightarrow F_3 \\ a &\mapsto a \\ b &\mapsto ba \\ c &\mapsto ca^2\end{aligned}$$

$$\text{Fix } \phi = \langle a, bab^{-1}, cac^{-1} \rangle$$

$$\begin{aligned}\varphi: F_4 &\rightarrow F_4 \\ a &\mapsto dac \\ b &\mapsto c^{-1}a^{-1}d^{-1}ac \\ c &\mapsto c^{-1}a^{-1}b^{-1}ac \\ d &\mapsto c^{-1}a^{-1}bc\end{aligned}$$

$$\text{Fix } \varphi = \langle w \rangle, \text{ where...}$$

$$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdada c d c d b c d a^{-1}a^{-1}d^{-1}a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}.$$

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# What is known about fixed subgroups ?

## Theorem (Dyer-Scott, 75)

*Let  $G \leq \text{Aut}(F_n)$  be a finite group of automorphisms of  $F_n$ . Then,  $\text{Fix}(G) \leq_{\text{ff}} F_n$ ; in particular,  $r(\text{Fix}(G)) \leq n$ .*

## Conjecture (Scott)

*For every  $\phi \in \text{Aut}(F_n)$ ,  $r(\text{Fix}(\phi)) \leq n$ .*

## Theorem (Gersten, 83 (published 87))

*Let  $\phi \in \text{Aut}(F_n)$ . Then  $r(\text{Fix}(\phi)) < \infty$ .*

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# Train-tracks

Main result in this story:

Theorem (Bestvina-Handel, 88 (published 92))

*Let  $\phi \in \text{Aut}(F_n)$ . Then  $r(\text{Fix}(\phi)) \leq n$ .*

introducing the theory of train-tracks for graphs.

After Bestvina-Handel, live continues ...

Theorem (Imrich-Turner, 89)

*Let  $\phi \in \text{End}(F_n)$ . Then  $r(\text{Fix}(\phi)) \leq n$ .*

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*Let  $\phi \in \text{End}(F_n)$ . If  $\phi$  is not bijective then  $r(\text{Fix}(\phi)) \leq n - 1$ .*

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# Description of fixed subgroups

There are three easy ways of building fixed points:

## (Construction-1)

Let  $\phi: F_n \rightarrow F_n$  be an automorphism and  $\text{Fix}(\phi)$  its fixed subgroup. Then, there are many ways of extending  $\phi$  to  $\phi': F_n * F_m \rightarrow F_n * F_m$  such that  $\text{Fix}(\phi') = \text{Fix}(\phi)$  (for example, invert all generators of  $F_m$ ).

## (Construction-2)

Let  $\phi_1: F_n \rightarrow F_n$  and  $\phi_2: F_m \rightarrow F_m$  be two automorphisms and  $\text{Fix}(\phi_1)$  and  $\text{Fix}(\phi_2)$  their fixed subgroups. Then,  $\phi_1 * \phi_2: F_n * F_m \rightarrow F_n * F_m$  has  $\text{Fix}(\phi_1 * \phi_2) = \text{Fix}(\phi_1) * \text{Fix}(\phi_2)$ .

## (Construction-3)

Let  $\phi: F_n \rightarrow F_n$  be an automorphism and  $\text{Fix}(\phi)$  its fixed subgroup. Let  $h, h' \in F_n$  be such that  $h\phi = h'hh'^{-1}$ . Then, the extension  $\phi': F_n * \langle z \rangle \rightarrow F_n * \langle z \rangle$  defined by  $z \mapsto h'h'z$  satisfies  $\text{Fix}(\phi') = \text{Fix}(\phi) * \langle z^{-1}hz \rangle$ .

