# Most groups are hyperbolic... or trivial ? It depends on the glasses in use...

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# Outline

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# Gromov's claim

**Claim (Gromov '87)**

*Most finite presentations of groups, present an hyperbolic infinite group.*

- Stated in his influential paper on hyperbolic groups: "Essays in group theory", 75-263, Springer, 1987,
- no proof, only the idea,
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# The meaning of “most”

Let $X$ be an infinite set. What is the meaning of sentences like “most elements in $X$ have property $\mathcal{P}$” ?

- Define a notion of size, $|\cdot| \colon X \to \mathbb{N}$, with finite preimages.
- Define the balls: $B(n) = \{x \in X \mid |x| \leqslant n\}$ (which are finite).
- Count the proportion $\rho_n = \frac{|\{x \in B(n) \mid x \text{ satisfies } \mathcal{P}\}|}{|B(n)|} = \frac{|\mathcal{P} \cap B(n)|}{|B(n)|}$.
- Define the density of $\mathcal{P}$ as $\rho = \lim_{n\to\infty} \rho_n \quad (\in [0, 1]$ if it exists).
- $\mathcal{P}$ is generic (or generically many elements satisfy $\mathcal{P}$) if $\rho = 1$.
- $\mathcal{P}$ is negligible if $\rho = 0$.

Of course, everything depends on the chosen size function, i.e. on the direction to infinity inside $X$.

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# Classical example: visible points

**Definition**

*A point* $(x_1, \dots, x_k) \in \mathbb{Z}^k$ *is visible if* $\gcd(x_1, \dots, x_k) = 1$.

**Theorem (Mertens, 1874 (case $k = 2$))**

*The density of visible points in* $\mathbb{Z}^k$ *is* $1/\zeta(k)$*, where* $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$ *is the Riemann zeta-function (with respect to* $||\cdot||_\infty$*).*

*In particular, visible points in the plane have density* $\frac{6}{\pi^2}$.

With artificial definitions of size, one can force it to be any $\alpha \in [0, 1]$.

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# Arzhantseva-Ol'shanskii's proof

- Fix $r \geqslant 2$ and $k \geqslant 1$.
- Consider the free group $F_A = \langle a_1, \ldots, a_r \mid - \rangle$.
- In $F_A$ we have the natural notion of size and balls.
- For $w_1, \ldots, w_k \in F_A$, let $G_{w_1,\ldots,w_k} = \langle a_1, \ldots, a_r \mid w_1, \ldots, w_k \rangle$.

**Theorem (Arzhantseva-Ol'shanskii, '96)**

$$\exists \quad \lim_{n \to \infty} \frac{|\{(w_1, \ldots, w_k) \in B(n)^k \mid G_{w_1,\ldots,w_k} \text{ is infinite hyperbolic } \}|}{|B(n)|^k} = 1.$$

- Hence, generically many presentations present an infinite hyperbolic group.
- The proof is a detailed counting, using the notion of small cancelation.

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# Comments

- This fits the algebraic intuition: the longer the relations are, the closest will the group be to a free group.
- Problem-1: this counts $r$-generated, $k$-related groups, with $r$ and $k$ fixed.
- Problem-2: this counts presentations, not really groups !
- maybe different $k$-tuples $(w_1, \ldots, w_k) \neq (w'_1, \ldots, w'_k)$ generate the same subgroup $\langle w_1, \ldots, w_k \rangle = \langle w'_1, \ldots, w'_k \rangle$.
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# A new point of view

## Observation

*Let* $N = \langle w_1, \ldots, w_k \rangle \leqslant F_A$. *Then,*

$$\langle a_1, \ldots, a_r \mid w_1, \ldots, w_k \rangle \simeq \langle a_1, \ldots, a_r \mid N \rangle.$$

and let us count f.g. subgroups $N$ of $F_A$, instead of counting $k$-tuples of words.

Advantages:

- $r$ still fixed, but not $k$.
- less redundancy.
- it will be an equally natural way of counting.

