DEGREES OF COMPRESSION AND INERTIA

RESEARCH PLAN

By

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1. Introduction

For any group G, we write $\mathrm{rk}(G)$ to denote the rank of G, which is the minimum cardinal of a generating set for G, i.e., $\mathrm{rk}(G) = \min\{|S| \mid S \subseteq G = \langle S\rangle\}$ and, $\tilde{\mathrm{rk}}(G) = \mathrm{rk}(G) - 1$. We say that G is finitely generated if $\mathrm{rk}(G)$ is finite. Finitely generated free groups, namely F_n have been extensively studied in the literature of group theory since more than a hundred years ago, in spite of their wider and complicated structure. On the other hand, free-abelian groups, namely $\mathbb{Z}^m$, are classical and very well known. All of my results during the first year is based on the group which is the direct product $\mathbb{Z}^m \times F_n$, namely free-abelian times free groups. And the group looks like the following: Let $\{x_1, x_2, \ldots, x_n\}$ is the free basis of F_n, and $\{t_1, t_2, \ldots, t_m\}$ is a basis of $\mathbb{Z}^m$.

$$\mathbb{Z}^m \times F_n = \langle t_1, ..., t_m, x_1, ..., x_n \mid t_i t_j = t_j t_i,\ t_i x_k = x_k t_i\rangle, \tag{1}$$

where $i, j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. The normal form of any element g in this group is,

$$t^a u = t_1^{a_1} \ldots t_m^{a_m} u(x_1, \ldots, x_n) = t^{(a_1,\ldots,a_m)} u(x_1, \ldots, x_n),$$

where $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ is a row integral vector, and $u = u(x_1, \ldots, x_n)$ is a reduced word in F_n. Note that the symbol t by itself has no real meaning, but it allows us to convert the notation for the abelian group $\mathbb{Z}^m$ from additive into multiplicative, by moving up the vectors (i.e. the entries of the vectors) to the level of exponents; this will be especially convenient when working in $\mathbb{Z}^m \times F_n$, a non-commutative group in general.

If we think about the behavior of $\mathbb{Z}^m \times F_n$, at a first glance, it seems very elementary consequences of behaviors of $\mathbb{Z}^m$ and F_n. In fact one problem or question concerning $\mathbb{Z}^m \times F_n$ will easily reduce to the corresponding question or problems for $\mathbb{Z}^m$ and F_n, if it is very easy or rigid enough. However, some other naive looking questions have a considerably more elaborated answer in $\mathbb{Z}^m \times F_n$ rather than in $\mathbb{Z}^m$ or F_n. This is the case, for example, when one considers automorphisms: $\mathrm{Aut}(\mathbb{Z}^m \times F_n)$ naturally contains $GL(m,\mathbb{Z}) \times \mathrm{Aut}(F_n)$, but there are many more automorphisms other than those preserving the factors $\mathbb{Z}^m$ and F_n. This fact causes potential complications when studying problems involving automorphisms: apart from understanding the problem in both the free-abelian and the free parts, one has to be able to control how is it affected by the interaction between the two parts. And in the paper [2] Delgado and Ventura gave a description that how any general automorphism looks like in this group $\mathbb{Z}^m \times F_n$.

Definition 1.1. Let G be any finitely generated group and $H \leqslant_{fg} G$. We say that H is *compressed* if $\mathrm{rk}(H) \leqslant \mathrm{rk}(K)$, for every $H \leqslant K \leqslant_{fg} G$. And H is *inert* if $\mathrm{rk}(H \cap K) \leqslant \mathrm{rk}(K)$, for every $K \leqslant_{fg} G$.

The notion of compression and inertia was first introduced by Dicks and Ventura in 1996 [3]. And they proved that fixed subgroups of families of monos of free group are inert in free group. And later in 2004, Martino and Ventura (see in [8]) proved that the fixed subgroup of any arbitrary family of endomorphisms of F_n, is compressed in F_n. As we already have the above results and the general structure of automorphism of $\mathbb{Z}^m \times F_n$, my goal was to give some positive results about the compression or inertia property of any subgroup and any auto-fixed subgroup of $\mathbb{Z}^m \times F_n$.

We define the degrees of compression and inertia as two parameters measuring how close is a subgroup of $\mathbb{Z}^m \times F_n$ to be compressed or inert. The extremal value 1 will indicate compression/inertia, and the bigger the coresponding degree is from 1, the farther will the subgroup be from being compressed/inert. For degree of compression we obtain no particularly explicit formula, but it is algorithmically computable; so, it is decidable if a subgroup is compressed or not. And we have concise formula for degree of inertia of any finitely generated subgroup H of $\mathbb{Z}^m \times F_n$.

First we will introduce the notion of degrees of compression and inertia and prove their general properties. In Theorems 2.10 and 3.1 the main results are proven. And in theorem 2.3 we proof the little bit modified version of compression for auto-fixed subgroup of $\mathbb{Z}^m \times F_n$.

Definition 1.2. Let G be any finitely generated group and $H \leqslant_{fg} G$. The *degree of compression* of H is $\mathrm{dc}(H) = \sup\limits_{K} \frac{\tilde{\mathrm{rk}}(H)}{\tilde{\mathrm{rk}}(K)}$, for every non-cyclic K and $H \leqslant K \leqslant_{fg} G$.

Observation 1.3. $\mathrm{dc}(H) = \sup\limits_{K} \frac{\tilde{\mathrm{rk}}(H)}{\tilde{\mathrm{rk}}(K)} = \frac{\tilde{\mathrm{rk}}(H)}{\inf\limits_{K} \tilde{\mathrm{rk}}(K)} = \frac{\tilde{\mathrm{rk}}(H)}{\min\limits_{K} \tilde{\mathrm{rk}}(K)} = \max\limits_{K} \frac{\tilde{\mathrm{rk}}(H)}{\tilde{\mathrm{rk}}(K)}$ *, for every non-cyclic K and $H \leqslant K \leqslant_{fg} G$.*

Definition 1.4. Let H be any finitely generated subgroup of any group G.The *degree of inertia* of H is $\mathrm{di}(H) = \sup\limits_{\substack{K \neq \langle x \rangle \\ H \cap K \leqslant_{fg} G}} \frac{\tilde{\mathrm{rk}}(H \cap K)}{\tilde{\mathrm{rk}}(K)}$, for every $K \leqslant_{fg} G$.

Remark 1.5. Let $H \leqslant_{fg} G$. Then, $\mathrm{di}(H) = \alpha \Rightarrow \tilde{\mathrm{rk}}(H \cap K) \leqslant \alpha\, \tilde{\mathrm{rk}}(K)$ for every non-cyclic K satisfying $H \leqslant K \leqslant_{fg} G$. And, $\forall 1 > \epsilon > 0$, there exists K(with above conditions) such that $\tilde{\mathrm{rk}}(H \cap K) > (\alpha - \epsilon)\, \tilde{\mathrm{rk}}(K)$.

Note 1.6. If H is cyclic, $\mathrm{dc}(H) = \mathrm{di}(H) = 1$

Lemma 1.7. *G be a finitely generated group and $\phi : G \to G$ be an automorphism, $H \leqslant_{fg} G$, then the following holds.*
(a) $\mathrm{di}(H\phi) = \mathrm{di}(H)$.
(b) $\mathrm{dc}(H\phi) = \mathrm{dc}(H)$.

Proof. (a) From the definition of *Degree of Inertia*, we have, $\mathrm{di}(H\phi) = \underset{\substack{K \leqslant_{fg} G \\ K \neq \langle x \rangle \\ H\phi \cap K \leqslant_{fg} G}}{sup} \frac{\tilde{\mathrm{rk}}(H\phi \cap K)}{\tilde{\mathrm{rk}}(K)}$.

We take one such finitely generated K which is not cyclic and $H\phi \cap K$ is also finitely generated. As ϕ is an automorphism,

$$\frac{\tilde{\mathrm{rk}}(H\phi \cap K)}{\tilde{\mathrm{rk}}(K)} = \frac{\tilde{\mathrm{rk}}((H\phi \cap K)\phi^{-1})}{\tilde{\mathrm{rk}}(K\phi^{-1})} = \frac{\tilde{\mathrm{rk}}(H \cap K\phi^{-1})}{\tilde{\mathrm{rk}}(K\phi^{-1})}$$

Since, ϕ is auto, $K\phi^{-1}$ and $H \cap K\phi^{-1}$ are finitely generated and $K\phi^{-1}$ is non-cyclic. Therefore,

$$\frac{\tilde{\mathrm{rk}}(H \cap K\phi^{-1})}{\tilde{\mathrm{rk}}(K\phi^{-1})} \leqslant \underset{\substack{L \leqslant_{fg} G \\ L \neq \langle x \rangle \\ H \cap L \leqslant_{fg} G}}{sup} \frac{\tilde{\mathrm{rk}}(H \cap L)}{\tilde{\mathrm{rk}}(L)} = \mathrm{di}(H)$$

Therefore, $\mathrm{di}(H\phi) \leqslant \mathrm{di}(H)$ for all ϕ and for every $H \leqslant_{fg} G$.
And replacing H by $H\phi$ and ϕ by ϕ^{-1}, we have, $\mathrm{di}(H\phi\phi^{-1}) \leqslant \mathrm{di}(H\phi) \Rightarrow \mathrm{di}(H) \leqslant \mathrm{di}(H\phi)$.

(b) same as (a). □

Corollary 1.8. *G be a finitely generated group and $H \leqslant_{fg} G$, then for any $x \in G$*
(a) $\mathrm{di}(H^x) = \mathrm{di}(H)$.
(b) $\mathrm{dc}(H^x) = \mathrm{dc}(H)$.

Observation 1.9. (a) *If G be any finitely generated free group,*

$$1 \leqslant \mathrm{dc}(H) \leqslant \mathrm{di}(H) \leqslant \tilde{\mathrm{rk}}(H),\ \forall H \leqslant_{fg} G.$$

(b) *If G be any finitely generated group,* $1 \leqslant \mathrm{dc}(H) \leqslant \mathrm{di}(H) \leqslant \infty$, $\forall H \leqslant_{fg} G$.
(c) *H is compressed if and only if* $\mathrm{dc}(H) = 1$.
(d) *H is inert if and only if* $\mathrm{di}(H) = 1$.

Proof. (a) By definition $1 \leqslant \mathrm{dc}(H)$, as we can take $K = H$.

$\mathrm{di}(H) = \sup\limits_{K \neq \langle x \rangle} \frac{\tilde{\mathrm{rk}}(H \cap K)}{\tilde{\mathrm{rk}}(K)}$, $\forall K \leqslant_{fg} G$

Let, $\mathrm{di}(H) = p$, therefore $\tilde{\mathrm{rk}}(H \cap K) \leqslant p\,\tilde{\mathrm{rk}}(K)$, for every $K \leqslant_{fg} G$.
Applying this inequality to any K containing H we get,

$$\begin{aligned} \tilde{\mathrm{rk}}(H) &\leqslant p\,\tilde{\mathrm{rk}}(K) \\ \Rightarrow \mathrm{dc}(H) &\leqslant p = \mathrm{di}(H). \end{aligned}$$

If G is a finitely generated free group, HNC holds, so,

$$\begin{array}{rrcll} & \tilde{\mathrm{rk}}(H\cap K) & \leqslant & \tilde{\mathrm{rk}}(H)\,\tilde{\mathrm{rk}}(K), & \quad \textit{for every } H \textit{ and } K. \\ \Rightarrow & \mathrm{di}(H) & \leqslant & \tilde{\mathrm{rk}}(H) & \end{array}$$

And hence, $1 \leqslant \mathrm{dc}(H) \leqslant \mathrm{di}(H) \leqslant \tilde{\mathrm{rk}}(H)$.

