Stallings sections and virtually free groups

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(Joint work with P. Silva, X. Soler-Escrivà)

Outline

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1. The bijection between subgroups of F_A and Stallings automata

2. Many applications

3. Moving out of free groups

4. Stallings sections

5. Virtually free groups

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $\widetilde{A} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
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Stallings automata

Definition

A Stallings automata is a finite A-labeled oriented graph with a distinguished vertex, (X, v), such that:

1- *X is connected,*

2- *no vertex of degree 1 except possibly v (X is a core-graph),*

3- *no two edges with the same label go out of (or in to) the same vertex.*

NO :

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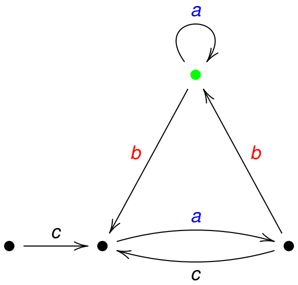

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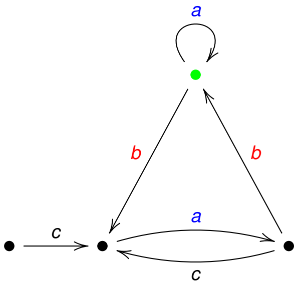

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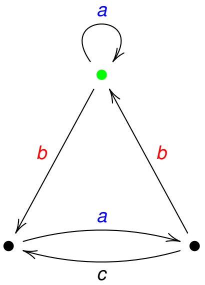

Stallings automata

In the influent paper

J. R. Stallings, Topology of finite graphs, Inventiones Math. 71 (1983), 551-565,

Stallings (building on previous works) gave a bijection between finitely generated subgroups of F_A and Stallings automata:

$$\{\text{f.g. subgroups of } F_A\} \quad \longleftrightarrow \quad \{\text{Stallings automata}\},$$

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Reading the subgroup from the automata

Definition

To any given (Stallings) automaton (X, v), we associate its fundamental group:

$$\pi(X, v) = \{ \text{labels of closed paths at } v\} \leqslant F_A,$$

clearly, a subgroup of F_A.

$X=$

$$\pi(X, \bullet) = \{1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \ldots\}$$

$$\pi(X, \bullet) \not\ni bc^{-1}bcaa$$

Membership problem in $\pi(X, \bullet)$ is solvable.

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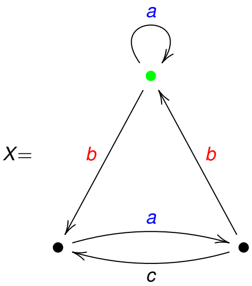

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Constructing the automata from the subgroup

In any automaton containing the following situation, for $x \in A^{\pm 1}$,

$$\bullet \xrightarrow{x} u, \qquad \bullet \xrightarrow{x} v$$

we can fold and identify vertices u and v to obtain

$$\bullet \xrightarrow{x} u = v \, .$$

This operation, $(X, v) \rightsquigarrow (X', v)$, is called a Stallings folding.

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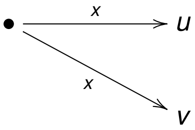

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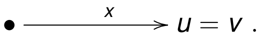

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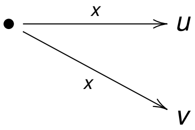

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Lemma (Stallings)

If $(X, v) \rightsquigarrow (X', v')$ is a Stallings folding then $\pi(X, v) = \pi(X', v')$.

Given a f.g. subgroup $H = \langle w_1, \ldots, w_m \rangle \leqslant F_A$ (we assume w_i are reduced words), do the following:

1- *Draw the flower automaton,*

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Well defined?

Need to see that the output does not depend on the process...

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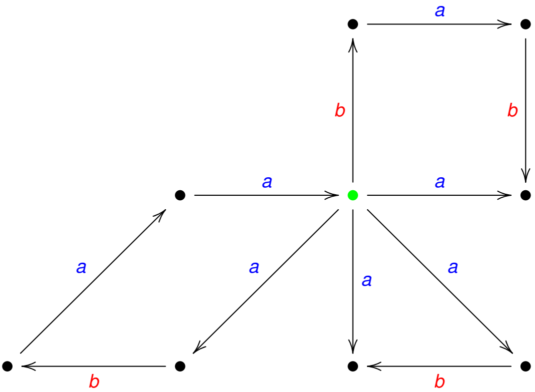

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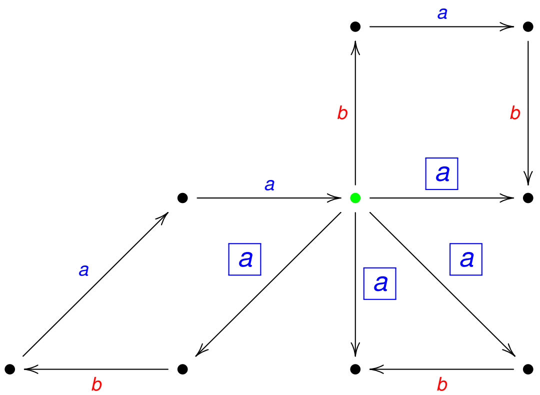

