A recursive presentation for Mihailova's subgroup

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&

CRM - Montreal

Montreal, August 19, 2010

(joint with O. Bogopolski)

Outline

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Mihailova's subgroup

Let $n \geqslant 2$, and F_n be the free group with basis $X = \{x_1, \ldots, x_n\}$.

Consider a finite presentation $H = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_m \rangle$.

Associated to it K.A. Mihailova in 1958 considered

$$M(H) = \{(w_1, w_2) \in F_n \times F_n \mid w_1 =_H w_2\} \leqslant F_n \times F_n,$$

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It has two interesting properties:

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Finite generation

Observation

$M(H)$ is generated by $\{(x_1, x_1), \ldots, (x_n, x_n), (1, R_1), \ldots, (1, R_m)\}$.

Proof. Given $(w_1, w_2) \in M(H)$,

- $w_1 =_H w_2$
- $w_1^{-1} w_2$ is a product of conjugates of the R_j's
- For every $z \in F_n$, $(1, z^{-1} R_j z) = (z^{-1}, z^{-1})(1, R_j)(z, z)$
- $(1, w_1^{-1} w_2) \in \langle (x_1, x_1), \ldots, (x_n, x_n), (1, R_1), \ldots, (1, R_m) \rangle$
- and $(w_1, w_2) = (w_1, w_1)(1, w_1^{-1} w_2)$. $\square$

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Membership problem

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$MP(M(H), F_n \times F_n)$ is solvable $\Longleftrightarrow$ $WP(H)$ is solvable

Proof. Obvious.

$MP(M(H), F_n \times F_n)$: given $(w_1, w_2) \in F_n \times F_n$ decide whether $(w_1, w_2) \in M(H)$ or not.

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$M(H)$ is recursively presented

- $F_n \times F_n$ has solvable word problem
- so, $M(H)$ has solvable word problem
- so, $M(H)$ is recursively presented.
- F. Grunewald, 1978: $M(H)$ is finitely presented $\Longleftrightarrow$ H is finite.
- Later results by Baumslag-Roseblade, Short, and Bridson-Wise also imply Grunewald's result.

Question (Grigorchuk, 2005)

Find explicit presentations for Mihailova's subgroups of $F_n \times F_n$.

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Main result

Theorem (O. Boopolski, E.V.)

Let F_n be the free group on $x_1, \ldots, x_n$, and let $H = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_m \rangle$ be a finite, concise *and* Peiffer aspherical *presentation. Then Mihailova's group $M(H) \leqslant F_n \times F_n$ admits the following presentation*

$$M(H) \simeq \langle d_1, \ldots, d_n, t_1, \ldots, t_m \mid [t_j, d^{-1} t_i^{-1} r_i\, d],\ [t_i,\ \mathrm{root}(r_i)] \rangle$$

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Outline

Conciseness

Let $H = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_m \rangle$ be an arbitrary finite presentation (here, R_j are words on $x_1, \ldots, x_n$ which we'll assume reduced).

Definition

$H = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_m \rangle$ is concise if every R_j is non-trivial, and every two relations R_i, R_j, $i \neq j$, are not conjugate to each other or to the inverse of each other.

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Identities among relations

Definition

Let $U_1, \ldots, U_l \in F_n$*, and* $R_{i_1}, \ldots, R_{i_l} \in \{R_1, \ldots, R_n\}$*, and suppose*

$$(U_1 R_{i_1}^{\varepsilon_1} U_1^{-1}) \cdots (U_l R_{i_l}^{\varepsilon_l} U_l^{-1}) = 1$$

holds in F_n*. In this situation, the sequence of elements of* F_n

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is called an identity among relations of length l*.*

For $l = 0$ *we have the empty identity among relations,* $(\,)$*.*

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Peiffer transformations

Define the following elementary transformations of an identity among relations $(U_1 R_{i_1}^{\varepsilon_1} U_1^{-1}, \ldots, U_l R_{i_l}^{\varepsilon_l} U_l^{-1})$:

• deletion/insertion: if $(U_p R_{i_p}^{\varepsilon_p} U_p^{-1}) \cdot (U_{p+1} R_{i_{p+1}}^{\varepsilon_{p+1}} U_{p+1}^{-1}) = 1$, delete them.

