Whitehead's classical algorithm for subgroups

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McGill seminar, Montreal

Oct. 20th, 2010.

Outline

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Definitions and notation

- $A = \{a_1, \ldots, a_k\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_k, a_k^{-1}\}$.
- Usually, $A = \{a, b, c\}$.
- $(A^{\pm 1})^*$ the free monoid on $A^{\pm 1}$ (words on $A^{\pm 1}$).
- $F_A = (A^{\pm 1})^* / \sim$ is the free group on A (words on $A^{\pm 1}$ modulo reduction).
- Every $w \in A^*$ has a unique reduced form,
- 1 denotes the empty word, and $|\cdot|$ the (shortest) length in F_A:
  $|1| = 0, \quad |aba^{-1}| = |abbb^{-1}a^{-1}| = 3, \quad |uv| \leqslant |u| + |v|$.
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Whitehead problem

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For a group G, find an algorithm s.t. given $u, v \in G$ decides whether there exists $\varphi \in Aut(G)$ such that $\varphi(u) = v$.

Theorem (Whitehead, 30's)

Whitehead problem is solvable in F_A.

"Proof":
First part: reduce $\|u\|$ and $\|v\|$ as much as possible by applying autos:

$$u \to u_1 \to u_2 \to \cdots \to u',$$

$$v \to v_1 \to v_2 \to \cdots \to v'.$$

Second part: analyze who is image of who by some auto, in the (finite!) sphere of given radius n, $S_n = \{w \in F_k \mid \|w\| = n\}$. □

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Whitehead minimization in polynomial time

Definition

Whitehead automorphisms are those of the form:

$$\begin{array}{rcll} F_A & \to & F_A & \\ a_i & \mapsto & a_i & \textit{(the multiplier)} \\ a_i \neq a_j & \mapsto & a_i^{\epsilon_j} a_j a_i^{\delta_j} & \end{array}$$

where $\epsilon_j = 0, -1$ *and* $\delta_j = 0, 1$ *(there are* $\sim k \cdot 4^k$ *many, where* $k = |A|$*).*

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Exactly the same can be done for finitely generated subgroups...
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There is an algorithm which solves Whitehead Minimization Problem for subgroups $H \leqslant F_k$, in time $O((n^2k^4 + n^3k^2)\log(nk))$, where $n = \|H\|$.

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Definition

A Stallings automaton *is a finite A-labeled oriented graph with a distinguished vertex, (X, v), such that:*

1- *X is connected,*

2- *no vertex of degree 1 except possibly v (X is a* core-graph*),*

3- *no two edges with the same label go out of (or in to) the same vertex.*

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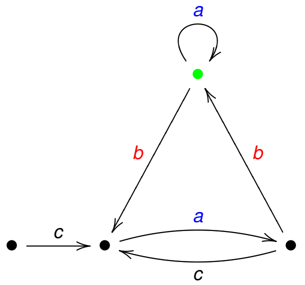

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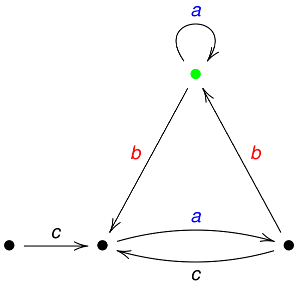

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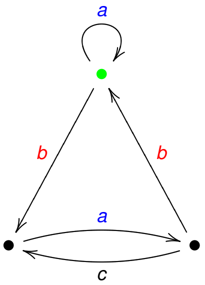

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Stallings (building on previous works) gave a bijection between finitely generated subgroups of F_A and Stallings automata:

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Reading the subgroup from the automata

Definition

To any given (Stallings) automaton (X, v)*, we associate its fundamental group:*

$$\pi(X, v) = \{ \textit{ labels of closed paths at } v \} \leqslant F_A,$$

clearly, a subgroup of F_A*.*

$$\pi(X, \bullet) = \{1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \ldots\}$$

$$\pi(X, \bullet) \not\ni bc^{-1}bcaa$$

Membership problem in $\pi(X, \bullet)$ is solvable.

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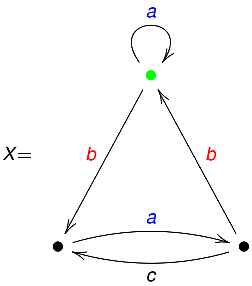

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A basis for $\pi(X, v)$

Proposition

For every Stallings automaton (X, v), the group $\pi(X, v)$ is free of rank $rk(\pi(X, v)) = 1 - |VX| + |EX|$.

