On the difficulty of inverting automorphisms of free groups

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Outline

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Motivation

(Joint work with P. Silva and M. Ladra.)

Find a group G where $\cdot$ is “easy" but $(\,)^{-1}$ is “difficult".

Natural candidate: $\mathrm{Aut}\,(F_n)$, where $F_r = \langle a_1, \ldots, a_r \mid \rangle$.

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(composing)

$F_3 = \langle a, b, c \mid \rangle.$

$$\begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ a & \mapsto & ab \\ b & \mapsto & ab^2c \\ c & \mapsto & bc^2 \end{array} \qquad \begin{array}{rcl} \psi\colon F_3 & \to & F_3 \\ a & \mapsto & bc^{-1} \\ b & \mapsto & a^{-1}bc \\ c & \mapsto & c^{-1}. \end{array}$$

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Let G be a group with a finite set of generators $A = \{a_1, \dots, a_r\}$. We have the word metric: for $g \in G$,

$$|g| = \min\{n \mid g = a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n}\}.$$

Definition

For $\theta \in Aut(G)$, note θ is determined by $a_1\theta, \dots, a_r\theta$ and define

$$||\theta||_1 = |a_1\theta| + \cdots + |a_r\theta|,$$

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$$\alpha_A(n) = \max\{||\theta^{-1}||_1 \mid \theta \in Aut(G),\ ||\theta||_1 \leqslant n\}.$$

Clearly, $\alpha_A(n) \leqslant \alpha_A(n+1)$.

The bigger is α_A, the more "difficult" will be to invert automorphisms of G (with respect to the given set of generators A).

Question

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Proposition

Let G be a group and $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_s\}$ be two finite sets of generators. Then, $\exists\ C > 0$ s. t. $\forall \theta \in Aut(G)$

$$\frac{1}{C}\|\theta\|_B \leqslant \|\theta\|_A \leqslant C\|\theta\|_B$$

Proof. Take $|b_i|_A \leqslant M$, $|a_i|_B \leqslant N$ and let $C = MNrs$.

$$\begin{aligned}
\|\theta\|_B &= |b_1\theta|_B + \cdots + |b_s\theta|_B \\
&\leqslant |b_1\theta|_A N + \cdots + |b_s\theta|_A N \\
&\leqslant N\big(|b_1|_A\|\theta\|_A + \cdots + |b_s|_A\|\theta\|_A\big) \\
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By symmetry, $\|\theta\|_A \leqslant C\|\theta\|_B$, so $\frac{1}{C}\|\theta\|_B \leqslant \|\theta\|_A$.

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Corollary

$$\frac{1}{C}\cdot\alpha_B\left(\frac{n}{C}\right)\leqslant\alpha_A(n)\leqslant C\cdot\alpha_B(Cn).$$

Proof.

$$\begin{aligned}\alpha_A(n) &= \max\{\|\theta^{-1}\|_A \mid \theta\in Aut(G),\ \|\theta\|_A\leqslant n\}\\ &\leqslant \max\{\|\theta^{-1}\|_A \mid \theta\in Aut(G),\ \|\theta\|_B\leqslant Cn\}\\ &\leqslant \max\{C\|\theta^{-1}\|_B \mid \theta\in Aut(G),\ \|\theta\|_B\leqslant Cn\}\\ &= C\cdot\max\{\|\theta^{-1}\|_B \mid \theta\in Aut(G),\ \|\theta\|_B\leqslant Cn\}\\ &= C\cdot\alpha_B(Cn).\end{aligned}$$

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Hence, $\alpha_A(n)$ is independent from A (up to a multiplicative constant in the domain and in the range).

Denote it by $\alpha_G(n)$.

Question

Are there groups G with $\alpha_G(n)$ linear ? quadratic? ... exponential?

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The same for outer autos

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For $\Theta \in Out(G)$, define

$$\|\Theta\|_1 = \min\{\|\theta\|_1 \mid \theta \in \Theta\},$$

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For a finitely generated group G,

$$\beta(n) = \max\{\|\Theta^{-1}\|_1 \mid \Theta \in Out(G),\ \|\Theta\|_1 \leqslant n\}.$$

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For the rest of the talk, $G = F_r = \langle a_1, \ldots, a_r \mid \rangle$.

For every $w \in F_r$, $|w|$ *is its free length.*
$|vw| \leqslant |v| + |w|$,
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$|w^n| \leqslant |n||w|$.