... but with very different results... this is a very different direction to infinity.

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*Let* $N = \langle w_1, \ldots, w_k \rangle \leqslant F_A$. *Then,*

$$\langle a_1, \ldots, a_r \mid w_1, \ldots, w_k \rangle \simeq \langle a_1, \ldots, a_r \mid N \rangle.$$

and let us count f.g. subgroups $N$ of $F_A$, instead of counting $k$-tuples of words.

Advantages:

- $r$ still fixed, but not $k$.
- less redundancy.
- it will be an equally natural way of counting.

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# Outline

# Stallings automata

## Definition

*A Stallings automaton is a finite A-labeled oriented graph with a distinguished vertex, $(X, v)$, such that:*

1- *$X$ is connected,*

2- *no vertex of degree 1 except possibly $v$ ($X$ is a core-graph),*

3- *no two edges with the same label go out of (or in to) the same vertex.*

NO :




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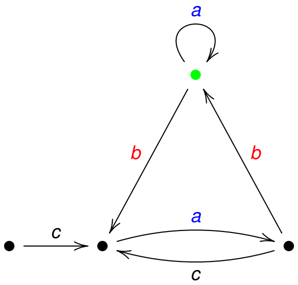


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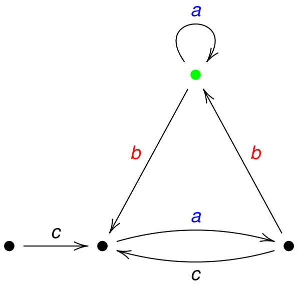


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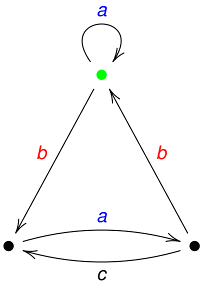

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Stallings (building on previous works) gave a bijection between finitely generated subgroups of $F_A$ and Stallings automata:

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# Reading the subgroup from the automata

## Definition

*To any given (Stallings) automaton $(X, v)$, we associate its fundamental group:*

$$\pi(X, v) = \{ \text{labels of closed paths at } v\} \leqslant F_A,$$

*clearly, a subgroup of $F_A$.*




$\pi(X, \bullet) = \{1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \ldots\}$

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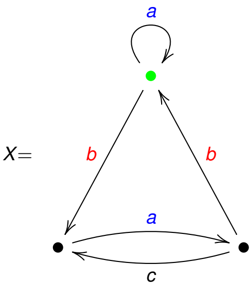


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# A basis for $\pi(X, v)$

## Proposition

*For every Stallings automaton* $(X, v)$*, the group* $\pi(X, v)$ *is free of rank* $rk(\pi(X, v)) = 1 - |VX| + |EX|$.

**Proof:**

- Take a maximal tree $T$ in $X$.
- Write $T[p, q]$ for the geodesic (i.e. the unique reduced path) in $T$ from $p$ to $q$.
- For every $e \in EX - ET$, $x_e = label(T[v, \iota e] \cdot e \cdot T[\tau e, v])$ belongs to $\pi(X, v)$.
- Not difficult to see that $\{x_e \mid e \in EX - ET\}$ is a basis for $\pi(X, v)$.
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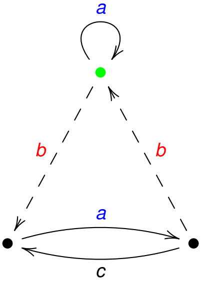


$H = \langle \quad \rangle$

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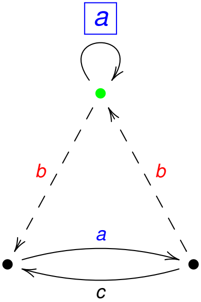


$H = \langle a, \quad \rangle$

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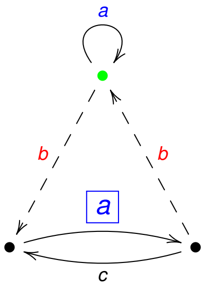