• exchange: replace two consecutive terms, say

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Peiffer asphericity

Definition

The presentation $H = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_m \rangle$ is Peiffer aspherical if every identity among relations can be carried to the empty one by a sequence of Peiffer transformations.

I. Chiswell, D. Collins, J. Huebschmann (1981): asphericity is preserved under free products, certain HNN extensions, and Tietze transformations.

In the literature other related concepts (diagrammatical asphericity, topological asphericity, ...).

Theorem (Collins, Miller, 1999)

There exists a finite, concise and Peiffer aspherical presentation $H = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_m \rangle$ with unsolvable word problem.

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1. Mihailova's subgroup
2. Asphericity
3. The recursive presentation
4. Orbit decidability

Main result

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Recall that $M(H) = \{(w_1, w_2) \in F_n \times F_n \mid w_1 =_H w_2\} \leqslant F_n \times F_n$,

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The proof

Proof. Consider $F_{n+m} = \langle d_1, \ldots, d_n, t_1, \ldots, t_m \rangle$, and the morphism

$$\begin{array}{rcl} \pi \colon F_{n+m} & \to & M(H) \\ d_i & \mapsto & (x_i, x_i) \\ t_j & \mapsto & (1, R_j). \end{array}$$

- π is clearly onto,
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The association

Let $w \in \ker(\pi)$ and write it as $w = u_1 t_{i_1}^{\varepsilon_1} u_2 \cdots u_l t_{i_l}^{\varepsilon_l} u_{l+1}$ (where $l \geqslant 0$ and $u_1, \ldots, u_{l+1}$ are words in the $d_i's$).

Projecting $\pi(w)$ to each coordinate, we have

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1) It is clear that $w \neq 1 \Rightarrow l > 0 \Rightarrow iar(w)$ non-empty.

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Outline

Theorem (Bogopolski-Martino-V., 2008)

Let

$$1 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$$

be an algorithmic short exact sequence of groups such that

(i) *$TCP(F)$ is solvable,*

(ii) *$CP(H)$ is solvable,*

(iii) *there is an algorithm which, given an input $1 \neq h \in H$, computes a finite set of elements $z_{h,1}, \ldots, z_{h,t_h} \in H$ such that*

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

Then,

$$CP(G) \text{ is solvable} \iff A_G = \left\{ \begin{array}{rccc} \gamma_g : & F & \to & F \\ & x & \mapsto & g^{-1}xg \end{array} \,\middle|\, g \in G \right\}$$

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Particularizing to the case where F and H are free, we obtain:

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$$CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m) \text{ is solvable } \Leftrightarrow \langle \varphi_1, \ldots, \varphi_m \rangle \leqslant Aut(F_n) \text{ is O.D.}$$

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Theorem (Miller, 70's)

There are free-by-free groups with unsolvable conjugacy problem.

So, there must be orbit undecidable subgroups in $\mathrm{Aut}(F_n)$, for $n \geqslant 3$.
Where are them ?

Proposition (Bogopolski-Martino-V., 2008)

Let F be a group, and let $A \leqslant B \leqslant Aut(F)$ and $w \in F$ be such that $B \cap Stab^*(w) = 1$. Then,

$$OD(A) \text{ solvable} \quad \Rightarrow \quad MP(A, B) \text{ solvable.}$$

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(and $w = qaqbq$) one obtains precisely the orbit undecidable subgroups corresponding to Miller's examples.

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But this is more general: any other way of embedding $F_2 \times F_2$ in $\mathrm{Aut}(F_3)$ will provide new examples of orbit undecidability, i.e. of free-by-free groups with unsolvable CP.

And also interesting for other groups apart from free:

Take $F = \mathbb{Z}^n$ with $n \geqslant 4$.

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(and $TCP(\mathbb{Z}^4)$ is solvable).

Corollary (Bogopolski-Martino-V., 2008)

There exists $\mathbb{Z}^4$-by-free groups with unsolvable conjugacy problem.

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THANKS