(b) If G is any group, we can not bound the rank of $H \cap K$, hence the result follows.

(c) and (d) follow immediately from the definition. $\square$

2. Degree of Compression

Definition 2.1. A finitely generated subgroup H of finitely generated group G is $(\,m, l)$ *-compressed* if $\tilde{\mathrm{rk}}(H) \leqslant m + l\,\tilde{\mathrm{rk}}(K)$, for every $H \leqslant K \leqslant_{fg} G$.

Remark 2.2. A subgroup is compressed if and only if it is $(0, 1)$-compressed.

From now through out the document mother group $G = (\mathbb{Z}^m \times F_n)$. Any automorphism of the group $(\mathbb{Z}^m \times F_n)$ is of the following type (see in [2]). Let $\Psi \in \mathrm{Aut}(\mathbb{Z}^m \times F_n)$ and it is defined as the following

$$\begin{array}{rcl} \Psi\colon \mathbb{Z}^m \times F_n & \longrightarrow & \mathbb{Z}^m \times F_n \\ t^a u & \longmapsto & t^{aQ+\mathbf{u}P}\phi(u), \end{array}$$

where $a \in Z^m$, $u \in F_n$, $\phi \in \mathrm{Aut}(\,F_n)$, $Q \in GL(\,m, \mathbb{Z})\,$, $P \in M_{n\times m}(\mathbb{Z})$ and $\mathbf{u}$ is the abelianization of u.

Theorem 2.3. *If Fix(Ψ) is f.g, then Fix(Ψ) is $(dimE_1(Q), l)$-compressed, where l depends on Ψ and l is computable.*

Proof. Letρ be the abelinazation map of F_n and M be the image of $(I_m - Q)$. ρ' is the restriction of ρ to Fixϕ (not to be confused with the abelianization map of the subgroup Fixϕ itself), and let P' be the restriction of P to image of ρ'. Let $N = M \cap \mathrm{Im}\,P'$,and consider its pre-image first by P' and then by ρ', see the following diagram:

$$\begin{tikzcd} & F_n \arrow[r, two heads, "\rho"] \arrow[d, phantom, sloped, "\geqslant"] & \mathbb{Z}^n \arrow[r, "P"] \arrow[d, phantom, sloped, "\geqslant"] & \mathbb{Z}^m \arrow[loop above, "\mathbf{I_m - Q}"] \geqslant M = \mathrm{Im}(I_m - Q) \arrow[d, phantom, sloped, "\geqslant"] \\ & \mathrm{Fix}\,\phi \arrow[r, two heads, "\rho'"] \arrow[d, phantom, sloped, "\geqslant"] & \mathrm{Im}\,\rho' \arrow[r, two heads, "P'"] \arrow[d, phantom, sloped, "\geqslant"] & \mathrm{Im}\,P' \arrow[d, phantom, sloped, "\geqslant"] \\ & (\mathrm{Fix}\,\Psi)\pi = N\mathbf{P'}^{-1}\rho'^{-1} & N\mathbf{P'}^{-1} \arrow[l, mapsto] & N = M \cap \mathrm{Im}\,P'. \arrow[l, mapsto] \end{tikzcd}$$

It is very easy to calculate that $\mathrm{Fix}(\Psi) \cap \mathbb{Z}^m = E_1(Q)$ and $Fix(\Psi)\pi = NP'^{-1}\rho'^{-1}$. Therefore we have,

$$\mathrm{rk}(Fix(\Psi)) = \dim(E_1(Q)) + \mathrm{rk}((Fix(\Psi)\pi) \tag{2}$$

Let, $l = [Fix(\phi) : NP'^{-1}\rho'^{-1}] = [Fix(\phi) : (Fix(\Psi))\pi]$. From the hypothesis Fix(Ψ) is finitely generated and $\dim(E_1(Q))$ is finite. Hence, from (2) $Fix(\Psi)\pi$ is finitely generated. We know that any non-trivial normal subgroup of free group is either of finite index or infinite generated. Therefore in our case l is finite. Since the index is finite, we can apply Schreier Index formula. Therefore, $\tilde{\mathrm{rk}}((Fix(\Psi))\pi) = l\,\tilde{\mathrm{rk}}(Fix(\phi))$. Let, K be any finitely generated subgroup of $\mathbb{Z}^m \times F_n$ containing Fix(Ψ) .
Now,

$$(Fix(\Psi))\pi \quad \leqslant_l \quad Fix(\phi)$$

Therefore,

$$(Fix(\Psi))\pi \quad \leqslant_{l'} \quad Fix(\phi) \cap K\pi \quad \leqslant \quad Fix(\phi),\ where\ l' \leqslant l.$$

And since $\phi \in \mathrm{Aut}(F_n)$, Fix(ϕ) is inert. Hence, $\tilde{\mathrm{rk}}(Fix(\phi) \cap K\pi) \leqslant \tilde{\mathrm{rk}}(K\pi)$ So,

$$\begin{aligned}
\tilde{\mathrm{rk}}((Fix(\Psi))\pi) &= l' \quad \tilde{\mathrm{rk}}(Fix(\phi) \cap K\pi) \\
&\leqslant l' \quad \tilde{\mathrm{rk}}(K\pi) \\
&\leqslant l' \quad \tilde{\mathrm{rk}}(K) \\
&\leqslant l \quad \tilde{\mathrm{rk}}(K)
\end{aligned}$$

Now from Equation (2),

$$\tilde{\mathrm{rk}}(Fix(\Psi)) \leqslant dim E_1(Q) + l\,\tilde{\mathrm{rk}}(K) \tag{3}$$

And from Equation (3) $\mathrm{rk}(Fix\Psi) \leqslant \dim(E_1(Q)) + l\,\mathrm{rk}(K)$. And this holds for any $K \leqslant_{fg} \mathbb{Z}^m \times F_n$. This completes the proof □

Note 2.4. As K is arbitrary, we can choose K in a way such that $K\pi = Fix\phi$. And then no $l' < l$ will satisfy (3) So, this l is the best minimal value.

Definition 2.5. Let $H \leqslant K$ be an extension of free groups and let $x \in K$. We say that x is *K-algebraic* over H if every free factor of K containing H, $H \leqslant L \leqslant_{ff} K$, satisfies $x \in L$. We say that an extension of free groups $H \leqslant K$ is algebraic, and we write $H \leqslant_{alg} K$, if every element of K is *K-algebraic* over H.

Theorem 2.6 (M. Takahasi, [9])**.** *Let $H \leqslant_{fg} F_n$. Then, there exists a finite computable collection of extensions of H, say $\mathcal{AE}(H) = \{H = H_0, H_1, \ldots, H_r\}$, all finitely generated and satisfying $H \leqslant H_i \leqslant F_n$, such that every extension K of H, $H \leqslant K \leqslant F_n$, contains one of them as a free factor, say $H \leqslant H_i \leqslant_* K = H_i * L$.*

Lemma 2.7. *Let $H \leqslant_{fg} \mathbb{Z}^m \times F_n$.*
$\mathrm{dc}(H) = \max\limits_{H \leqslant K} \frac{\tilde{\mathrm{rk}}(H)}{\tilde{\mathrm{rk}}(K)} = \max\limits_{\substack{H \leqslant K \\ H\pi \leqslant_{alg} K\pi}} \frac{\tilde{\mathrm{rk}}(H)}{\tilde{\mathrm{rk}}(K)}$, *where $K \leqslant_{fg} \mathbb{Z}^m \times F_n$.*

Proof. Let, $H = \langle t^{a_1}u_1, t^{a_2}u_2, \ldots, t^{a_r}u_r, t^{b_1}, t^{b_2}, \ldots, t^{b_s} \rangle \leqslant \mathbb{Z}^m \times F_n$, where $\{u_1, \ldots, u_r\}$ is a free basis of $H\pi$, and $a_i \in \mathbb{Z}^m$ for $i = 1, 2, \ldots, r$, and $\{b_1, \ldots, b_s\}$ is a free-abelian basis of $L_H = H \cap \mathbb{Z}^m$. And let K be any subgroup of $\mathbb{Z}^m \times F_n$ containing H and $K \cap \mathbb{Z}^m = L_K = \langle d_1, d_2, \ldots, d_{s'} \rangle$. Clearly, $L_H \leqslant L_K$, therefore, $\dim(L_H) \leqslant \dim(L_K)$ and $H\pi \leqslant K\pi$.
Therefore, there exists $M \in \mathcal{AE}(H\pi)$ such that $H\pi \leqslant_{alg} M \leqslant_{ff} K\pi$. Now we will take a basis for $M = \langle v_1, v_2, \ldots, v_p \rangle$ and extend it to a basis of $K\pi = \langle v_1, v_2, \ldots, v_p, v_{p+1}, \ldots, v_q \rangle$. We will choose $c_1, c_2, \ldots, c_p, c_{p+1}, \ldots, c_q$ in such a way that $H \leqslant \langle t^{c_1}v_1, t^{c_2}v_2, \ldots, t^{c_p}v_p, t^{c_{p+1}}v_{p+1}, \ldots, t^{c_q}v_q, L_K \rangle$ where $c_i \in \mathbb{Z}^m; i = 1, 2, \ldots, q$. Let, $\tilde{M} = \langle t^{c_1}v_1, t^{c_2}v_2, \ldots, t^{c_p}v_p, t^{d_1}, t^{d_2}, \ldots, t^{d_{s'}} \rangle, i.e., L_{\tilde{M}} \equiv L_K$.

Therefore, $H\pi \leqslant_{alg} \tilde{M}\pi \equiv M \leqslant_{ff} K\pi$ and $L_K \leqslant L_{\tilde{M}}$. Now, we have to show that $t^{a_i}_{u_i} \in \tilde{M}$. Since, $H\pi \leqslant_{alg} \tilde{M}\pi$, $u_i = w_i(v_1, v_2, \dots, v_p); i = 1, 2, \dots r$, and it is unique. For this unique word, we will compute $w_i(t^{c_1}v_1, t^{c_2}v_2, \dots, t^{c_p}v_p)$ in place of $v_1, v_2, \dots v_p$.

$$w_i(t^{c_1}v_1, t^{c_2}v_2, \dots, t^{c_p}v_p) = t^{\tilde{c}_i} w_i(v_1, v_2, \dots, v_p) = t^{\tilde{c}_i} u_i \tag{4}$$

where, $\tilde{c}_i = c_1|u_i|_{v_1} + c_2|u_i|_{v_2} + \dots + c_p|u_i|_{v_p}$.
From the construction, $t^{\tilde{c}_i}u_i \in K$ and $t^{a_i}u_i \in K$ as $H \leqslant K$. Therefore from (4) we have $t^{\tilde{c}_i - a_i} \in L_K$ and so also in $L_{\tilde{M}}$. Again, $t^{\tilde{c}_i}u_i \in \tilde{M}$, hence we have,

$$[t^{\tilde{c}_i - a_i}]^{-1} t^{\tilde{c}_i} u_i \in \tilde{M} \Rightarrow t^{a_i}u_i \in \tilde{M}. \tag{5}$$

Since, i is arbitrary (5) holds for every $i = 1, 2, \dots, r$.