Proof:

- Take a maximal tree T in X.
- Write $T[p, q]$ for the geodesic (i.e. the unique reduced path) in T from p to q.
- For every $e \in EX - ET$, $x_e = label(T[v, \iota e] \cdot e \cdot T[\tau e, v])$ belongs to $\pi(X, v)$.
- Not difficult to see that $\{x_e \mid e \in EX - ET\}$ is a basis for $\pi(X, v)$.
- And, $$\begin{aligned} |EX - ET| &= |EX| - |ET| \\ &= |EX| - (|VT| - 1) = 1 - |VX| + |EX|. \ \square \end{aligned}$$

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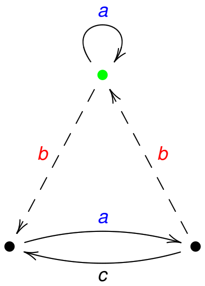
Example



$H = \langle \quad \rangle$

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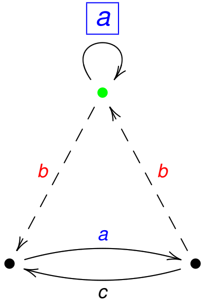

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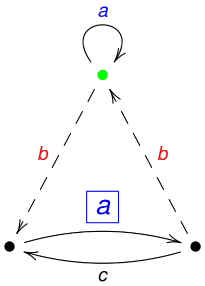

$H = \langle a,\ bab,\ \quad \rangle$

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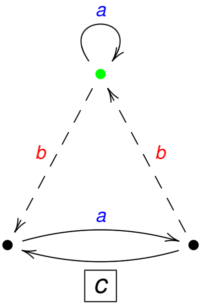

$H = \langle a,\ bab,\ b^{-1}cb^{-1} \rangle$

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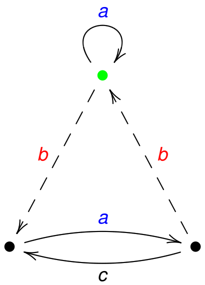
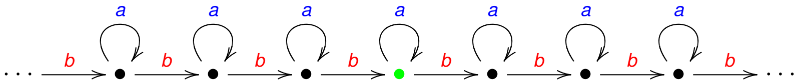


$H = \langle a,\ bab,\ b^{-1}cb^{-1} \rangle$
$rk(H) = 1 - 3 + 5 = 3.$

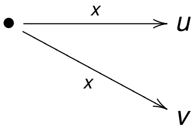
Example-2



$$F_{\aleph_0} \simeq H = \langle \ldots, b^{-2}ab^2, b^{-1}ab, a, bab^{-1}, b^2ab^{-2}, \ldots \rangle \leqslant F_2.$$

Constructing the automata from the subgroup

In any automaton containing the following situation, for $x \in A^{\pm 1}$,



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$$\bullet \xrightarrow{\ x\ } u = v \ .$$

This operation, $(X, v) \rightsquigarrow (X', v)$, is called a Stallings folding.

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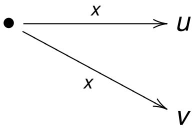

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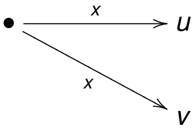

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Lemma (Stallings)

If $(X, v) \rightsquigarrow (X', v')$ is a Stallings folding then $\pi(X, v) = \pi(X', v')$.

Given a f.g. subgroup $H = \langle w_1, \ldots, w_m \rangle \leqslant F_A$ (we assume w_i are reduced words), do the following:

1- *Draw the flower automaton,*

2- *Perform successive foldings until obtaining a Stallings automaton, denoted $\Gamma(H)$.*

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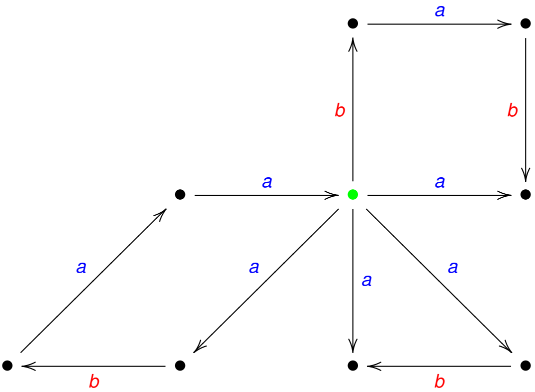