For $\theta \in Aut(F_r)$ and $\Theta \in Out(F_r)$,

$$||\theta||_1 = |a_1\theta| + \cdots + |a_r\theta|, \qquad ||\theta||_\infty = \max\{|a_1\theta|, \ldots, |a_r\theta|\}$$
$$||\Theta||_1 = \min\{||\theta||_1 \mid \theta \in \Theta\} \qquad ||\Theta||_\infty = \min\{||\theta||_\infty \mid \theta \in \Theta\}.$$

For $r \geqslant 2$,

$$\alpha_r(n) = \max\{||\theta^{-1}||_1 \mid \theta \in Aut\, F_r,\ ||\theta||_1 \leqslant n\},$$

$$\beta_r(n) = \max\{||\Theta^{-1}||_1 \mid \Theta \in Out\, F_r,\ ||\Theta||_1 \leqslant n\}.$$

Main results

Theorem

For rank $r = 2$ we have

(i) *for* $n \geqslant 4$, $\alpha_2(n) \leqslant \frac{(n-1)^2}{2}$,

(ii) *for* $n \geqslant n_0$, $\frac{n^2}{16} \leqslant \alpha_2(n)$,

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For $r \geqslant 3$ there exist $K = K(r)$, $K' = K'(r)$, and $M = M(r)$ such that, for $n \geqslant 1$,

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A lower bound for β_r

Theorem

For $r \geqslant 2$, and $n \geqslant n_0$, we have $\frac{1}{2r^{r-1}} n^{r-1} \leqslant \beta_r(n)$.

Proof: For $r \geqslant 2$ and $n \geqslant 1$, consider

$$\begin{array}{rcl} \psi_{r,n}\colon F_r & \to & F_r \\ a_1 & \mapsto & a_1 \\ a_2 & \mapsto & a_1^n a_2 \\ a_3 & \mapsto & a_2^n a_3 \\ & \vdots & \\ a_r & \mapsto & a_{r-1}^n a_r \end{array} \qquad \begin{array}{rcl} \psi_{r,n}^{-1}\colon F_r & \to & F_r \\ a_1 & \mapsto & a_1 \\ a_2 & \mapsto & a_1^{-n} a_2 \\ & \vdots & \\ a_i & \mapsto & (a_{i-1}^{-n})\psi_{r,n}^{-1} \cdot a_i \\ & (2 \leqslant i \leqslant r) & \end{array}$$

A straightforward calculation shows that $||\psi_{r,n}||_1 = (r-1)n + r$, and $||\psi_{r,n}^{-1}||_1 = n^{r-1} + 2n^{r-2} + \cdots + (r-1)n + r \geqslant n^{r-1}$.

An detailed argument shows that these are $||[\psi_{r,n}]||_1$ and $||[\psi_{r,n}^{-1}]||_1$.

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Hence, for $n \geqslant r$,

$$\beta_r(rn) \geqslant \beta_r((r-1)n + r) \geqslant n^{r-1}.$$

Now, for n big enough, take the closest multiple of r below,

$$n \geqslant rm > n - r,$$

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$$\beta_r(n) \geqslant \beta_r(rm) \geqslant m^{r-1} > \left(\frac{n-r}{r}\right)^{r-1} = \left(\frac{n}{r} - 1\right)^{r-1} \geqslant \frac{1}{2r^{r-1}} n^{r-1}. \qquad \square$$

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Outline

Outer space

To prove the upper bound

(ii) $\beta_r(n) \leqslant Kn^M$,

we'll need to use the recently discovered metric in the outer space $\mathcal{X}_r$.

Definition

- *By graf* Γ *we mean a finite, connected graph of rank* r*, with no vertices of degree 1 or 2.*
- *A metric on* Γ *is a map* $\ell \colon E\Gamma \to [0,1]$ *such that* $\sum_{e \in E\Gamma} \ell(e) = 1$*, and* $\{e \in E\Gamma \mid \ell(e) = 0\}$ *is a forest.*
- *For a graph* Γ*,* $\Sigma_\Gamma = \{\text{metrics on } \Gamma\}$ *= a simplex with missing faces.*
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The outer space $\mathcal{X}_r$ is

$$\mathcal{X}_r = \{ (\Gamma, f, \ell) \} / \sim$$

(where $\sim$ is an equivalence relation).