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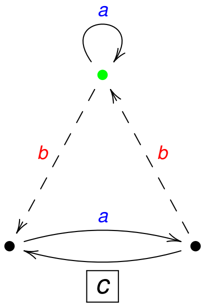


$H = \langle a,\ bab,\ b^{-1}cb^{-1} \rangle$

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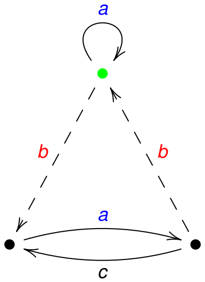


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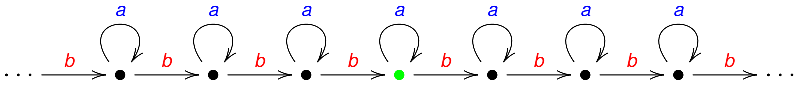
$rk(H) = 1 - 3 + 5 = 3.$

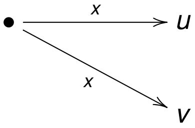
# Example-2




$$F_{\aleph_0} \simeq H = \langle \ldots, b^{-2}ab^2, b^{-1}ab, a, bab^{-1}, b^2ab^{-2}, \ldots \rangle \leqslant F_2.$$

# Constructing the automata from the subgroup

In any automaton containing the following situation, for $x \in A^{\pm 1}$,

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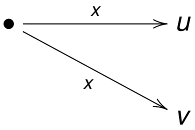


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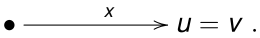


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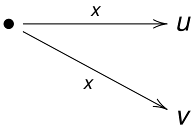


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**Lemma (Stallings)**

*If $(X, v) \rightsquigarrow (X', v')$ is a Stallings folding then $\pi(X, v) = \pi(X', v')$.*

*Given a f.g. subgroup $H = \langle w_1, \ldots w_m \rangle \leqslant F_A$ (we assume $w_i$ are reduced words), do the following:*

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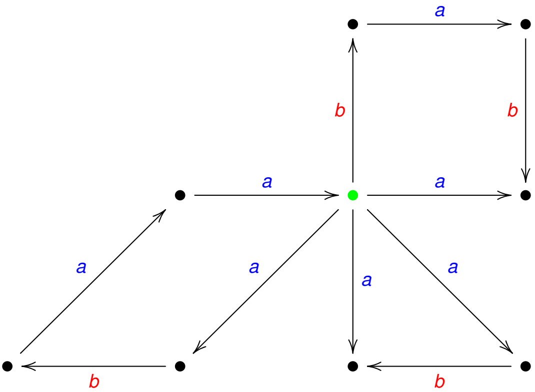


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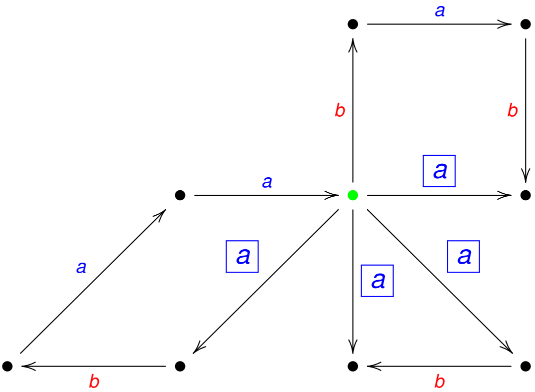


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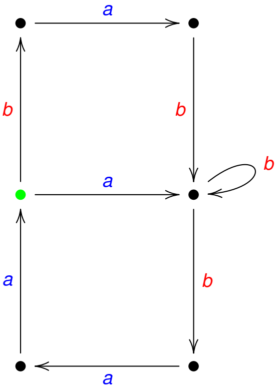


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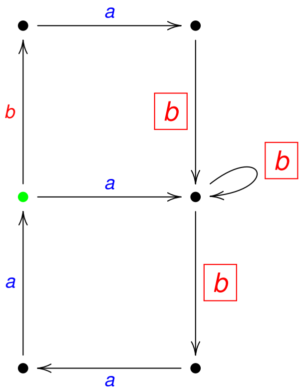