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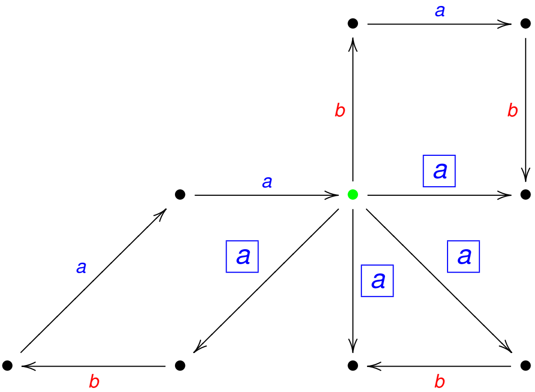

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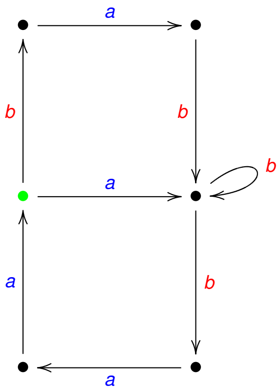

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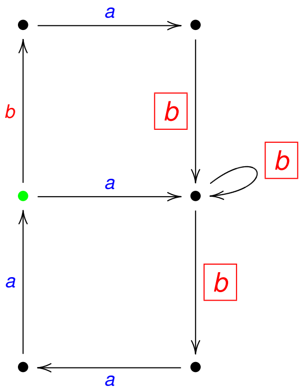

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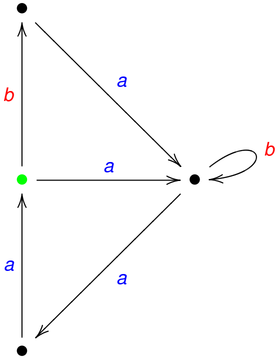

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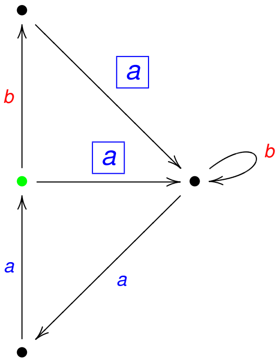

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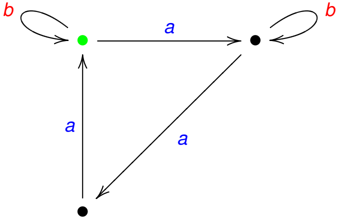

Folding #3. $\Gamma(H)$

By Stallings Lemma, $\pi(\Gamma(H), \bullet) = \langle baba^{-1}, aba^{-1}, aba^2 \rangle$

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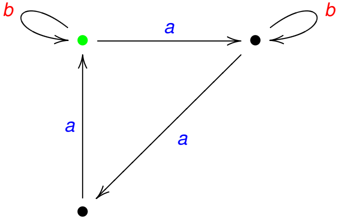

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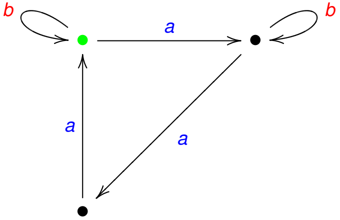

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By Stallings Lemma, $\pi(\Gamma(H), \bullet) = \langle baba^{-1}, aba^{-1}, aba^2 \rangle = \langle b, aba^{-1}, a^3 \rangle$

Local confluence

It can be shown that

Proposition

The automaton $\Gamma(H)$ does not depend on the sequence of foldings

Proposition

The automaton $\Gamma(H)$ does not depend on the generators of H.

Theorem

The following is a bijection:

$$\begin{array}{rcl} \{\text{f.g. subgroups of } F_A\} & \longleftrightarrow & \{\text{Stallings automata}\} \\ H & \to & \Gamma(H) \\ \pi(X,v) & \leftarrow & (X,v) \end{array}$$

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Every subgroup of F_A is free.

- Finite automata work for the finitely generated case, but everything extends easily to the general case (using infinite graphs).
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Outline

Peak reduction for subgroups

Definition

For $H \leqslant F_A$*, we define* $\|H\| = \#V(\Gamma'(H))$.