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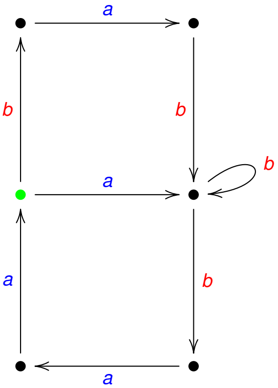

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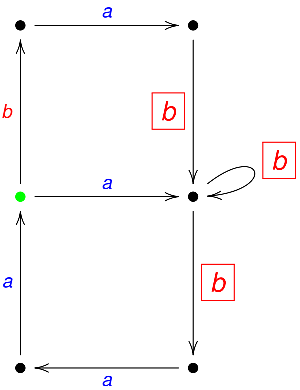

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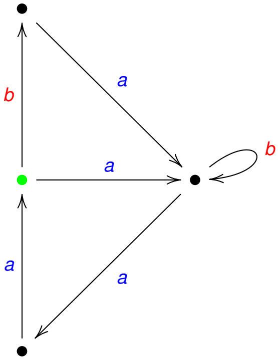

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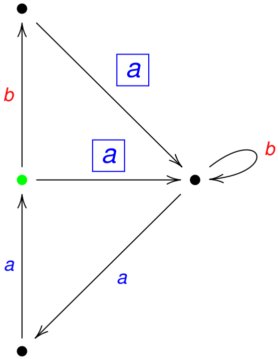

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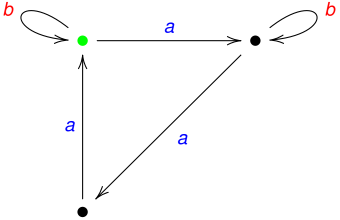

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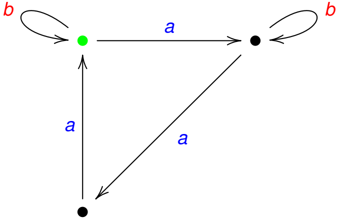

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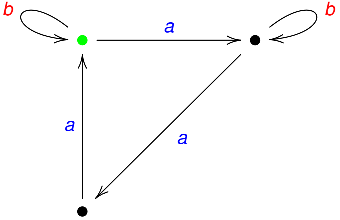

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The bijection

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The automaton $\Gamma(H)$ does not depend on the sequence of foldings.

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Theorem

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Corollary (Nielsen-Schreier)

Every subgroup of F_A is free.

- We have proved the finitely generated case, but everything extends easily to the general case.
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Membership & containment

(Membership)

Does w belong to $H = \langle w_1, \ldots, w_m \rangle$?

- Construct $\Gamma(H)$,
- Check whether w is readable as a closed path in $\Gamma(H)$ (at the basepoint).

(Containment)

Given $H = \langle w_1, \ldots, w_m \rangle$ and $K = \langle v_1, \ldots, v_n \rangle$, is $H \leqslant K$?

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Given $H = \langle w_1, \ldots, w_m \rangle$ and $K = \langle v_1, \ldots, v_n \rangle$, is $H \leqslant K$?

- Construct $\Gamma(K)$,
- Check whether all the w_i's are readable as closed paths in $\Gamma(H)$ (at the basepoint).

Membership & containment

(Membership)

Does w belong to $H = \langle w_1, \ldots, w_m \rangle$?

- Construct $\Gamma(H)$,
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Basis & conjugacy

(Computing a basis)

Given $H = \langle w_1, \ldots, w_m \rangle$, find a basis for H.

- Construct $\Gamma(H)$,
- Choose a maximal tree,
- Read the corresponding basis.

(Conjugacy)

Given $H = \langle w_1, \ldots, w_m \rangle$ and $K = \langle v_1, \ldots, v_n \rangle$, are they conjugate (i.e. $H^x = K$ for some $x \in F_A$) ?

- Construct $\Gamma(H)$ and $\Gamma(K)$,
- Check whether the are "equal" up to the basepoint.
- Every path between the two basepoints spells a valid x.

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Finite index subgroups

(Finite index)

Given $H = \langle w_1, \ldots, w_m \rangle$, we can decide whether $H \leqslant_{f.i.} F_A$; and, if yes, compute a set of coset representatives.

(Schreier index formula)

If $H \leqslant_{f.i.} F_A$ is of index $[F : H]$, then $r(H) = 1 + [F : H] \cdot (r(F_A) - 1)$.