$$\begin{array}{rcccc} \tilde{\mathrm{rk}}(\tilde{M}) = & \tilde{\mathrm{rk}}(\tilde{M}\pi) & + & \dim(L_{\tilde{M}}) \\ = & (p-1) & + & \dim(L_K) \\ \leqslant & (q-1) & + & \dim(L_K) \\ = & \tilde{\mathrm{rk}}(K). & & \end{array}$$

This proves that $\forall H \leqslant K \, \exists \, \tilde{M}$ in between, i.e., $H \leqslant \tilde{M} \leqslant K$ such that $\tilde{M}\pi \in \mathcal{AE}(H\pi)$ and $\tilde{\mathrm{rk}}(\tilde{M}) \leqslant \tilde{\mathrm{rk}}(K)$.
So there are infinitely many containing H, but all of them are built over finitely many $K\pi$'s, precisely those are in $\mathcal{AE}(H\pi)$.
Hence, $\mathrm{dc}(H) = \max_{H \leqslant K} \frac{\tilde{\mathrm{rk}}(H)}{\tilde{\mathrm{rk}}(K)} = \max_{\substack{H \leqslant K \\ H\pi \leqslant_{alg} K\pi}} \frac{\tilde{\mathrm{rk}}(H)}{\tilde{\mathrm{rk}}(K)}$, where $H, K \leqslant_{fg} \mathbb{Z}^m \times F_n$. $\square$

Given, $H = \langle t^{a_1}u_1, t^{a_2}u_2, \dots, t^{a_r}u_r, t^{b_1}, t^{b_2}, \dots, t^{b_s} \rangle \leqslant \mathbb{Z}^m \times F_n$, let,

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix} \in M_{r \times m}(\mathbb{Z}) \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_s \end{pmatrix} \in M_{s \times m}(\mathbb{Z}).$$

And let $K = \langle t^{c_1}v_1, t^{c_2}v_2, \dots, t^{c_p}v_p, L \rangle$ such that $L = K \cap \mathbb{Z}^m$ and $K\pi = \langle v_1, v_2, \dots, v_p, \rangle \underset{alg}{\geqslant} H\pi = \langle u_1, u_2, \dots, u_r, \rangle$. Therefore there exists an unique word $w_i(v_1, v_2, \dots, v_p) = u_i$ and $u_i^{ab} = (|u_i|_{v_1}, |u_i|_{v_2}, \dots, |u_i|_{v_p}) \in \mathbb{Z}^p$. Clearly,

$$w_i(t^{c_1}v_1, t^{c_2}v_2, \dots, t^{c_p}v_p) = t^{c_1|u_i|_{v_1} + c_2|u_i|_{v_2} + \dots + c_p|u_i|_{v_p}} w_i(v_1, v_2, \dots, v_p) = t^{c_1|u_i|_{v_1} + c_2|u_i|_{v_2} + \dots + c_p|u_i|_{v_p}} u_i$$

And this holds for every $u_i; i = 1, 2, \dots, r$. Let, $\mathcal{U} \in M_{r \times p}(\mathbb{Z})$ and $\mathcal{C} \in M_{p \times m}(\mathbb{Z})$,

$$\mathcal{U} = \begin{pmatrix} |u_1|_{v_1} & |u_1|_{v_2} & \dots & |u_1|_{v_p} \\ \vdots & \vdots & & \vdots \\ |u_i|_{v_1} & |u_i|_{v_2} & \dots & |u_i|_{v_p} \\ \vdots & \vdots & & \vdots \\ |u_r|_{v_1} & |u_r|_{v_2} & \dots & |u_r|_{v_p} \end{pmatrix} \text{ and } \mathcal{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix}.$$

Definition 2.8. For given $A \in M_{r\times m}(\mathbb{Z})$ and $\mathcal{U} \in M_{r\times p}(\mathbb{Z})$ and $B \in M_{s\times m}(\mathbb{Z})$, $d(A,\mathcal{U},B) = \min\{\dim L | L \leqslant \mathbb{Z}^m, \exists\, \mathcal{C} \in M_{p\times m}(\mathbb{Z})$ *such that* $row(A - \mathcal{U}\mathcal{C}) \leqslant L$ *and* $row(B) \leqslant L\}$

Proposition 2.9. *$d(A,\mathcal{U},B)$ is algorithmically computable.*

Proof. We first diagonalize the given matrix $\mathcal{U}$.

$$\mathcal{U} \longrightarrow \dots \underset{operations}{row/column} \dots \longrightarrow \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_r \end{pmatrix} \;\; or, \;\; \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_p \end{pmatrix}.$$

such that, $d_i \mid d_{i+1}; i = 1, 2, \dots, r-1(/p-1)$, depending upon $r > p$ or $p > r$ and let call this diagonal matrix D. Therefore, there exists $P \in M_{r\times r}(\mathbb{Z})$ and $Q \in M_{p\times p}(\mathbb{Z})$ (both P and Q are invertible) such that $D = P\mathcal{U}Q$. and so $\mathcal{U} = P^{-1}DQ^{-1}$ and let $Q^{-1}\mathcal{C} = \mathcal{C}', PA = A'$.
Hence, $rowspace(A - \mathcal{U}\mathcal{C}) = rowspace(A - P^{-1}DQ^{-1}\mathcal{C})$. Since Q is invertible, therefore, $rowspace(A - P^{-1}DQ^{-1}\mathcal{C}) = rowspace(A - P^{-1}D\mathcal{C}')$. As P is also invertible, therefore we have,

$$\begin{aligned} rowspace(A - P^{-1}D\mathcal{C}') &= rowspace(P^{-1}(PA - D\mathcal{C}')) \\ &= rowspace(PA - D\mathcal{C}') \\ &= rowspace(A' - D\mathcal{C}'). \end{aligned}$$

In D there are may be several numbers of 1 and 0. Without loss of generality we can assume that,

$$A' = \begin{pmatrix} a_1{}' \\ a_2{}' \\ a_3{}' \\ a_4{}' \\ a_5{}' \end{pmatrix}, \quad D = \begin{pmatrix} 1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & d_4 & \\ & & & & 0 \end{pmatrix}, \quad and \;\; \mathcal{C}' = \begin{pmatrix} c_1{}' \\ c_2{}' \\ \vdots \\ c_p{}' \end{pmatrix}.$$

Now we have to check whether the following holds or not.

$$\begin{aligned} a_1{}' - c_1{}' &\in L \\ a_2{}' - d_2c_2{}' &\in L \\ a_3{}' - d_3c_3{}' &\in L \\ a_4{}' - d_4c_4{}' &\in L \\ a_5{}' &\in L, \end{aligned} \tag{6}$$

for some choices of $c_1{}', c_2{}', c_3{}', c_4{}'$. Let $L_0 = \langle b_1, b_2, \dots, b_s\rangle$. Now if we take, $c_1{}' = a_1{}'$, clearly, $a_1{}' - c_1{}' = 0 \in L_0$. Let, $L_1 = \langle b_1, b_2, \dots, b_s, a_5{}'\rangle$. We have to check (6) for L_1 i.e., whether $a_2{}' - d_2c_2{}', a_3{}' - d_3c_3{}', a_4{}' - d_4c_4{}'$ are in L_1 or not for some choices of $c_2{}', c_3{}', c_4{}'$. If all of them are not in L_1 we will define a map,

$$\begin{aligned} \phi_d : \;\; \mathbb{Z}^m &\longrightarrow \mathbb{Z}^m \\ v &\longmapsto dv \end{aligned}$$

Clearly, ϕ_d is a homomorphism. Furthermore, $Im(\phi_d) = \langle (d, 0, \dots, 0), (0, d, \dots, 0), \dots, (0, 0, \dots, d)\rangle = d\mathbb{Z}^m \leqslant_{d^m} \mathbb{Z}^m$.

From the construction, $d_2 \mid d_3 \mid d_4$. Therefore,

$$\mathbb{Z}^m \;\geqslant\; d_2\mathbb{Z}^m \;\geqslant\; d_3\mathbb{Z}^m \;\geqslant\; d_4\mathbb{Z}^m$$

Let us consider the maps $\Pi_i : \;\; \mathbb{Z}^m \longrightarrow \mathbb{Z}^m/d_i\mathbb{Z}^m$, $\mathbb{Z}^m/d_4\mathbb{Z}^m \xrightarrow{\alpha} \mathbb{Z}^m/d_3\mathbb{Z}^m \xrightarrow{\beta} \mathbb{Z}^m/d_2\mathbb{Z}^m$, such that $\Pi_3 = \Pi_4\alpha$ and $\Pi_2 = \Pi_3\beta = \Pi_4\alpha\beta$. For the appropriate L, we have $\langle a_4{}'\Pi_4, L_1\Pi_4\rangle \leqslant L\Pi_4$, $\langle a_4{}'\Pi_4\alpha, a_3{}'\Pi_3, L_1\Pi_3\rangle = \langle a_4{}'\Pi_3, a_3{}'\Pi_3, L_1\Pi_3\rangle \leqslant L\Pi_3$, and $\langle a_4{}'\Pi_4\alpha\beta, a_3{}'\Pi_3\beta, a_2{}'\Pi_2, L_1\Pi_2\rangle =$

$\langle {a_4}'\Pi_2, {a_3}'\Pi_2, {a_2}'\Pi_2, L_1\Pi_2\rangle \leqslant L\Pi_2$. Let, $d_4 = \mu d_3$ and $d_3 = \lambda d_2$, i.e., $d_4 = \mu\lambda d_2$. Therefore if we consider, any L with $L\Pi_4 = \langle {a_4}'\Pi_4, ({a_3}'\Pi_3)\alpha^{-1}, ({a_2}'\Pi_2)\beta^{-1}\alpha^{-1}, L_1\Pi_4\rangle$ it will satisfy all the conditions of 6. Now ${a_3}'\Pi_3$ has μ^m pre-images in $L\Pi_4$ and ${a_2}'\Pi_2$ has $(\mu\lambda)^m$ pre-images in $L\Pi_4$. And we will choose the suitable pre-images of ${a_3}'\Pi_3$ and ${a_2}'\Pi_2$ among these $\mu^m \times (\mu\lambda)^m$ choices such that the dimension of $L\Pi_4$ is minimal. Let, $L\Pi_4 = \langle \tilde{e_1}, \tilde{e_2}, \dots, \tilde{e_r}\rangle$ is the best possible solution having the minimum dimension. Then, $L = \langle e_1, e_2, \dots, e_r\rangle$, attains the smallest value for $d(A,\mathcal{U})$, where e_i is any pre-image of $\tilde{e_i}$ for $i = 1, 2, \dots, r$ of the map Π_4. □

Theorem 2.10. *For given* $H \leqslant_{fg} \mathbb{Z}^m \times F_n$,
(a) *if* $\mathrm{rk}(H) \leqslant 1$, $\mathrm{dc}(H) = 1$.
(b) *And if* $\mathrm{rk}(H) \geqslant 2$, $\mathrm{dc}(H) = \frac{\tilde{\mathrm{rk}}(H)}{\min\{d(A,\mathcal{U}_M,B)+\tilde{\mathrm{rk}}(M) | M\in\mathcal{AE}(H\pi)\}}$.
In particular, $\mathrm{dc}(H)$ *is algorithmically computable.*

Proof. (a) Let for the given H, $\mathrm{rk}(H) \leqslant 1$. Then,

$$\begin{aligned}\mathrm{rk}(H) \leqslant 1 \quad &\Rightarrow \quad H \ \textit{is abelian} \\ &\Rightarrow \quad \tilde{\mathrm{rk}}(H) \leqslant \tilde{\mathrm{rk}}(K), \qquad \forall H \leqslant K \leqslant_{fg} \mathbb{Z}^m \times F_n \\ &\Rightarrow \quad \tfrac{\tilde{\mathrm{rk}}(H)}{\tilde{\mathrm{rk}}(K)} \leqslant 1\end{aligned}$$

And, this holds for any arbitrary K. If, $H = K$, we have the equality. Therefore, $\mathrm{dc}(H) = 1$.