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These are essentially the only possibilities:

## Observation

*A cyclic subgroup  $\langle w \rangle \leq F_n$  is the fixed subgroup of some  $\phi \in \text{Aut}(F_n)$  if and only if  $w$  is not a proper power.*

## Theorem (Martino-V., 04)

*Every automorphism  $\phi: F_n \rightarrow F_n$  and its fixed subgroup  $\text{Fix}(\phi)$  can be built from finitely many automorphisms  $\phi_i: F_{m_i} \rightarrow F_{m_i}$  ( $m_i \leq n$ ),  $i = 1, \dots, r$ , with cyclic fixed subgroup,  $r(\text{Fix}(\phi_i)) = 1$ , by finitely many applications of Constructions 1, 2 and 3.*

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# Inertia

## Definition

A subgroup  $H \leq F_n$  is called *inert* if  $r(H \cap K) \leq r(K)$  for every  $K \leq F_n$ .

## Theorem (Dicks-V, 96)

Let  $G \subseteq \text{Mon}(F_n)$  be an arbitrary set of monomorphisms of  $F_n$ . Then,  $\text{Fix}(G)$  is inert; in particular,  $r(\text{Fix}(G)) \leq n$ .

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A subgroup  $H \leq F_n$  is called *inert* if  $r(H \cap K) \leq r(K)$  for every  $K \leq F_n$ .

## Theorem (Dicks-V, 96)

Let  $G \subseteq \text{Mon}(F_n)$  be an arbitrary set of monomorphisms of  $F_n$ . Then,  $\text{Fix}(G)$  is inert; in particular,  $r(\text{Fix}(G)) \leq n$ .

## Theorem (Bergman, 99)

Let  $G \subseteq \text{End}(F_n)$  be an arbitrary set of endomorphisms of  $F_n$ . Then,  $r(\text{Fix}(G)) \leq n$ .

## Conjecture (V.)

Let  $\phi \in \text{End}(F_n)$ . Then  $\text{Fix}(\phi)$  is inert.

# The four families

## Definition

A subgroup  $H \leq F_n$  is said to be

- **1-auto-fixed** if  $H = \text{Fix}(\phi)$  for some  $\phi \in \text{Aut}(F_n)$ ,
- 1-endo-fixed if  $H = \text{Fix}(\phi)$  for some  $\phi \in \text{End}(F_n)$ ,
- auto-fixed if  $H = \text{Fix}(S)$  for some  $S \subseteq \text{Aut}(F_n)$ ,
- endo-fixed if  $H = \text{Fix}(S)$  for some  $S \subseteq \text{End}(F_n)$ ,

Easy to see that 1-mono-fixed = 1-auto-fixed.

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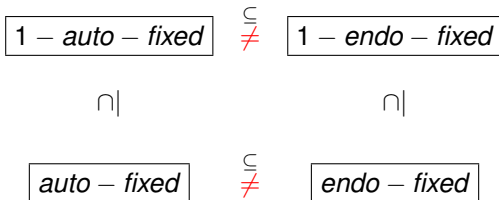
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Easy to see that 1-mono-fixed = 1-auto-fixed.

# Relations between them

$$\boxed{1 - \text{auto} - \text{fixed}} \subseteq \boxed{1 - \text{endo} - \text{fixed}}$$
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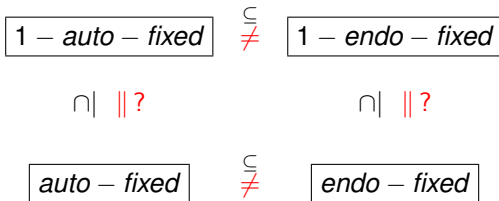


Example (Martino-V., 03; Ciobanu-Dicks, 06)

Let  $F_3 = \langle a, b, c \rangle$  and  $H = \langle b, \text{cacbab}^{-1}c^{-1} \rangle \leq F_3$ . Then,  $H = \text{Fix}(a \mapsto 1, b \mapsto b, c \mapsto \text{cacbab}^{-1}c^{-1})$ , but  $H$  is **NOT** the fixed subgroup of any set of automorphism of  $F_3$ .