Theorem (M. Hall)

Every f.g. subgroup $H \leqslant_{fg} F_A$ is a free factor of a finite index one, $H \leqslant_{ff} H * L \leqslant_{f.i.} F_A$.

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Theorem (Howson)

The intersection of finitely generated subgroups of F_A is again finitely generated.

Theorem

We can effectively compute a basis for $H \cap K$ from a set of generators for H and from K.

Theorem (H. Neumann)

$\tilde{r}(H \cap K) \leqslant 2\tilde{r}(H)\tilde{r}(K)$, *where* $\tilde{r}(H) = \max\{0, r(H) - 1\}$.

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Outline

Our goal

Can we extend this to other families of groups $G = \langle A \mid R \rangle$?

- *f.g. subgroups $H \leqslant G$ are not free in general,*
- *there exist subgroups $H \leqslant F_2 \times F_2$ with unsolvable membership problem,*
- *... for general G this is asking too much.*

(Goal 1)

Put conditions to the presentation $G = \langle A \mid R \rangle$ to recreate the bijection with f.g. subgroups and the membership problem, algorithmically.

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Identify which are the groups admitting such a presentation.

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The Schreier graph

Definition

The Schreier graph $\Gamma(G, H, A)$ *of a subgroup* $H \leqslant G = \langle A \mid R \rangle$ *w.r.t.* A *is:*

- *vertices: left cosets of* G *modulo* H, $V = \{Hg \mid g \in G\}$,
- *edges:* $Hg \xrightarrow{a} Hga$, *for* $g \in G$ *and* $a \in A$,
- *basepoint:* $H \cdot 1$.

Note that $\Gamma(G, H, A)$ *is finite if and only if* $H \leqslant_{f.i.} G$.

Definition

The core of a graph (Γ, v) *is the smallest subgraph containing* v *and having the same fundamental group; i.e.* $c(\Gamma)$ *is* Γ *after deletion of all "pending trees".*

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The key observation

Observation

$\Gamma(H)$ is the core of the Schreier graph $\Gamma(F_A, H, A)$, for $H \leqslant F_A$.

(Key observation)

In the free case, $\Gamma(H)$ is the "central" part of $\Gamma(F_A, H, A)$, i.e. it is a part of $\Gamma(F_A, H, A)$ such that

- *it is finite,*
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(Finite groups)

If $G = \langle A \mid R \rangle$ is finite and $H \leqslant G$, then we can take $\Gamma(H)$ to be the whole $\Gamma(G, H, A)$...

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For all the talk, $G = \langle A \mid R \rangle$ *and* $\pi : \widetilde{A}^* \twoheadrightarrow G$.

Definition

A section of π *is a subset* $S \subseteq \widetilde{A}^*$ *such that* $S\pi = G$ *and* $S^{-1} = S$.

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Given a section $S \subseteq \widetilde{A}^*$ *and* $H \leqslant_{f.g.} G$, *define* $\Gamma(G, H, A) \sqcap S$ *to be the smallest subgraph of* $\Gamma(G, H, A)$ *where you can read all* $w \in S$ *as closed paths at the basepoint.*

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In the free case, $\pi : \widetilde{A}^* \twoheadrightarrow F_A$, $S = R_A$ *is a section, and* $\Gamma(F_A, H, A) \sqcap S = \Gamma(H)$.

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Stallings sections

$$G = \langle A \mid R \rangle \quad \text{and} \quad \pi : \widetilde{A}^* \twoheadrightarrow G.$$

Definition

A section $S \subseteq \widetilde{A}^*$ is a Stallings section if

(S0) *S is a regular language and effectively computable,*

(S1) $\forall\, g \in G,\quad S_g = g\pi^{-1} \cap S$ *is rational and effectively computable,*

(S2) $\forall\, g, h \in G,\quad S_{gh} \subseteq \overline{S_g S_h}$.

Observation

If $\mathcal{A}$ is an automaton and $L \subseteq \widetilde{A}^*$ is regular and effectively computable then $\mathcal{A} \sqcap L$ is effectively computable too.

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Free groups

Proposition

For the free group $F_A = \langle A \mid - \rangle$, $S = R_A$ is a Stallings section.

Proof. $R_A\pi = F_A$ and $R_A^{-1} = R_A$.

(S0) R_A is rational and effectively computable by

Theorem (Benois)

$L \subseteq \widetilde{A}^*$ rational $\Rightarrow \overline{L} \subseteq \widetilde{A}^*$ is rational and effectively computable.

(S1) $\forall g \in F_A$, $S_g = \{\overline{g}\}$ rational and effectively computable.

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$G = \langle A \mid R \rangle$ *finite, and* $\pi\colon \widetilde{A}^* \twoheadrightarrow G$.

Proposition

For a finite group $G = \langle A \mid R \rangle$, $S = R_A$ is a Stallings section.