(b) As from 2.7, we know that one of the members of $\mathcal{AE}(H\pi)$ attains the $\mathrm{dc}(H)$, so we first compute $\mathcal{AE}(H\pi)$. And we know that $\mathcal{AE}(H\pi)$ is finite and computable. We will abelianize every $u_i \in H\pi$ with respect to the basis of $M \in \mathcal{AE}(H\pi)$ to have the matrix $\mathcal{U}_M$. Then we compute $d(A,\mathcal{U}_M,B) + \tilde{\mathrm{rk}}(M)$ and it is computable from 2.9. We will continue this procedure for every $M \in \mathcal{AE}(H\pi)$. Let $\tilde{M}$ attains the minimum sum, i.e., $\min\{d(A,\mathcal{U}_M,B) + \tilde{\mathrm{rk}}(M) | M \in \mathcal{AE}(H\pi)\}$. Therefore $\tilde{\mathrm{rk}}(\tilde{M})$ is the minimum among all subgroups whose free part is in $\mathcal{AE}(H\pi)$ and containing H. Then from 2.7, $\mathrm{dc}(H) = \frac{\tilde{\mathrm{rk}}(H)}{\tilde{\mathrm{rk}}(\tilde{M})}$. and this completes the proof. □

Remark 2.11. There are examples of H, for which let $M \in \mathcal{AE}(H\pi)$ has the minimum reduced rank among all the members of $\mathcal{AE}(H\pi)$ and let $\tilde{M} \geqslant H$ such that $\tilde{M}\pi = M$, but the $\mathrm{dc}(H)$ is not attained by $\tilde{M}$. In general, it is not true that $d(A,\mathcal{U}_M,B) + \tilde{\mathrm{rk}}(M) \leqslant d(A,\mathcal{U}_{M'},B) + \tilde{\mathrm{rk}}(M')$ if $\tilde{\mathrm{rk}}(M) \leqslant \tilde{\mathrm{rk}}(M')$, $\forall M, M' \in \mathcal{AE}(H\pi)$.

Here we will illustrate one of such examples. Let, $H = \langle t^{(-1,0)}b^2, t^{(1,0)}ac^{-1}ac^{-1}, t^{(0,1)}bac^{-1}\rangle$, therefore, $H\pi = \langle b^2, ac^{-1}ac^{-1}, bac^{-1}\rangle$. The Figure 1 presents the Stalling's graph for $H\pi$ as a subgroup of F_3 and with respect to the ambient basis $A = \{a, b, c\}$.

Successively identifying pairs of vertices of $\Gamma_A(H\pi)$ and reducing the resulting A-labeled graph in all possible ways, one concludes that $\Gamma_A(H\pi)$ has five congruences, whose corresponding quotient graphs are depicted in Figure 2 Therefore, $\mathcal{AE}(H\pi) = \{H\pi, \langle b, ac^{-1}\rangle\}$. Let $M = \langle b, ac^{-1}\rangle$, clearly,

$$\tilde{\mathrm{rk}}(M) \leqslant \tilde{\mathrm{rk}}(H\pi) \tag{7}$$

Now, $A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathcal{U}_M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$.
Now we first diagonalize the matrix $\mathcal{U}_M$.

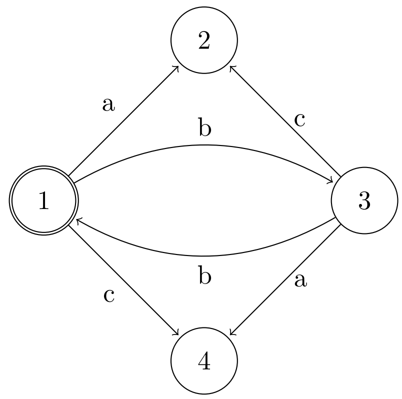

FIGURE 1. Graph of $H\pi$

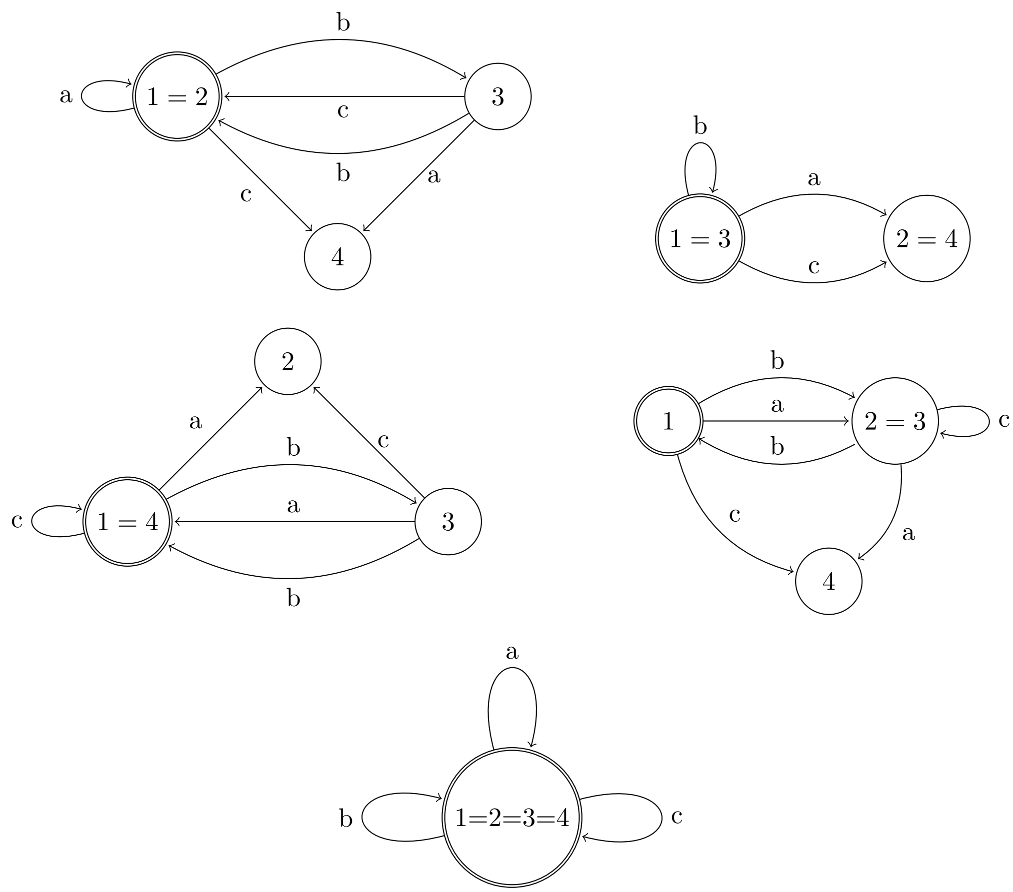

FIGURE 2. five quotients of $\Gamma_A(H\pi)$

$$\mathcal{U}_M \underset{C_2-C_1}{\longrightarrow} \begin{pmatrix} 2 & -2 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \underset{R_1+R_2}{\longrightarrow} \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \underset{R_1-2R_3}{\longrightarrow} \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \underset{R_{13}}{\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} = D(\text{say}).$$

$$I_3 \underset{R_1+R_2}{\longrightarrow} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{R_1-2R_3}{\longrightarrow} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{R_{13}}{\longrightarrow} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix} = P(\text{say}), \text{ and}$$

$$I_2 \underset{C_2-C_1}{\longrightarrow} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = Q(\text{say}).$$

Let, $H \leqslant \tilde{M} \leqslant \mathbb{Z}^2 \times F^3$ such that $\tilde{M}\pi = M$ and $L_{\tilde{M}} = \tilde{M} \cap \mathbb{Z}^2$, i.e., $\tilde{M} = \langle t^{c_1}b, t^{c_2}ac^{-1}, L_{\tilde{M}} \rangle$, where $c_1, c_2 \in \mathbb{Z}^2$. Therefore,

$$row(A - \mathcal{U}_M\mathcal{C}) \leqslant L_{\tilde{M}} \Rightarrow row(PA - D\mathcal{C}') \leqslant L_{\tilde{M}}. \tag{8}$$

Let, $\mathcal{C}' = \begin{pmatrix} c_1' \\ c_2' \end{pmatrix}$. From (8) $row(PA - D\mathcal{C}') = row\{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_1' \\ c_2' \end{pmatrix}\} = row\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -2 \end{pmatrix} - \begin{pmatrix} c_1' \\ 2c_2' \\ 0 \end{pmatrix}\} \leqslant L_{\tilde{M}}$.

Therefore,

$$\begin{array}{rcl} (0,1) - c_1' & \in & L_{\tilde{M}} \\ (1,0) - 2c_2' & \in & L_{\tilde{M}} \\ (0,-2) & \in & L_{\tilde{M}} \end{array} \tag{9}$$

And all three conditions of (9) are not satisfied if $\dim(L_{\tilde{M}}) = 1$. Because, if $\dim(L_{\tilde{M}}) = 1$, $L_{\tilde{M}} = \langle (0,1) \rangle$ as from (9), $(0,-2) \in L_{\tilde{M}}$ and $c_1{}' = (0,1)$. Let, $c_2{}' = (x,y)$, therefore, $(1,0) - 2(x,y) \in \langle (0,1) \rangle$, which is not possible for any choice of x, y. So, $\dim(L_{\tilde{M}}) = 2$ and as $L_{\tilde{M}} \leqslant \mathbb{Z}^2$, $L_{\tilde{M}} = \mathbb{Z}^2$. Hence, $\tilde{\mathrm{rk}}(\tilde{M}) = 2 + 1 = 3$. And for the group H itself $H\pi \in \mathcal{AE}(H\pi)$ and it satisfies (7), but $\tilde{\mathrm{rk}}(H) = 3 - 1 = 2 \leqslant \tilde{\mathrm{rk}}(\tilde{M})$ Therefore, not $\tilde{M}$, H itself attains the $\mathrm{dc}(H)$ and $\mathrm{dc}(H) = \frac{\tilde{\mathrm{rk}}(H)}{\tilde{\mathrm{rk}}(H)} = 1$.

3. Degree of Inertia

Theorem 3.1. *Let* $H \leqslant_{fg} \mathbb{Z}^m \times F_n$.
(a) $\mathrm{rk}(H\pi) \leqslant 1 \;\Rightarrow\; \mathrm{di}(H) = 1$.
(b) $H \cap \mathbb{Z}^m \leqslant_\infty \mathbb{Z}^m$, $\mathrm{rk}(H\pi) \geqslant 2 \;\Rightarrow\; \mathrm{di}(H) = \infty$.
(c) $H \cap \mathbb{Z}^m \leqslant_l \mathbb{Z}^m$, $\mathrm{rk}(H\pi) \geqslant 2 \;\Rightarrow\; \mathrm{di}(H) = l\,\mathrm{di}(H\pi)$.

First we proof part (a) and (b) of the theorem. And before proving part (c), we need to prove several lemmas.