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There is a natural action of $Aut(F_r)$ on $\mathcal{X}_r$, given by

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Metric on $\mathcal{X}_r$

Definition

Let $x, x' \in \mathcal{X}_r$, $x = (\Gamma, f, \ell)$, $x' = (\Gamma', f', \ell')$. A difference of markings is a map $\alpha \colon \Gamma \to \Gamma'$, which is linear over edges and $f\alpha \simeq f'$.
For such an α, define $\sigma(\alpha)$ to be its maximum slope over edges.

Definition

$\mathcal{X}_r$ admits the following "metric":

$$d(x, x') = \min\{\log(\sigma(\alpha)) \mid \alpha \text{ diff. markings}\}.$$

This minimum is achieved by Arzela-Ascoli's theorem.

This is Bestvina-AlgomKfir version of Martino-Francaviglia's original metric.

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(i) $d(x,y) \geqslant 0$, *and* $= 0 \Leftrightarrow x = y$.

(ii) $d(x,z) \leqslant d(x,y) + d(y,z)$.

(iii) *$Out(F_r)$ acts by isometries, i.e.* $d(\phi \cdot x, \phi \cdot y) = d(x,y)$.

(iv) *But...* $d(x,y) \neq d(y,x)$ *in general.*

Definition

For $\epsilon > 0$, *the* ϵ*-thick part of* $\mathcal{X}_r$ *is*

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Bestvina-AlgomKfir theorem

Theorem (Bestvina-AlgomKfir)

For any $\epsilon > 0$ there is constant $M = M(r, \epsilon)$ such that for all $x, y \in \mathcal{X}_r(\epsilon)$,

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Proof

Remind $\beta_r(n) = \max\{||\Theta^{-1}||_1 \mid \theta \in Aut\, F_r,\ ||\Theta||_1 \leqslant n\}$.

Proof. Given $\theta \in \Theta \in \mathrm{Out}\,(F_r)$, consider $x = (R_r, id, \ell_0) \in \mathcal{X}_r$, and $\theta \cdot x = (R_r, \theta, \ell_0) \in \mathcal{X}_r$, where ℓ_0 is the uniform metric.

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d(x,\ \theta \cdot x) &= \min\{\log(\sigma(\alpha)) \mid \alpha \text{ diff. markings}\} \\
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Now, using Bestvina-AlgomKfir theorem,

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Outline

The rank 2 case

These functions for $\mathrm{Aut}(F_2)$ are much easier to understand due to the following technical lemmas.

Lemma

Let $\varphi \in Aut(F_2)$ be positive. Then φ^{-1} is cyclically reduced and $||\varphi^{-1}||_1 = ||\varphi||_1$.

Lemma

For every $\theta \in Aut(F_2)$, there exist two letter permuting autos $\psi_1, \psi_2 \in Aut(F_2)$, a positive one $\varphi \in Aut^+(F_2)$, and an element $g \in F_2$, such that $\theta = \psi_1 \varphi \psi_2 \lambda_g$ and $||\varphi||_1 + 2|g| \leqslant ||\theta||_1$.

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The rank 2 case: β_2

Theorem

For every $\theta \in Aut(F_2)$, $||[\theta^{-1}]||_1 = ||[\theta]||_1$. Hence, $\beta_2(n) = n$.

Proof. Let $\theta \in \mathrm{Aut}(F_2)$, decomposed as above, $\theta = \psi_1 \varphi \psi_2 \lambda_g$. Then,

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For $n \geqslant 4$ we have $\alpha_2(n) \leqslant \frac{(n-1)^2}{2}$.

Proof. Let $\theta \in \mathrm{Aut}(F_2)$, decomposed as above, $\theta = \psi_1 \varphi \psi_2 \lambda_g$. Then, $\theta^{-1} = \lambda_{g^{-1}} \psi_2^{-1} \varphi^{-1} \psi_1^{-1}$ and

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Now from $||\varphi||_1 + 2|g| \leqslant ||\theta||_1 = n$, we deduce $|g| \leqslant \frac{n-||\varphi||_1}{2}$ and so,

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Summarizing

Theorem

For rank $r = 2$ we have

(i) for $n \geqslant 4$, $\alpha_2(n) \leqslant \frac{(n-1)^2}{2}$,

(ii) for $n \geqslant n_0$, $\frac{n^2}{16} \leqslant \alpha_2(n)$,

(iii) for $n \geqslant 1$, $\beta_2(n) = n$.

Theorem

For $r \geqslant 3$ there exist $K = K(r)$, $K' = K'(r)$, and $M = M(r)$ such that, for $n \geqslant 1$,

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