# Relations between them



## Conjecture

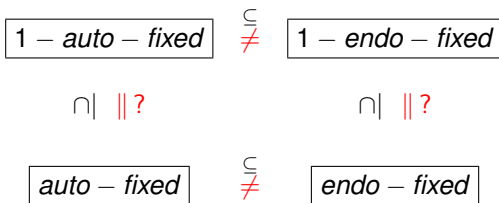
*For every  $S \subseteq \text{End}(F_n)$  ( $S \subseteq \text{Aut}(F_n)$ ) there exists  $\phi \in \text{End}(F_n)$  ( $\phi \in \text{Aut}(F_n)$ ) such that  $\text{Fix}(S) = \text{Fix}(\phi)$ .*

## Theorem (Martino-V., 00)

*Let  $S \subseteq \text{End}(F_n)$ . Then,  $\exists \phi \in \langle S \rangle$  such that  $\text{Fix}(S) \leq_{\text{ff}} \text{Fix}(\phi)$ .*

But... free factors of 1-endo-fixed (1-auto-fixed) subgroups need not be even endo-fixed (auto-fixed).

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# Outline

- 1 Some history
- 2 Algorithmic results**
- 3 Needed tools
- 4 The proof

# Computing fixed subgroups

Proposition (Turner, 86)

*There exists a **pseudo-algorithm** to compute fix of an endo.*

Easy but is **not** an algorithm...

Theorem (Maslakova, 03)

*Fixed subgroups of automorphisms of  $F_n$  are computable.*

Difficult but it **is** an algorithm!

Conjecture

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# Deciding fixedness

In this talk, I'll solve the two dual problems:

## Theorem

*Given  $H \leq_{\text{fg}} F_n$ , one can algorithmically decide whether*

- i)  $H$  is auto-fixed or not,*
- ii)  $H$  is endo-fixed or not,*

*and in the affirmative case, find a finite family,  $S = \{\phi_1, \dots, \phi_m\}$ , of automorphisms (endomorphisms) of  $F_n$  such that  $\text{Fix}(S) = H$ .*

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# Fixed closures

## Definition

Given  $H \leq_{\text{fg}} F_n$ , we define the (*auto-* and *endo-*) *stabilizer of  $H$* , respectively, as

$$\text{Aut}_H(F_n) = \{\phi \in \text{Aut}(F_n) \mid H \leq \text{Fix}(\phi)\} \leq \text{Aut}(F_n)$$

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$$\text{End}_H(F_n) = \{\phi \in \text{End}(F_n) \mid H \leq \text{Fix}(\phi)\} \leq \text{End}(F_n)$$

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Given  $H \leq F_n$ , we define the *auto-closure* and *endo-closure* of  $H$  as

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# Main result

## Theorem

*For every  $H \leq_{\text{fg}} F_n$ ,  $a\text{-Cl}(H)$  and  $e\text{-Cl}(H)$  are finitely generated and one can algorithmically compute bases for them.*

## Corollary

*Auto-fixedness and endo-fixedness are decidable.*

Observe that  $e\text{-Cl}(H) \leq a\text{-Cl}(H)$  but, in general, they are not equal.

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# Retracts

## Definition

A subgroup  $H \leq F_n$  is a **retract** if there exists a **retraction**, i.e. a morphism  $\rho: F_n \rightarrow H$  which restricts to the identity of  $H$ .

Free factors are retracts, but there are more.

## Observation

If  $H \leq F_n$  is a retract then  $r(H) \leq n$  (and,  $r(H) = n \Leftrightarrow H = F_n$ ).

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# The stable image

## Definition

Let  $\phi \in \text{End}(F_n)$ . The *stable image* of  $\phi$  is  $F_n\phi^\infty = \bigcap_{i=1}^{\infty} F_n\phi^i$ .

## Theorem (Imrich-Turner, 89)

For every endomorphism  $\phi: F_n \rightarrow F_n$ ,

- i)  $F_n\phi^\infty$  is  $\phi$ -invariant,
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**Example:** For  $\phi: F_2 \rightarrow F_2$ ,  $a \mapsto a$ ,  $b \mapsto b^2$ , we have  $F_2\phi = \langle a, b^2 \rangle$ ,  
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# Stallings' graphs and intersections

## Theorem (Stallings, 83)

*For any free group  $F_n = F(A)$ , there is an effectively computable bijection*

$$\{\text{f.g. subgroups of } F_n\} \longleftrightarrow \{\text{finite } A\text{-labeled core graphs}\}$$