Proof of (a). Let, $K \leqslant_{fg} \mathbb{Z}^m \times F_n$ and $L_H = H \cap \mathbb{Z}^m$.
$\tilde{\mathrm{rk}}(H\pi) \leqslant 1 \Rightarrow H = \langle t^a u, L_H \rangle \Rightarrow H$ is abelian $\Rightarrow \tilde{\mathrm{rk}}(H \cap K) \leqslant \tilde{\mathrm{rk}}(K) \Rightarrow \frac{\tilde{\mathrm{rk}}(H\cap K)}{\tilde{\mathrm{rk}}(K)} \leqslant 1$, and, equality holds if $H = K$. This holds for any arbitrary K. From the definition, $\mathrm{di}(H) \geqslant 1$, and this completes the proof.

Proof of (b). Let, $\mathrm{rk}(H\pi) = n_1$, from the hypothesis, $n_1 \geqslant 2$
Let, $H = \langle t^{a_1}u_1, t^{a_2}u_2, \ldots, t^{a_{n_1}}u_{n_1}, L_H \rangle$, where $L_H = H \cap \mathbb{Z}^m$ and $a_1, a_2, \ldots, a_{n_1} \in \mathbb{Z}^m$ and $\{u_1, u_2, \ldots, u_{n_1}\}$ be the free basis of $H\pi$.

As, $H\pi$ is a free subgroup of F_n, we can draw the Stalling's graph for $H\pi$, say $\Gamma(H\pi)$. For proving the statement that $\mathrm{di}(H) = \infty$ our goal is to construct a family of subgroups $K_N \leqslant_{fg} \mathbb{Z}^m \times F_n$ for every $N \in \mathbb{N}$, such that the denominator of the quotient in the formula of $\mathrm{di}(H)$, is always 2, i.e., $\tilde{\mathrm{rk}}(K_N) = 2$ but $\tilde{\mathrm{rk}}((H \cap K_N)\pi) = N$, for every $N \in \mathbb{N}$. Therefore the numerator of the quotient in the formula of $\mathrm{di}(H)$ depends on N and in particular the rank of $\tilde{\mathrm{rk}}(H \cap K_N)$ is increased as N is increased. Now we construct $K_N \leqslant_{fg} \mathbb{Z}^m \times F_n$, where $N \in \mathbb{N}$. But, $\mathrm{rk}(K_N)$ doesn't depend on N. Let, $K_N = \langle t^{{a_1}'}u_1, t^{{a_2}'}u_2, L_{K_N} \rangle$, using $n_1 \geqslant 2$. L_{K_N} and ${a_1}', {a_2}' \in \mathbb{Z}^m$, will be determined later. Therefore, $H\pi \cap K_N\pi = \langle u_1, u_2 \rangle$.

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_{n_1} \end{pmatrix} \in M_{n_1 \times m}(\mathbb{Z}), \quad A' = \begin{pmatrix} {a_1}' \\ {a_2}' \end{pmatrix} \in M_{2 \times m}(\mathbb{Z}).$$

Let $\rho_1 : H\pi \twoheadrightarrow \mathbb{Z}^{n_1}, \rho_2 : K_N\pi \twoheadrightarrow \mathbb{Z}^2, \rho_3 : H\pi \cap K_N\pi \twoheadrightarrow \mathbb{Z}^2$ be the corresponding abelianization maps. The rows of the matrices $P = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \end{pmatrix}$ and $P' = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (of sizes $2 \times n_1$ and 2×2, respectively) are describing the abelianizations of the inclusion maps,

$$\begin{tikzcd} H\pi & H\pi \cap K_N\pi \arrow[l, hook', "\iota"'] \arrow[r, hook, "\iota'"] & K_N\pi \end{tikzcd} .$$

See the diagram (10). Let, $R = PA - P'A' = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} = \begin{pmatrix} a_1 - a'_1 \\ a_2 - a'_2 \end{pmatrix}$.
And clearly, $\mathrm{rk}(K_N\pi) = \mathrm{rk}(H\pi \cap K_N\pi) = 2$.

$$\begin{tikzcd} & (H \cap K_N)\pi \arrow[d, phantom, "\rotatebox{90}{$\leqslant$}"] & \\ H\pi \arrow[d, two heads, "\rho_1"'] \arrow[dr, phantom, "///"] & H\pi \cap K_N\pi \arrow[l, hook', "\iota"'] \arrow[r, hook, "\iota'"] \arrow[d, two heads, "\rho_3"] \arrow[dr, phantom, "///"] & K_N\pi \arrow[d, two heads, "\rho_2"] \\ \mathbb{Z}^{n_1} \arrow[dr, "\mathbf{A}"] & \mathbb{Z}^2 \arrow[l, "\mathbf{P}"'] \arrow[r, "\mathbf{P'}"] \arrow[d, red, "\mathbf{R}"] & \mathbb{Z}^2 \arrow[dl, "\mathbf{A'}"'] \\ & \mathbb{Z}^m & \end{tikzcd} \tag{10}$$

From the hypothesis, $L_H \leqslant_\infty \mathbb{Z}^m, i.e., \mathrm{rk}(L_H) = r < m$, so there exists $\tilde{L}_H$ such that $L_H \leqslant_{fi} \tilde{L}_H \leqslant_\oplus \mathbb{Z}^m$.

Now take a basis $\{l_1, l_2, \ldots, l_r\}$ of $\tilde{L}_H$, such that $\{d_1 l_1, d_2 l_2, \ldots, d_r l_r\}$ is basis of L_H for appropriate choices of $d_1, d_2, \ldots, d_r \in \mathbb{Z}$ (there is always a basis like this by standard liniear algebra arguments) and $\{l_1, l_2, \ldots, l_r, \ldots, b'\}$ is a basis of $\mathbb{Z}^m$. And as we are free to choose A', i.e., a'_1, a'_2 we will choose $a'_1 = a_1 - b'$ and $a'_2 = a_2$. Therefore, $R = \begin{pmatrix} b' \\ 0 \end{pmatrix}$

Now, we construct, $K_N \cap \mathbb{Z}^m = L_{K_N} = \langle N b' \rangle$. Then,

$$\begin{array}{rcl} (L_H + L_{K_N})R^{-1} & = & \{(x_1, x_2) \in \mathbb{Z}^2 \mid (x_1, x_2)R \in L_H + L_{K_N}\} \\ & = & \{(x_1, x_2) \in \mathbb{Z}^2 \mid (x_1, x_2)\begin{pmatrix} b' \\ 0 \end{pmatrix} \in L_H + L_{K_N}\} \\ & = & \{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 b' \in L_H + L_{K_N}\} \\ & = & \{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 b' \in \langle N b' \rangle\} \\ & = & N\mathbb{Z} \times \mathbb{Z} \leqslant_N \mathbb{Z}^2 \end{array}$$

As, ρ_3 is onto, the index will be preserved. So, $(L_H + L_{K_N})R^{-1}{\rho_3}^{-1} = (H \cap K_N)\pi \leqslant_N H\pi \cap K_N\pi$. Therefore by Schreier Index formula,

$$\begin{array}{lcllcl} \tilde{\mathrm{rk}}((H \cap K_N)\pi) & = & N\,\tilde{\mathrm{rk}}(H\pi \cap K_N\pi) & = & N(2-1) & = \; N \\ \Rightarrow \tilde{\mathrm{rk}}(H \cap K_N) & = & N + \dim(L_H \cap L_{K_N}) & \geqslant & N & \\ \Rightarrow \frac{\tilde{\mathrm{rk}}(H \cap K_N)}{\tilde{\mathrm{rk}}(K_N)} & \geqslant & N/_2 & & & \end{array}$$

So, $\forall N \in \mathbb{N}$ there exists K_N such that $\frac{\tilde{\mathrm{rk}}(H \cap K_N)}{\tilde{\mathrm{rk}}(K_N)} \geqslant N/_2$. Therefore, $\sup\limits_{\substack{K\, is\, fg \\ H\cap K\, is\, fg \\ K \neq \langle x \rangle}} \frac{\tilde{\mathrm{rk}}(H \cap K)}{\tilde{\mathrm{rk}}(K)} = \infty$.

Hence, $\mathrm{di}(H) = \infty$.

Proof of (c). We first prove that $\mathrm{di}(H) \leqslant l\,\mathrm{di}(H\pi)$. For any arbitrary $K = \langle t^{a'_1} v_1, \ldots, t^{a'_{n_2}} v_{n_2}, L_K \rangle \leqslant_{fg} \mathbb{Z}^m \times F_n$ and given $H = \langle t^{a_1} u_1, \ldots, t^{a_{n_1}} u_{n_1}, L_H \rangle$ we have

$$\frac{\tilde{\mathrm{rk}}(H \cap K)}{\tilde{\mathrm{rk}}(K)} = \frac{\dim(L_H \cap L_K) + \tilde{\mathrm{rk}}((H \cap K)\pi)}{\dim L_K + \tilde{\mathrm{rk}}(K\pi)} \tag{11}$$

From the hypothesis, $L_H \leqslant_l \mathbb{Z}^m \;\Rightarrow\; L_H + L_K \leqslant_{l'} \mathbb{Z}^m$, where $l' \leqslant l$. As before, let $\rho_1 : H\pi \twoheadrightarrow \mathbb{Z}^{n_1}$, $\rho_2 : K\pi \twoheadrightarrow \mathbb{Z}^{n_2}$, $\rho_3 : H\pi \cap K\pi \twoheadrightarrow \mathbb{Z}^{n_3}$ be the corresponding abelianization maps. And let the

$$\begin{tikzcd}
 & (H\cap K)\pi \arrow[d, phantom, "\rotatebox{90}{=}"] & \\
H\pi \arrow[d, two heads, "\rho_1"'] & H\pi\cap K\pi \arrow[l, hook', "\iota"'] \arrow[r, hook, "\iota'"] \arrow[d, two heads, "\rho_3"] \arrow[dl, phantom, "///"] \arrow[dr, phantom, "///"] & K\pi \arrow[d, two heads, "\rho_2"] \\
\mathbb{Z}^{n_1} \arrow[dr, "\mathbf{A}"] & \mathbb{Z}^{n_3} \arrow[l, "\mathbf{P}"'] \arrow[r, "\mathbf{P'}"] \arrow[d, "\mathbf{R}"] & \mathbb{Z}^{n_2} \arrow[dl, "\mathbf{A'}"'] \\
 & \mathbb{Z}^m &
\end{tikzcd} \tag{12}$$

rows of the matrices P and P' (of sizes $n_3 \times n_1$ and $n_3 \times n_2$ respectively) describe the abelianizations of the inclusion maps; see diagram (12). Let $R = PA - P'A'$, where $A \in M_{n_1 \times m}(\mathbb{Z})$, $A' \in M_{n_2 \times m}(\mathbb{Z})$ are the coefficient matrices with the rows $\{a_1, a_2, \dots, a_{n_1}\}$ and $\{a'_1, a'_2, \dots, a'_{n_1}\}$, respectively.

