Whitehead's classical algorithm and a modern version in polynomial time

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Outline

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Definitions and notation

- $A = \{a_1, \ldots, a_k\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_k, a_k^{-1}\}$.
- Usually, $A = \{a, b, c\}$.
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- $F_A = (A^{\pm 1})^* / \sim$ is the free group on A (words on $A^{\pm 1}$ modulo reduction).
- Every $w \in A^*$ has a unique reduced form,
- 1 denotes the empty word, and $|\cdot|$ the (shortest) length in F_A:
  $|1| = 0, \quad |aba^{-1}| = |abbb^{-1}a^{-1}| = 3, \quad |uv| \leqslant |u| + |v|$.
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Whitehead problem

Whitehead Problem

For a group G, find an algorithm s.t. given $u, v \in G$ decides whether there exists $\varphi \in Aut(G)$ such that $\varphi(u) = v$.

Theorem (Whitehead, 30's)

Whitehead problem is solvable in F_A.

"Proof":
First part: reduce $\|u\|$ and $\|v\|$ as much as possible by applying autos:

$$u \to u_1 \to u_2 \to \cdots \to u',$$

$$v \to v_1 \to v_2 \to \cdots \to v'.$$

Second part: analyze who is image of who by some auto, in the (finite!) sphere of given radius n, $S_n = \{w \in F_k \mid \|w\| = n\}$. $\square$

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Whitehead minimization problem

Let us concentrate in the first part:

Whitehead Minimization Problem (WMP)

Given $u \in F_A$, find $\varphi \in Aut(F_A)$ such that $\|\varphi(u)\|$ is minimal.

Lemma (Whitehead)

Let $u \in F_A$. If $\exists\, \varphi \in Aut(F_A)$ such that $\|\varphi(u)\| < \|u\|$ then $\exists$ a Whitehead automorphism α such that $\|\alpha(u)\| < \|u\|$.

Definition

Whitehead automorphisms are those of the form:

$$\begin{array}{rcll} F_A & \to & F_A & \\ a_i & \mapsto & a_i & \quad \text{(the multiplier)} \\ a_i \neq a_j & \mapsto & a_i^{\epsilon_j} a_j a_i^{\delta_j} & \end{array}$$

where $\epsilon_j = 0, -1$ and $\delta_j = 0, 1$ (there are $\sim k \cdot 4^k$ many, where $k = |A|$).

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Classical Whitehead's algorithm (first part)

Classical whitehead algorithm is:

- *Keep applying whitehead automorphisms to given u until finding one that decreases its cyclic length.*
- *Repeat until all whiteheads are non-decreasing.*

This is polynomial on $\|u\|$, but exponential on the ambient rank, k.

There are several recent results (theoretical, heuristic, probabilistic) suggesting that Whitehead algorithm is faster in practice.

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Improvement

Theorem (Roig, V., Weil, 2007)

There is an algorithm which solves Whitehead Minimization Problem for F_k in time $O(n^2 k^3)$.

main idea: given $u \in F_k$, we find in polynomial time one of the whiteheads that decreases $\|u\|$ the most possible.

Key point: How does a given Whitehead automorphism α affect the length of a given word u ?

Three ingredients:

1) Codify u as its Whitehead's graph (classic in Group Theory),
2) Codify α as a cut in this graph ($\approx$ classic in Group Theory),
3) Use max-flow min-cut algorithm (classic in Computer Science),
4) ... put together and mix (new!).

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Whitehead's graph

First ingredient: Whitehead's graph of a word.

Definition

Given $u \in F_k$ (cyclically reduced), its (unoriented) Whitehead graph, denoted $Wh(u)$, is:

- *vertices: $A^{\pm 1}$,*
- *edges: for every pair of (cycl.) consecutive letters $u = \cdots xy \cdots$ put an edge between x and y^{-1}.*

Example

$u = aba^{-1}c^{-1}bbabc^{-1}$,

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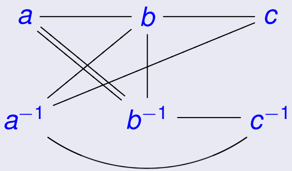

Cut in a graph

Second ingredient: Cut in a graph.

Definition

Given a Whitehead's automorphism α, we represent it as the (a, a^{-1})-cut

$$(T = \{a\} \cup \{\text{letters that go multiplied on the right by } a\}, a)$$

of the set $A^{\pm 1}$.

Example

$$\begin{array}{rcl} \alpha\colon \langle a, b, c\rangle = F_3 & \to & F_3 \\ a & \mapsto & ab \\ b & \mapsto & b \\ c & \mapsto & b^{-1}cb \end{array}$$

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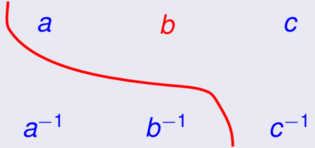

Rephrasement of Wh. Lemma

Lemma (Whitehead)

Given a word $u \in F_k$ and a Whitehead automorphism α, think α as a cut in $Wh(u)$, say $\alpha = (T, a)$, and then

$$\|\alpha(u)\| - \|u\| = \mathrm{cap}(T) - \deg(a).$$

Proof: Analyzing combinatorial cases (see Lyndon-Schupp).