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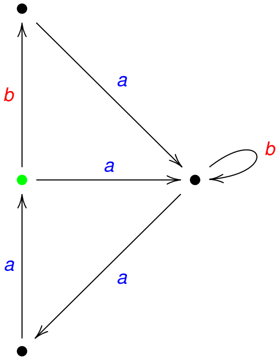


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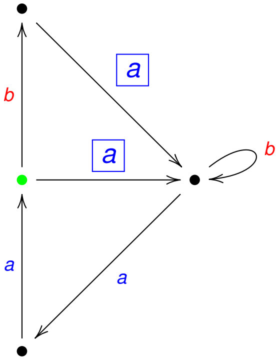


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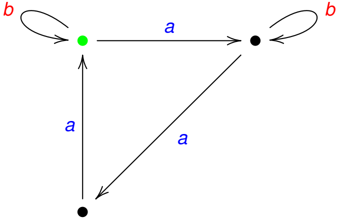


Folding $\#3$.

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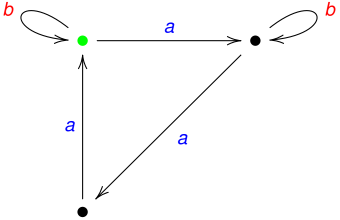


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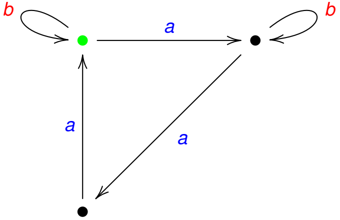


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# Local confluence

It can be shown that

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*The automaton $\Gamma(H)$ does not depend on the sequence of foldings.*

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*The following is a bijection:*

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**Corollary (Nielsen-Schreier)**

*Every subgroup of $F_A$ is free.*

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# Counting Stallings graphs

*From now on, let us think presentations as*

$$\langle a_1, \ldots, a_r \mid \Gamma \rangle,$$

*where* $\Gamma$ *is a Stallings graph.*

*The natural size function to consider is the number of vertices:*

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# Stallings' graphs as partial injections

## Definition

*Let $\Gamma$ be a Stallings graph. Every letter in $A$ determines a partial injection of the set of vertices $V\Gamma$: $a(i) = j$ iff $i \xrightarrow{a} j$ .*

$\Gamma =$




| $a: V$ | $\to$ | $V$ | $b: V$ | $\to$ | $V$ | $c: V$ | $\to$ | $V$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $\mapsto$ | 1 | 1 | $\mapsto$ | 2 | 1 | | |
| 2 | $\mapsto$ | 3 | 2 | | | 2 | | |
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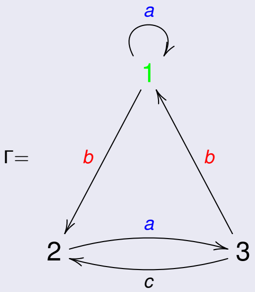


| $a: V$ | $\to$ | $V$ | $b: V$ | $\to$ | $V$ | $c: V$ | $\to$ | $V$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $\mapsto$ | 1 | 1 | $\mapsto$ | 2 | 1 | | |
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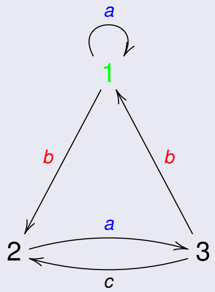


| | | | | | | | | |
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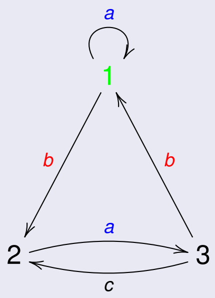
$\Gamma =$ 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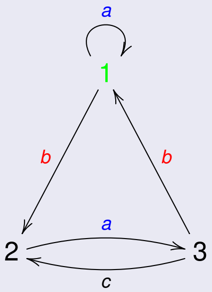
$\Gamma =$ 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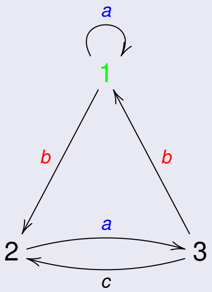
$\Gamma=$ 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*Let $I_n$ be the set of partial injections of $[n] = \{1, 2, \ldots, n\}$ (this is a monoid containing the symmetric group $S_n$).*