As in general R is not onto, $(L_H + L_K)R^{-1} \leqslant_{l''} \mathbb{Z}^{n_3}$, where $l'' \leqslant l'$. As, ρ_3 is onto $(L_H + L_K)R^{-1}{\rho_3}^{-1} \leqslant_{l''} (H\pi \cap K\pi)$. And, $(L_H + L_K)R^{-1}{\rho_3}^{-1} = (H \cap K)\pi$, hence we have, $(H \cap K)\pi \leqslant_{l''} (H\pi \cap K\pi)$. Therefore,

$$\tilde{\mathrm{rk}}((H \cap K)\pi) = l''\, \tilde{\mathrm{rk}}(H\pi \cap K\pi) \quad = \quad l'' \frac{\tilde{\mathrm{rk}}(H\pi \cap K\pi)}{\tilde{\mathrm{rk}}(K\pi)}\, \tilde{\mathrm{rk}}(K\pi) \quad \leqslant \quad l''\, \mathrm{di}(H\pi)\, \tilde{\mathrm{rk}}(K\pi)$$

Using (11) we have,

$$\frac{\tilde{\mathrm{rk}}(H \cap K)}{\tilde{\mathrm{rk}}(K)} \leqslant \frac{\dim(L_H \cap L_K) + l''\, \mathrm{di}(H\pi)\, \tilde{\mathrm{rk}}(K\pi)}{\dim(L_K) + \tilde{\mathrm{rk}}(K\pi)} \tag{13}$$

Now, $\frac{dim\,(L_H \cap L_K)}{dim\,(L_K)} \leqslant 1 \leqslant \frac{l''\, \mathrm{di}(H\pi)\, \tilde{\mathrm{rk}}(K\pi)}{\tilde{\mathrm{rk}}(K\pi)} = l''\, \mathrm{di}(H\pi)$

Using Bad Boy's Lemma from (13) we deduce that,

$\frac{\tilde{\mathrm{rk}}(H \cap K)}{\tilde{\mathrm{rk}}(K)} \quad \leqslant \quad l''\, \mathrm{di}(H\pi) \quad \leqslant \quad l'\, \mathrm{di}(H\pi) \quad \leqslant \quad l\, \mathrm{di}(H\pi)$

Therefore,

$$\mathrm{di}(H) = \sup_{\substack{K\, is\, fg \\ H \cap K\, is\, fg \\ K \neq \langle x \rangle}} \frac{\tilde{\mathrm{rk}}(H \cap K)}{\tilde{\mathrm{rk}}(K)} \leqslant l\, \mathrm{di}(H\pi) \tag{14}$$

To prove the other inequality, $\mathrm{di}(H) \geqslant l\, \mathrm{di}(H\pi)$, we will see that, for any given $\epsilon > 0$, there exists $K \leqslant_{fg} \mathbb{Z}^m \times F_n$ such that $\frac{\tilde{rk}(H \cap K)}{\tilde{rk}(K)} > l\, \mathrm{di}(H\pi) - \epsilon$; this way, $\mathrm{di}(H) > l\, \mathrm{di}(H\pi) - \epsilon$ and, if this works for every $\epsilon > 0$, we get $\mathrm{di}(H) \geqslant l\, \mathrm{di}(H\pi)$. Looking at the steps of the above proof, we will obtain such a K by finding s subgroup with the following properties:

(1) $\frac{\tilde{\mathrm{rk}}(H\pi \cap K\pi)}{\tilde{\mathrm{rk}}(K\pi)} > \mathrm{di}(H\pi) - \epsilon$;
(2) $L_K = \{0\}$ (and, this way, $l' = l$);
(3) R is onto by choosing appropriate A' (and, this way, $l'' = l'$).

Following the arguments above with such a subgroup K we get what we want:

$$\frac{\tilde{rk}(H \cap K)}{\tilde{rk}(K)} = \frac{\dim(L_H \cap L_K) + \tilde{rk}((H \cap K)\pi)}{\dim(L_K) + \tilde{rk}(K\pi)} = \frac{\tilde{rk}((H \cap K)\pi)}{\tilde{rk}(K\pi)} =$$

$$= \frac{l''\tilde{rk}(H\pi \cap K\pi)}{\tilde{rk}(K\pi)} = \frac{l'\tilde{rk}(H\pi \cap K\pi)}{\tilde{rk}(K\pi)} = \frac{l\tilde{rk}(H\pi \cap K\pi)}{\tilde{rk}(K\pi)} > l(\mathrm{di}(H\pi) - \epsilon).$$

In order to see that a subroup K satisfying $(1) - (3)$ always exists, we need the following lemmas.

Lemma 3.2. *If G be any group and $N, M \leqslant G$. Then, $[N : N \cap M] \leqslant [G : M]$, with equality if $MN = G$. (If $[N : N \cap M]$ is finite, then equality holds if and only if $MN = G$.)*

Proof. Let $G = Mx_1 \sqcup Mx_2 \sqcup ... \sqcup Mx_i = \bigsqcup_{i\in I} Mx_i$; $|I| \leqslant \infty$. Intersecting with N, $N = \bigsqcup_{i\in I}(N \cap Mx_i) = \bigsqcup_{i\in I'}(N\cap M)y_i$, for some $|I'| \leqslant |I|$. Because, either $N \cap Mx_i$ is empty or a coset of $N \cap M$. Furthermore, $\forall x_i \in G$, $i \in I$, and for some $n \in N$, $m \in M$,

$$\begin{aligned} MN = G \quad &\Rightarrow \quad x_i = mn \\ &\Rightarrow \quad n = m^{-1}x_i. \end{aligned}$$

Therefore, $N \cap Mx_i \neq \emptyset \;\; \forall i \in I$. □

Corollary 3.3. *Let $M' \leqslant M \leqslant F_n$ and $N \leqslant F_n$, then $[(N \cap M) : (N \cap M')] \leqslant [M : M']$, and equality holds if $M'(N \cap M) = M$.*

Proof. $(N \cap M), M'$ are subgroups of M. By Lemma 3.2,
$[(N\cap M) : (N\cap M)\cap M'] = [(N\cap M) : (N\cap M')] \leqslant [M : M']$ and $[(N\cap M) : (N\cap M')] = [M : M']$ if $M'(N \cap M) = M$. □

Lemma 3.4. *Given, $M = \langle e_1, e_2, \dots, e_r\rangle \leqslant F_n$, where $\{e_1, e_2, \dots, e_r\}$ is a free basis of M and $X \leqslant M$ with $\mathrm{rk}(X) \geqslant 2$. Then, $\exists w_1, w_2, \dots, w_r \in X$ such that, $M' = \langle e_1w_1, e_2w_2, \dots, e_rw_r\rangle \leqslant_\infty M$, or, $M' = \langle w_1e_1, w_2e_2, \dots, w_re_r\rangle \leqslant_\infty M$; furthermore in both cases, $M'X = M$.*

Proof. Let $\Gamma(X)$ be the Stalling's graph for X with respect to $\{e_1, e_2, ..., e_r\}$, the basis of M. We will choose an edge e^ϵ in (there exists at least one) of $\Gamma(X)$ such that $\iota e^\epsilon = \odot$. Let ${e_i}^{\epsilon_i} = \ell(e^\epsilon)$ where $\epsilon_i = \pm 1$. And let u be non-trivial reduced word in X and u reads ${e_i}^{\epsilon_i}$ as the end label. And the existence of such word u is explained in the following argument:
Let $p = \tau e^\epsilon$ and γ be a closed path at $p \in V\Gamma(X) \setminus \{\odot\}$ in $\Gamma(X) \setminus \{e^\epsilon\}$. If e^ϵ is a bridge, all the vertices in the connected component which is not containing the $\odot$ are of degree at least 2, and so we can find some γ. On the other hand, if e^ϵ is not a bridge, $\Gamma(X) \setminus \{e^\epsilon\}$ is connected and of $rank \geqslant 1$ which assures the existence of γ.
Therefore, $u = e_iu'{e_i}^{-1} \in X$, or, $u = {e_i}^{-1}u'e_i \in X$, where $u' = \ell(\gamma)$.

Case-1: $u = e_iu'{e_i}^{-1}, \epsilon_i = +1$

Let $w_j = 1, j \neq i$ *and* $w_i = u = e_iu'{e_i}^{-1}$, so $M' = \langle e_1, e_2, \dots, e_iw_i, e_{i+1}, \dots, e_r\rangle \leqslant_\infty M$. Because, the Stalling's graph for M' with respect to the basis $\{e_1, e_2, \dots, e_r\}$ is not complete.

Case-2: $u = {e_i}^{-1}u'e_i, \epsilon_i = -1$

Let $w_j = 1, j \neq i$ *and* $w_i = u = {e_i}^{-1}u'e_i$, so $M' = \langle e_1, e_2, \dots, w_ie_i, e_{i+1}, \dots, e_r\rangle \leqslant_\infty M$. Because, in this case, the Stalling's graph for M' with respect to the same basis set $\{e_1, e_2, \dots, e_r\}$ is also not complete.

In both the cases of choosing $w_i's$ we have, $M' \leqslant_\infty M$ and either, $M'X \geqslant M$ $(As,\ (e_iw_i){w_i}^{-1} = e_i)$, or, $XM' \geqslant M$ $(As\ {w_i}^{-1}(w_ie_i) = e_i)$. Therefore, in both cases we have $M'X = XM' = M$. □

Corollary 3.5. *Given $N, M \leqslant F_n$, with $\mathrm{rk}(N \cap M) \geqslant 2$. Then, there exists $M' \leqslant_\infty M$ such that for all finite index subgroups L satisfying $M' \leqslant L \leqslant_d M$, we have $[(N \cap M) : (N \cap L)] = [M : L] = d$.*

Proof. Applying the Lemma 3.4 on M and $X = N \cap M \leqslant M$ $(with\,\mathrm{rk}(N \cap M) \geqslant 2)$, there exists $M' \leqslant_\infty M$ with $M'(N \cap M) = M$. Then any L with $M' \leqslant L \leqslant_d M$, also satisfies $L(N \cap M) = M$ and so $[(N \cap M) : (N \cap L)] = [M : L] = d$. □

Definition 3.6. For $N \leqslant F_n$, $\Gamma(N)$ denotes the corresponding Stallings graph (whose base point is denoted $\odot$). On it, for a vertex $p \in V\Gamma(N)$ and an element $w \in F_n$, we define $pw := \tau\gamma$, where γ is the (unique) path starting from p and labeled w, in case it exists; otherwise, pw is undefined.

For $N, M \leqslant F_n$, we denote by$\Gamma(N) \times \Gamma(M)$ the pull-back of $\Gamma(N)$ and $\Gamma(M)$ (not necessarily connected, nor a core graph in general) and by $\Gamma(N) \wedge \Gamma(M)$ the core of the connected component of $\Gamma(N) \times \Gamma(M)$ containing $(\odot, \odot)$. It is well known that $\Gamma(N) \wedge \Gamma(M) = \Gamma(N \cap M)$.