Proof. $R_A\pi = G$ and $R_A^{-1} = R_A$.

(S0) R_A is rational and effectively computable by Benois Theorem.

(S1) $\forall g \in G$, $S_g = g\pi^{-1} \cap R_A = \overline{g\pi^{-1}}$ is rational (because $|G| < \infty$) and effectively computable.

(S2) for $u \in S_{gh}$, take $v \in S_h$ and we have $u = \overline{uv^{-1}v} = \overline{\overline{uv^{-1}}\,v} \in \overline{S_g S_h}$. □

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Independence of the presentation

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Suppose $\langle A \mid R\rangle \simeq G \simeq \langle A' \mid R'\rangle$. Then, there exists a Stallings section for $\pi\colon \widetilde{A}^* \twoheadrightarrow G$ if and only if there exists a Stallings section for $\pi'\colon \widetilde{A'}^* \twoheadrightarrow G$.

Proof. *Take a monoid morphism $\varphi\colon \widetilde{A}^* \to \widetilde{A'}^*$ such that $\varphi\pi' = \pi$.*
If S is a Stallings section for $\pi\colon \widetilde{A}^* \twoheadrightarrow G$, then $\overline{S\varphi}$ will be a Stallings section for $\pi'\colon \widetilde{A'}^* \twoheadrightarrow G$, and viceversa. □

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Constructing $\Gamma(G, H, A) \sqcap S$

Lemma

Let S be a Stallings section for $\pi\colon \widetilde{A}^* \twoheadrightarrow G$, let $H = \langle h_1, \ldots, h_r \rangle \leqslant_{f.g.} G$, and let $\mathcal{A}$ be an inverse automaton such that

(1) $S_H \subseteq L(\mathcal{A}) \subseteq H\pi^{-1}$,

(2) *there is no path $p \xrightarrow{w} q$ with $p \neq q$ and $w\pi = 1$.*

Then, $\Gamma(G, H, A) \sqcap S = \mathcal{A} \sqcap S$.

Constructing such an $\mathcal{A}$ is possible:

- *For every letter $a \in A$: apply (S1) to obtain a finite automaton recognizing $S_{a\pi}$, and then identify all terminal vertices to get a uniterminal automaton $\mathcal{C}_{a\pi}$ such that $S_{a\pi} \subseteq \overline{L(\mathcal{C}_{a\pi})} \subseteq a\pi\pi^{-1}$.*
- *Identifying the terminal vertex of each with the initial vertex of the following one, we get uniterminal automata $\mathcal{C}_i$ such that $S_{h_i} \subseteq \overline{L(\mathcal{C}_i)} \subseteq h_i\pi^{-1}$; these are the "petals".*

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- *Identify all terminal vertices with q_0 and fold, to obtain $\mathcal{B}_1$ satisfying (1) $S_H \subseteq L(\mathcal{B}_1) \subseteq H\pi^{-1}$.*
- *Find all pairs of vertices $p \neq q$ in $\mathcal{B}_1$ for which there is a path $p \xrightarrow{w} q$ with $w\pi = 1$; identify all such pairs of vertices, to get $\mathcal{B}_2$. This new automaton satisfies (1) and (2) there is no path $p \xrightarrow{w} q$ with $p \neq q$ and $w\pi = 1$.*

Theorem

Let S be a Stallings section for $\pi \colon \widetilde{A}^* \twoheadrightarrow G$.
For every $H \leqslant_{f.g.} G$, $\Gamma(G, H, A) \sqcap S$ is effectively computable and satisfies $S_H \subseteq L(\Gamma(G, H, A) \sqcap S) \subseteq H\pi^{-1}$.

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Membership

Corollary

Let S be a Stallings section for $\pi \colon \widetilde{A}^* \twoheadrightarrow G$, $H \leqslant_{f.g.} G$, and $g \in G$. TFAE:

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(b) $S_g \subseteq L(\Gamma(G, H, A) \sqcap S)$,

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Hence, the membership problem is solvable in G.

Proof.
(a) $\Rightarrow$ (b). If $g \in H$ then $S_g \subseteq S_H \subseteq L(\Gamma(G, H, A) \sqcap S)$.
(b) $\Rightarrow$ (c). $S_g \neq \emptyset$ because S is a section.
(c) $\Rightarrow$ (a). Take $w \in S_g \cap L(\Gamma(G, H, A) \sqcap S)$ and we have $g = w\pi \in H$.
The decidability comes from computability of $\Gamma(G, H, A) \sqcap S$, and intersection of regular languages being regular and computable. $\square$

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Amalgamation and HNN

After several quite technical computations...

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If G_1 and G_2 are groups with Stallings sections, and H is a finite subgroup of both, then the amalgamated product $G_1 *_H G_2$ also admits a Stallings section.

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