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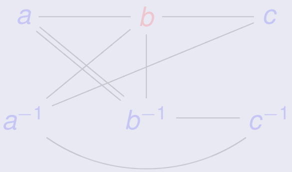
like before. We have $\alpha(u) = aba^{-1}b^{-1}c^{-1}bbbabc^{-1}b$. *Furthermore,*



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$$12 - 9 = \|\alpha(u)\| - \|u\| = \mathrm{cap}(T) - \deg(b) = 7 - 4.$$

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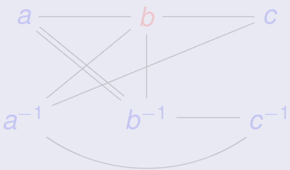

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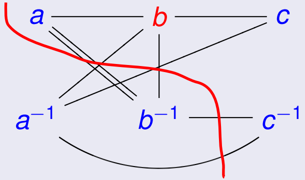

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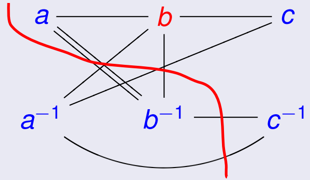

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Max-flow min-cut algorithm

Third ingredient: Max-flow min-cut algorithm.

Hence, Whitehead's Minimization Problem reduces to:

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Outline

Primitivity

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Observation

u is primitive $\Leftrightarrow$ the orbit of u contains a $\Leftrightarrow$ bottom of the orbit has length 1.

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Given a word $u \in F_k$, one can check whether u is primitive in F_k in time $O(n^2 k^3)$, where $n = \|u\|$.

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Deciding free-factorness

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A given subgroup $H \leqslant F_k$ with basis $\{h_1, \ldots, h_r\}$ ($r(H) = r \leqslant k$) is a free factor of F_k if and only if bottom of the orbit of $(h_1, \ldots, h_r)$ has length $1 + \overset{r}{\cdots} + 1 = r$.

Corollary (Roig, V., Weil, 2007)

Given a f.g. subgroup $H \leqslant F_k$, one can check whether H is a free factor of F_k in time $O((n^2k^4 + n^3k^2)\log(nk))$, where $n = \|H\|$.

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Given f.g. subgroups $H \leqslant K \leqslant F_k$, one can check whether H is a free factor of K in polynomial time w.r.t. the given generators of H and K.

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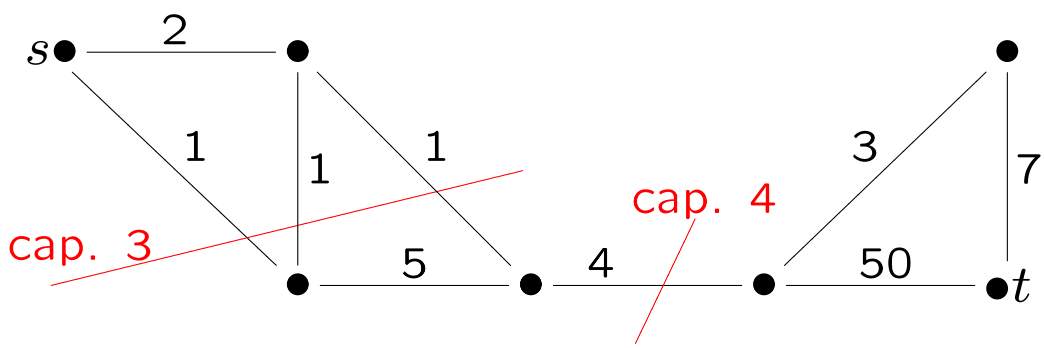
Outline

Third ingredient: max-flow min-cut algorithm.

Given a graph X (unoriented and with weights on edges), and two vertices $s, t \in VX$, find the max flow from s to t:



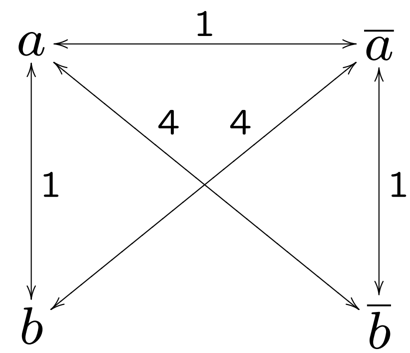
Observation:
maximal $(s \to t)$-flow $\leq$ cap. of any (s, t)-cut.

Theorem:
max. $(s \to t)$-flow $=$ cap. of min. (s, t)-cut,

and it is possible to find both in polynomial time w.r.t. the size of the graph.

Example: Find one of the best Whitehead autos for $u = ba\overline{b}\,\overline{a}\,\overline{b}\,\overline{a}\,\overline{a}baba$.