Lemma 3.7. *If $N \cap M$ has infinite index in N then, there exists $v \in N$ such that $\odot v$ is undefined in $\Gamma(M)$.*

Proof. Suppose the conclusion is not true, i.e., for every $v \in N$, $\odot v \in \{p_0 = \odot, p_1, \ldots, p_r\} \subseteq V\Gamma(M)$. Choosing a maximal tree T in $\Gamma(M)$ and defining $w_i = \ell(T[\odot, p_i]) \in F_n$ for $i = 0, \ldots, r$ (note that $w_0 = 1$), we have $N \subseteq M \sqcup Mw_1 \sqcup \cdots \sqcup Mw_r$. Intersecting with N, we get $N \subseteq (N \cap M) \sqcup (N \cap M)v_1 \sqcup \cdots \sqcup (N \cap M)v_s$ for some $v_i \in N$ and $s \leqslant r$. Since the other inclusion is immediate, we have $N = (N \cap M) \sqcup (N \cap M)v_1 \sqcup \cdots \sqcup (N \cap M)v_s$ and so, $N \cap M$ has finite index in N, a contradiction. $\square$

Lemma 3.8. *Let N, M be two finitely generated subgroups of F_n. If $\mathrm{rk}(N) \geqslant 2$, $\odot$ has degree at least 3 in $\Gamma(N)$, and $N \cap M$ has infinite index in N then, there exist infinitely many $v \in N$ such that $\odot v$ is undefined in $\Gamma(M)$.*

Proof. Let e_a, e_b, e_c be three different edges going out from $\odot$ in $\Gamma(N)$, say with (pairwise different) labels $a, b, c \in X^{\pm 1}$, respectively. By Lemma 3.7, there is $v_0 \in N$ such that $\odot v_0$ is undefined in $\Gamma(M)$. Realize v_0 as a reduced closed path γ_0 at $\odot$ in $\Gamma(N)$ and, without lost of generality, we can assume it finishes with e_a^{-1}. For $\alpha = a, b, c$, take a non-trivial reduced closed path η_α at τe_α in the graph $\Gamma(N) \setminus \{e_\alpha\}$ (there always exists one because $\mathrm{rk}(N) \geqslant 2$), and consider $\gamma_\alpha = e_\alpha \eta_\alpha e_\alpha^{-1}$, a reduced closed path at $\odot$ beginning with e_α and ending with e_α^{-1} (so, $v_\alpha = \ell(\gamma_\alpha) \in N$ is a reduced word begining with α and ending with α^{-1}). By construction, all the products $\gamma_0\gamma_b$,$\gamma_0\gamma_c$, and $\gamma_\alpha\gamma_\beta$ with $\alpha \neq \beta$ are reduced. Therefore, all the elements $v = \gamma_0$, $\gamma_0\gamma_b$, $\gamma_0\gamma_b\gamma_a$, $\gamma_0\gamma_b\gamma_a\gamma_b$, $\ldots$ belong to N and have $\odot v$ undefined in $\Gamma(M)$. $\square$

Lemma 3.9. *Let $N \leqslant_{fg} F_n$ and N is not cyclic. Then,*
$$\mathrm{di}(N) = \sup_{\substack{M \leqslant_{fg} F_n \\ M \neq \langle x \rangle}} \frac{\tilde{\mathrm{rk}}(N \cap M)}{\tilde{\mathrm{rk}}(M)} = \sup_{\substack{M \leqslant_{fg} F_n \\ M \neq \langle x \rangle \\ N \nleqslant M}} \frac{\tilde{\mathrm{rk}}(N \cap M)}{\tilde{\mathrm{rk}}(M)} = \sup_{\substack{M \leqslant_{fg} F_n \\ M \neq \langle x \rangle \\ \exists v \in N, \odot v \, is\, not\, defined\, in\, M}} \frac{\tilde{\mathrm{rk}}(N \cap M)}{\tilde{\mathrm{rk}}(M)}$$

Proof. Given $1 > \epsilon > 0$. Let, $M = \langle e_1, e_2, \ldots, e_r \rangle \leqslant_{fg} F_n$, which is not cyclic and satisfying $\frac{\tilde{\mathrm{rk}}(N \cap M)}{\tilde{\mathrm{rk}}(M)} > \mathrm{di}(N) - \epsilon/2$. $1 > \epsilon > 0 \Rightarrow \frac{\tilde{\mathrm{rk}}(N \cap M)}{\tilde{\mathrm{rk}}(M)} \geqslant \frac{1}{2} \Rightarrow \tilde{\mathrm{rk}}(N \cap M) \geqslant 1 \Rightarrow \mathrm{rk}(N \cap M) \geqslant 2$. Therefore by Corollary 3.5 there exists $M' \leqslant_\infty M$ such that $\forall M' \leqslant L \leqslant_d M$ satisfies $[(N \cap M) : (N \cap L)] = [M : L]$. And, $\frac{\tilde{\mathrm{rk}}(N \cap L)}{\tilde{\mathrm{rk}}(M)} = \frac{d\,\tilde{\mathrm{rk}}(N \cap M)}{d\,\tilde{\mathrm{rk}}(M)} > \mathrm{di}(N) - \epsilon/2$.

Now, $M' \leqslant_\infty M \leqslant F_n \Rightarrow (N \cap M') \leqslant_\infty (N \cap M) \leqslant N$. By Lemma 3.7 $\exists v \in N \cap M$ such that $\odot v$ is not defined in M'. That is as a path γ_v starting from the base point $(\odot)$ is not defined in $\Gamma_{\{x_i' s\}}(M')$, hence as a path it is also not defined in $\Gamma_{\{e_i' s\}}(M')$. Because, in $\Gamma_{\{e_i' s\}}(M')$ replacing every petal e_i by $e_i := e_i(x_i)$ and by foldings we get $\Gamma_{\{x_i' s\}}(M')$. As, $N \cap M, M'$ both are subgroups of M we can draw the pull-back of $\Gamma_{\{e_i' s\}}(N \cap M)$ and $\Gamma_{\{e_i' s\}}(M')$ and as a connected component of the pull-back, we have $\Gamma_{\{e_i' s\}}(N \cap M')$.

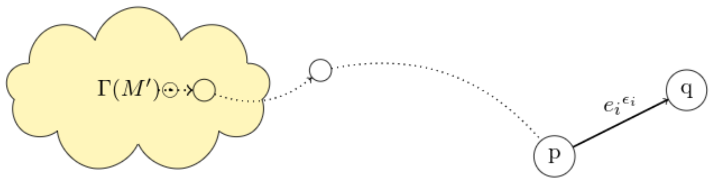

FIGURE 3. Diagram of Y

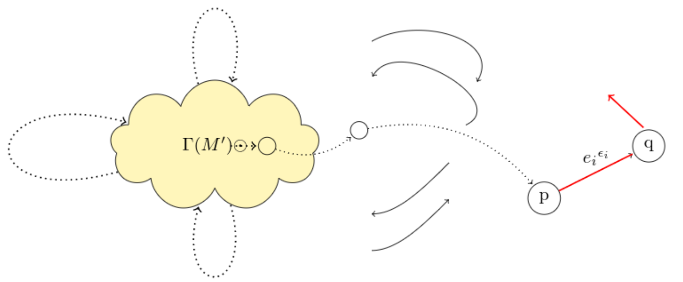

FIGURE 4. Diagram of $\tilde{Y} = \Gamma(L)$

Let $v := v(e_1, e_2, \ldots, e_r)$ which is not defined in M' ends with ${e_i}^{\epsilon_i}$. And let
$Y = \Gamma_{\{e'_i s\}}(M') + tail\ \ upto\ \ the\ \ end\ \ of\ \ \gamma_v$. See the figure 3.

Pairing all the missing edges in the graph of Y and adding them we will have the complete graph with respect to the basis of M. Let $\tilde{Y}$ be the completion of Y and $L = \pi(\tilde{Y})$, the fundamental group of $\tilde{Y}$. Thus we construct L (see the figure 4) as a finite index (say d) subgroup of M.

Therefore, $N \cap L \leqslant_d N \cap M$
Note that there exists another edge from q to a vertex $\neq q$ labeled ${e_i}^{\epsilon_i}$ so, the terminating edge of γ_v, i.e., ${e_i}^{\epsilon_i}$ is not a bridge.

Now we choose $s \in \mathbb{N}, s > \frac{2(|E_{e_i}\Gamma_{\{e'_i s\}}(N \cap M)| + \mathrm{rk}(N \cap L))}{\epsilon} + 1$

Then, we do the $s-explosion$ (see the figure 5 and compare with the figure 4) of the edge ${e_i}^{\epsilon_i}$ (last edge of γ_v outside of $\Gamma_{\{e'_i s\}}(M')$) to get $L_s \leqslant_s L$.

Hence from the construction we have, $M' \leqslant L_s \leqslant_s L \leqslant_d M \Rightarrow L_s \leqslant_{sd} M \Rightarrow (N \cap L_s) \leqslant_{sd} (N \cap M)$. Finally, let L'_s be constructed from L_s by deleting the edge ${e_i}^{\epsilon_i}$ ($\longrightarrow$ in the figure 5) from p_0 to q_1 (i.e., from the first block to the second block). Clearly, it is not a bridge, so, $\tilde{\mathrm{rk}}(L'_s) = \tilde{\mathrm{rk}}(L_s) - 1$.

The effect in the pull-back is that we loose at most $E := |E_{e_i}\Gamma_{\{e'_i s\}}(N \cap M)|$ edges from first block to second block. And as an effect the rank of the pull-back is decreased by $\leqslant E + \mathrm{rk}(N \cap L)$. Each rectangle in the figure 6 is $\Gamma(N \cap L) = \Gamma(N \cap M) \wedge \Gamma(L)$ which is the connected component of the pull-back $\Gamma(N \cap M) \times \Gamma(L)$ containing $(\odot, \odot)$. And the diagrams of $\Gamma(N \cap M), \Gamma(N \cap L)$ and $\Gamma(N \cap L_s)$ are depicted in the Figures 7, and 8 respectively.

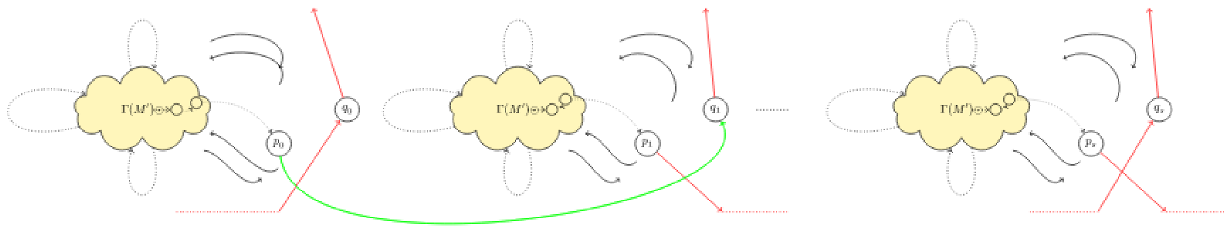

FIGURE 5. Diagram of $\Gamma(L_s)$

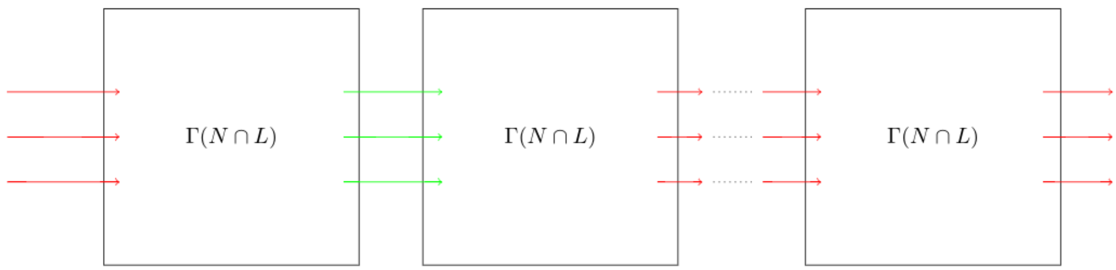

FIGURE 6. Block diagram of pull-back

It contains the pairing of $p \longrightarrow q \in L$ with $\odot \longrightarrow r \in N \cap M$ (*labeled* e_i) and with some other edges labeled e_i from $N \cap M$, some like $u_1 \longrightarrow u_2$ giving a non-bridge and some other like $u_3 \longrightarrow u_4$ giving a bridge.

(1) For those $(u_1, p) \longrightarrow (u_2, q) \in N \cap L$ not being bridges, the corresponding deleted edge $\longrightarrow$ is not a bridge. And so, this deletion decreases rank by 1 unit.
(2) For those $(u_3, p) \longrightarrow (u_4, q) \in N \cap L$ being a bridge, the corresponding deleted edge $\longrightarrow$ is a bridge. And so, for this deletion the rank is decreased by $t \geqslant 1$ units.