For a cyclic word w*, it is clear that* $\|w\| = \|\langle w\rangle\|$.

Definition

A peak in F_A *is a trio* (H, σ, τ) *where* $H \leqslant_{\mathrm{fg}} F_A$, $\sigma, \tau \in W_I \cup W_{II}$*, such that* $\|\sigma(H)\| \leqslant \|H\|$ *and* $\|\tau(H)\| \leqslant \|H\|$ *with at least one inequality strict.*

Lemma (Peak reduction for subgroups)

For every peak (H, σ, τ) *there exists* $s \geqslant 1$ *and* $\rho_1, \ldots, \rho_s \in W_I \cup W_{II}$ *such that*

- $\tau\sigma^{-1} = \rho_s \cdots \rho_1$,
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Whitehead algorithm for subgroups

Then all the arguments and algorithms extend naturally to this more general context...

Theorem ("Gersten", 1984)

Given two subgroups $H, K \leqslant_{fg} F_A$, it is decidable whether there exists $\varphi \in Aut(F_A)$ such that $\varphi(H) = K$.

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Let's do the corresponding improvement with "max-flow min-cut" techniques, to obtain a polynomial time algorithm in this last case ...

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Whitehead's hypergraph

Thinking a cyclically reduced word w as the circle $\Gamma'(\langle w \rangle)$, its Whitehead graph $Wh(u)$ just describes the in-links of the vertices.

Definition

Let $H \leqslant F_k$ be a f.g. subgroup, and let $\Gamma'(H)$ be its core graph. We define the Whitehead hyper-graph of H, denoted $Wh(H)$, as:

- *vertices: $A^{\pm 1}$,*
- *hyper-edges: for every vertex v in $\Gamma'(H)$, put a hyper-edge consisting on the in-link of v.*

Lemma (Roig, V., Weil, 2007)

Given a f.g. subgroup $H \leqslant F_k$ and a Whitehead automorphism α, think α as a cut in $Wh(H)$, say $\alpha = (T, a)$, and then

$$\|\alpha(H)\| - \|H\| = \mathrm{cap}(T) - \deg(a),$$

where $\mathrm{cap}(T)$ is the number of hyper-edges with at least one vertex in T and one outside T.

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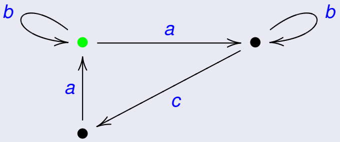
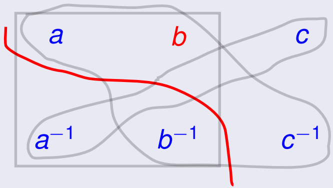
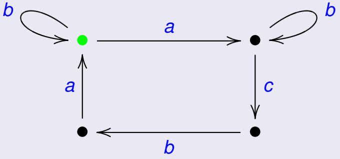
Consider $H = \langle b, aba^{-1}, aca \rangle \leqslant F_3$*. Its core graph* $\Gamma(H)$*, and Whitehead hyper-graph* $Wh(H)$ *are:*





In fact, $\alpha(H) = \langle b, aba^{-1}, acbab \rangle$ *and then*

$\Gamma(\alpha(H)) =$



and so, $4 - 3 = \|\alpha(H)\| - \|H\| = 3 - 2$.

So, Whitehead's Minimization Problem for subgroups reduces to:

- *run over all possible multipliers, say a, (there are $2k$),*
- *find an (a, a^{-1})-cut with minimal possible capacity in the given hyper-graph.*

Unfortunately, there is no analog of max-flow min-cut algorithm for hyper-graphs ...

...but it is still possible to find minimal cuts in polynomial time because of sub-modularity:

Observation

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A map $f\colon \mathcal{P}(V) \to \mathbb{N}$ *is called* ***sub-modular*** *if, for every* $A, B \subseteq V$, $f(A \cup B) + f(A \cap B) \leqslant f(A) + f(B)$.

Efficient minimization of sub-modular functions is an active research topic in computer science. One of the classical results is the following

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There exists a algorithm which, given a sub-modular function $f\colon \mathcal{P}(V) \to \mathbb{N}$ *computes its minimum with a number of queries to evaluate* f *bounded above by a polynomial on* $|V|$.

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