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## Theorem (Bassino, Nicaud, Weil, 2008)

a) $\frac{|\{\Gamma(\sigma) \mid \sigma \in I_n{}^r\}|}{|I_n|^r}$ tends to 1.

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*Generically, a Stallings graph (over A) with n vertices is just a r-tuple of partial injections on* $[n]$, $\sigma \in I_n{}^r$.

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With the word-based distribution malnormality is exponentially generic ...

## Proposition

$$\exists \quad \lim_{n\to\infty} \frac{|\{(w_1,\ldots,w_k) \in B(n)^k \mid \langle w_1,\ldots,w_k\rangle \textit{ is malnormal in } F(A)\}|}{|B(n)|^k} = 1$$

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# Outline

# Permutations and fragmented permutations

## Observation

*Any partial injection $\sigma \in I_n$ decomposes in orbits of two types: closed and open (i.e. cycles and segments).*

## Definition

*A partial injection $\sigma \in I_n$ is called a*

- *permutation if all its orbits are closed,*
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*Let $S_n$ and $J_n$, resp., be the sets of permutations and fragmented permutations in $I_n$.*

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*Every partial injection is the disjoint union of a permutation and a fragmented permutation. In particular, $|I_n| = \sum_{k=0}^{n} \binom{n}{k} |S_k||J_{n-k}| = \sum_{k=0}^{n} \frac{n!}{(n-k)!} |J_{n-k}|$.*

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*Hence*, $\frac{|J_n|}{|I_n|} = \mathcal{O}(\frac{1}{n^{1/2}})$.

# Permutations and fragmented permutations

## Definition

*a) The EGS for partial injections:* $I(z) = \sum_{n=0}^{\infty} \frac{|I_n|}{n!} z^n$.
*b) The EGS for permutations:* $S(z) = \sum_{n=0}^{\infty} \frac{|S_n|}{n!} z^n = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.
*c) The EGS for fragmented permutations:* $J(z) = \sum_{n=0}^{\infty} \frac{|J_n|}{n!} z^n$.

## Theorem

a) $I(z) = \frac{1}{1-z} e^{\frac{z}{1-z}} = 1 + 2z + \frac{7}{2} z^2 + \frac{17}{3} z^3 + \cdots$.
b) $\frac{|I_n|}{n!} = \frac{e^{2\sqrt{n}}}{2\sqrt{\pi e}} n^{-\frac{1}{4}} (1 + o(1))$.

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# Most groups are trivial

## Definition

*Let $\sigma \in I_n$. Define* $\mathrm{gcd}(\sigma)$ *as the* gcd *of the lengths of the closed orbits of $\sigma$ (if $\sigma \in J_n$, put* $\mathrm{gcd}(\sigma) = \infty$*).*

## Key observation

*Let $\sigma = (\sigma_1, \ldots, \sigma_r) \in {I_n}^r$, let $\Gamma(\sigma)$ be the corresponding (Stallings) graph, and let $G = \langle a_1, \ldots, a_r \mid \pi(\Gamma(\sigma))\rangle$. We have,*

- *if* $\mathrm{gcd}(\sigma_i) = 1$ *then $a_i = 1$ in $G$,*
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**Theorem (Bassino, Martino, Nicaud, V., Weil, 2010)**

$$\frac{|\{\sigma \in I_n \mid \gcd(\sigma) > 1\}|}{|I_n|} = \mathcal{O}(\frac{1}{n^{1/6}})$$

**Corollary (Bassino, Martino, Nicaud, V., Weil, 2010)**

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Thanks