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An extension of subgroups  $H \leq K \leq F_n$  is called *algebraic*, denoted  $H \leq_{\text{alg}} K$ , if  $H$  is not contained in any proper free factor of  $K$ . Write

$$\mathcal{AE}(H) = \{K \leq F_n \mid H \leq_{\text{alg}} K\}.$$

## Theorem (Takahasi, 51)

If  $H \leq_{\text{fg}} F_n$  then  $\mathcal{AE}(H)$  is finite and computable (i.e.  $H$  has finitely many algebraic extensions, all of them are finitely generated, and bases are computable from  $H$ ).

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Every extension of subgroups  $H \leq K \leq F_n$  factors in a unique way as  $H \leq_{\text{alg}} L \leq_{\text{ff}} K \leq F_n$ .



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## Theorem (McCool)

*Let  $H \leq_{\text{fg}} F_n$ . Then  $\text{Aut}_H(F_n)$  is finitely generated (in fact, finitely presented) and a finite set of generators (and relations) is algorithmically computable from  $H$ .*

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*For every  $H \leq_{\text{fg}} F_n$ ,  $a\text{-Cl}(H)$  is finitely generated and algorithmically computable.*

**Proof.**  $a\text{-Cl}(H) = \text{Fix}(\text{Aut}_H(F_n))$   
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For the endomorphism case, a similar approach **does not** work because:

- we **don't know** how to compute fix subgroups of endomorphisms
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## Example (Ciobanu-Dicks, 06)

Consider  $F_3 = \langle a, b, c \rangle$ , the element  $d = ba[c^2, b]a^{-1}$ , and the subgroup  $H = \langle a, d \rangle \leq F_3$ . Clearly, the morphisms

$$\begin{array}{ccc}
 \psi: F_3 \rightarrow F_3 & \phi: F_3 \rightarrow F_3 & \phi^n \psi: F_3 \rightarrow F_3 \\
 a \mapsto a & a \mapsto a & a \mapsto a \\
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 c \mapsto 1 & c \mapsto cb & c \mapsto d^n
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satisfy  $H \leq \text{Fix}(\phi^n \psi)$  for every  $n \in \mathbb{Z}$ .

With some computations, it can be shown that

$$\text{End}_H(F_3) = \{\text{Id}, \phi^n \psi \mid n \in \mathbb{Z}\}.$$

But,  $\phi^m \psi \cdot \phi^n \psi = \phi^m \psi$ . Hence,  $\text{End}_H(F_3)$  is not finitely generated.

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But,  $\phi^m \psi \cdot \phi^n \psi = \phi^m \psi$ . Hence,  $\text{End}_H(F_3)$  is not finitely generated.

Furthermore,  $a\text{-Cl}(H) = \text{Fix}(\text{Id}) = F_3$  and  $e\text{-Cl}(H) = \text{Fix}(\psi) = H$ .

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## Example (Ciobanu-Dicks, 06)

Consider  $F_3 = \langle a, b, c \rangle$ , the element  $d = ba[c^2, b]a^{-1}$ , and the subgroup  $H = \langle a, d \rangle \leq F_3$ . Clearly, the morphisms

$$\begin{array}{ccc}
 \psi: F_3 & \rightarrow & F_3 & \quad & \phi: F_3 & \rightarrow & F_3 & \quad & \phi^n \psi: F_3 & \rightarrow & F_3 \\
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## Theorem

*For every  $H \leq_{\text{fg}} F_n$ ,  $e\text{-Cl}(H)$  is finitely generated and algorithmically computable.*

**Proof.** Given  $H$  (in generators),

- Compute  $\mathcal{AE}(H) = \{H_1, H_2, \dots, H_q\}$ .
- Select those which are retracts,  $\mathcal{AE}_{\text{ret}}(H) = \{H_1, \dots, H_r\}$  ( $1 \leq r \leq q$ ).
- Write the generators of  $H$  as words on the generators of each one of these  $H_i$ 's,  $i = 1, \dots, r$ .
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