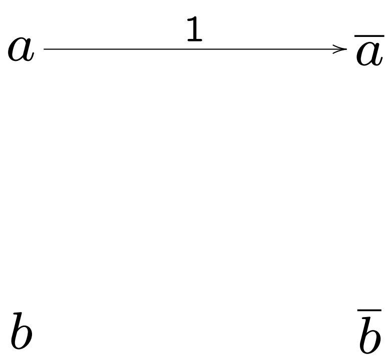
Wh. graph $=$

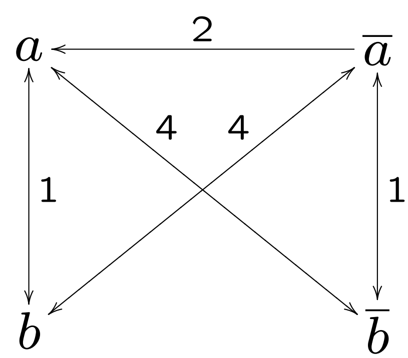


- Choose first multiplier, say a;
- Choose an augmenting path from a to $\overline{a}$:

$$a \xrightarrow{\ 1\ } \overline{a}\ ;$$

- Total flow: residual graph:





- Choose another augm. path from a to $\overline{a}$:

$$a \xrightarrow{\ 1\ } b \xrightarrow{\ 1\ } \overline{a}\ ;$$

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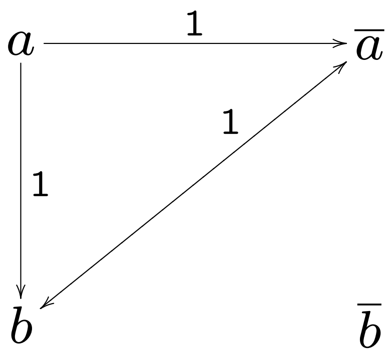

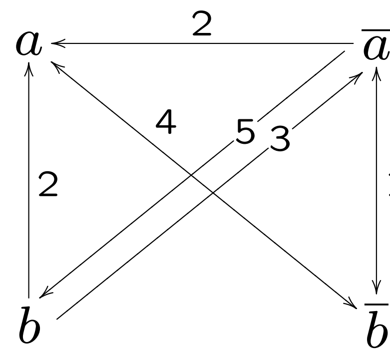

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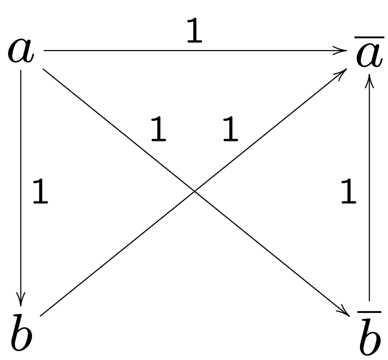
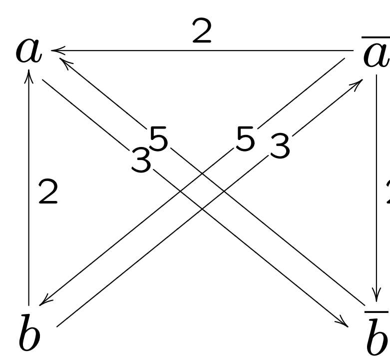
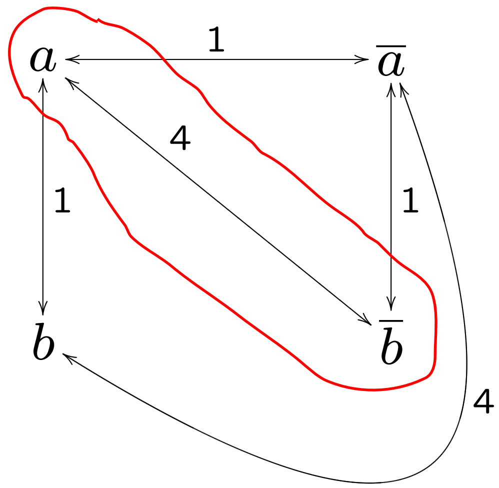




- No paths from a to $\overline{a}$, so STOP.

The total flow carried from a to $\overline{a}$ is 3 and corresponds to the cut

$$Y = \{v \mid \exists \text{ path } a \to v \text{ in res. graph}\}.$$



; $\quad$ cap($\{a, \overline{b}\}$) = 3.

So, the Whitehead auto

$$Y = \{a, \overline{b}\} \quad \equiv \quad \begin{array}{ccc} a & \overset{\alpha}{\mapsto} & a \\ b & \mapsto & \overline{a}b \end{array}$$

satisfies $\|u\alpha\| - \|u\| = 3 - 6 = -3$.

- Repeat for multiplier b (and get less).

$$\begin{array}{rcl} u = ba\overline{b}\,\overline{a}\,\overline{b}\,\overline{a}\,\overline{a}baba & \mapsto & (\not\overline{a}b)a(\overline{b}\not a)\not\overline{a}(\overline{b}\not a)\not\overline{a}\,\overline{a}(\overline{a}b)\not a(\not\overline{a}b)\not a \\ & \sim & ba\overline{b}\,\overline{b}\overline{a}\,\overline{a}bb \end{array}$$

$$\|u\| = 11 \quad , \quad \|u\alpha\| = 8.$$

THANKS