Now the total number of such edges in $N\cap L$ is at most $|E_{e_i}\Gamma_{\{e_i' s\}}(N\cap M)|$ and the sum of the $t's$ for those being bridges is at most $\mathrm{rk}(N\cap L)$. Hence, after the deletion of all $\longrightarrow$ from $N\cap L_s$ we loose the rank at most by $C := |E_{e_i}\Gamma_{\{e_i' s\}}(N\cap M)| + \mathrm{rk}(N\cap L)$. So, $\tilde{\mathrm{rk}}(L_s') = \tilde{\mathrm{rk}}(L_s) - 1$ *and* $\tilde{\mathrm{rk}}(N\cap L_s') \geqslant \tilde{\mathrm{rk}}(N\cap L_s) - C$.
Therefore, we have L_s' which is finitely generated, non-cyclic and satisfying the extra condition that there exists some element $v \in N$ such that $\odot v$ is not defined in L_s' and,

$$\begin{aligned}\frac{\tilde{\mathrm{rk}}(N\cap L_s')}{\tilde{\mathrm{rk}}(L_s')} \geqslant \frac{\tilde{\mathrm{rk}}(N\cap L_s)-C}{\tilde{\mathrm{rk}}(L_s)-1} &= \frac{sd\,\tilde{\mathrm{rk}}(N\cap M)-C}{sd\,\tilde{\mathrm{rk}}(M)-1}\\ &\geqslant \frac{\tilde{\mathrm{rk}}(N\cap M)}{\tilde{\mathrm{rk}}(M)} - \epsilon/2\\ &> \mathrm{di}(N) - \epsilon/2 - \epsilon/2\\ &= \mathrm{di}(N) - \epsilon\end{aligned} \tag{15}$$

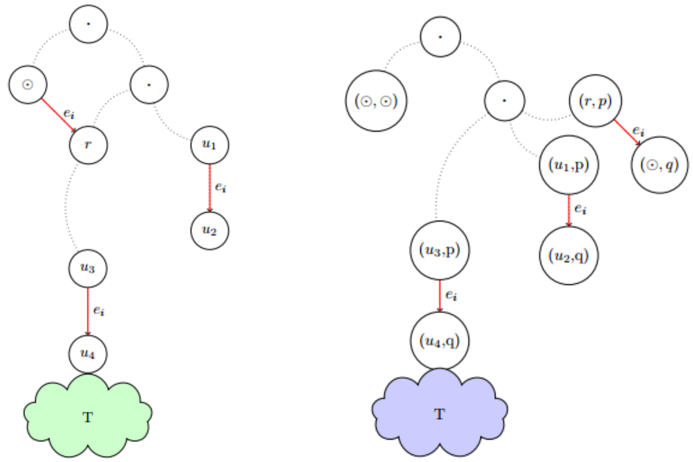

FIGURE 7. Diagram of $\Gamma(N \cap M)$ and $\Gamma(N \cap L)$

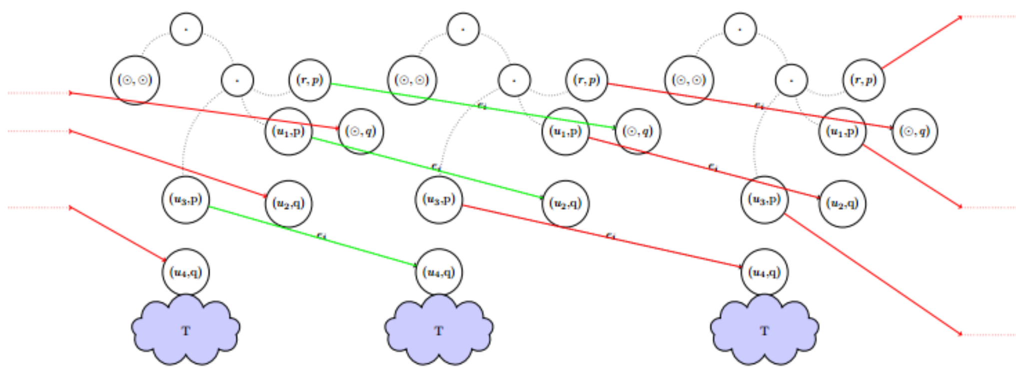

FIGURE 8. Diagram of $\Gamma(N \cap L_s)$

The second inequality of (15) is true because of the following: $s > \frac{2C}{\epsilon} + 1$ so, $\epsilon s > 2C + \epsilon$ and then

$$\epsilon sd \cdot \tilde{rk}(M)^2 \geqslant \epsilon s \cdot \tilde{rk}(M) > 2C \cdot \tilde{rk}(M) + \epsilon \cdot \tilde{rk}(M) > 2C \cdot \tilde{rk}(M) + \epsilon \cdot \tilde{rk}(M) - 2 \cdot \tilde{rk}(N \cap M)$$

and so,

$$-C \cdot \tilde{rk}(M) \geqslant -\tilde{rk}(N \cap M) - \frac{\epsilon sd \cdot \tilde{rk}(M)^2}{2} + \frac{\epsilon \cdot \tilde{rk}(M)}{2},$$

$$sd \cdot \tilde{rk}(M) \cdot \tilde{rk}(N \cap M) - C \cdot \tilde{rk}(M) \geqslant \tilde{rk}(N \cap M) \cdot \big(sd \cdot \tilde{rk}(M) - 1\big) - \frac{\epsilon}{2}\tilde{rk}(M) \cdot \big(sd \cdot \tilde{rk}(M) - 1\big),$$

and therefore, $\frac{sd \cdot \tilde{rk}(N \cap M) - C}{sd \cdot \tilde{rk}(M) - 1} \geqslant \frac{\tilde{rk}(N \cap M)}{\tilde{rk}(M)} - \frac{\epsilon}{2}$. Hence the proof is complete by Equation (15) $\square$

Now we are in position to show that $\mathrm{di}(H) \geqslant l\,\mathrm{di}(H\pi)$. Since, by Corollary 1.8, conjugating does not affect the degree of inertia of the subgroup, this is equivalent to showing that $\mathrm{di}(H^w) \geqslant l\,\mathrm{di}(H\pi^w)$, for every $w \in F_n$. Now, since $rk(H\pi) \geqslant 2$, there is at least one vertex in $\Gamma(H\pi)$ of degree three or more and, conjugating appropriately, we can assume this is the basepoint.

In order to see $\mathrm{di}(H) \geqslant l\,\mathrm{di}(H\pi)$, we fix $\epsilon > 0$ and will construct $K \leqslant_{fg} \mathbb{Z}^m \times F_n$, where K is non-cyclic, such that $H \cap K$ is also finitely generated and $\frac{\tilde{rk}(H\cap K)}{\tilde{rk}(K)} > l\,\mathrm{di}(H\pi) - \epsilon$. And will do this in three steps: declare $L_K = K \cap \mathbb{Z}^m = \{0\}$, construct the free part $K\pi$ and, finally, decide the vectors completing a basis from the free part.

Let us construct the free part $K\pi$ as follows: applying Lemma 3.9 to $H\pi$, there exists $M \leqslant_{fg} F_n$ not cyclic, with some $h \in H\pi$ such that $\odot h$ is not defined in $\Gamma(M)$, and such that $\frac{\tilde{rk}(H\pi\cap M)}{\tilde{rk}(M)} > \mathrm{di}(H\pi) - \epsilon$. If $H\pi \cap M$ had finite index in $H\pi$ then every element in $H\pi$ would have a power contained in $H\pi \cap M$ and so readable in $\Gamma(M)$, contradicting the existence of h. So, $H\pi \cap M \leqslant_\infty H\pi$. Now, applying Lemma 3.8 and the fact that $\odot$ has degree at least 3 in $\Gamma(H\pi)$, we deduce the existence of infinitely many such elements h. Take the first m of them (with the form coming from the proof of Lemma 3.8), and attach the corresponding paths going out of $\Gamma(M)$ to obtain a bigger Stallings, and take $K\pi$ to be its fundamental group. Clearly, $K\pi$ is a free multiple of M, more concretely, $K\pi = M * \langle w_0, w_1, \ldots, w_{m-1}\rangle$, where $w_i = h\gamma_a^i \gamma_b \gamma_a^{-i} h^{-1}$, $i = 0, \ldots, m-1$. Put $n_1 := rk(H\pi)$, $n_2 := \mathrm{rk}(K\pi) = \mathrm{rk}(M) + m$, and $n_3 := \mathrm{rk}(H\pi \cap K\pi)$. Note that, by construction, the pullback of $\Gamma(H\pi)$ and $\Gamma(K\pi)$ will contain as a subgraph $\Gamma(H\pi \cap K\pi)$ enlarged in the same way as in picture 9 and, possibly, with some other new closed paths; therefore, $H\pi \cap K\pi = (H\pi \cap M) * \langle w'_1, \ldots w'_{m'}\rangle * \langle w_0, w_1, \ldots, w_{m-1}\rangle$, and $rk(H\pi \cap K\pi) = \mathrm{rk}(H\pi \cap M) + m' + m$, for some $m' \geqslant 0$.

Finally we see now that, in our situation, it is possible to complete the construction of K by choosing the vectors completing a basis of $K\pi$ (i.e., the rows of P') in such a way that the map $R = PA - P'A'$ in the diagram becomes onto:

$$\begin{tikzcd}
 & (H\cap K)\pi \arrow[d, phantom, "\rotatebox{-90}{$\leqslant$}"] & \\
H\pi \arrow[d, "\rho_1"'] \arrow[dr, phantom, "///"] & H\pi\cap K\pi \arrow[l, hook', "\iota"'] \arrow[r, hook, "\iota'"] \arrow[d, "\rho_3"] \arrow[dr, phantom, "///"] & K\pi \arrow[d, "\rho_2"] \\
\mathbb{Z}^{n_1} \arrow[dr, "\mathbf{A}"] & \mathbb{Z}^{n_3} \arrow[l, "\mathbf{P}"'] \arrow[r, "\mathbf{P}'"] \arrow[d, red, "\mathbf{R}"] & \mathbb{Z}^{n_2} \arrow[dl, "\mathbf{A}'"'] \\
 & \mathbb{Z}^m &
\end{tikzcd}$$

Recall that, in this diagram, $P \in M_{n_3\times n_1}$, $P' \in M_{n_3\times n_2}$, $A \in M_{n_1\times m}$ and $A' \in M_{n_2\times m}$, where $n_3 = rk(H\pi\cap M) + m + m'$, and we write matrices with respect to the abelianizations of the natural free bases of $H\pi$, $K\pi$, and $H\pi \cap K\pi$ coming from the descriptions above.

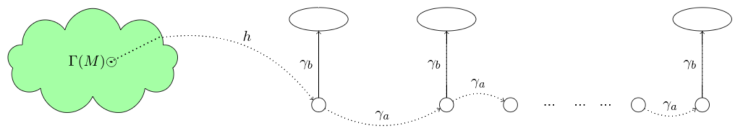

FIGURE 9. Diagram of $\Gamma(K\pi)$

Let, Q be the lower $m \times m$ part of the matrix PA and define

$$A' = \left(\begin{array}{c} 0 \\ \hline -I_m + Q \end{array} \right).$$

Then, $R = PA - P'A'$ equals

$$\left(\begin{array}{c} * \\ \hline * \\ \hline Q \end{array} \right) - \left(\begin{array}{c|c} * & 0 \\ \hline * & * \\ \hline 0 & I_m \end{array} \right) \left(\begin{array}{c} 0 \\ \hline -I_m + Q \end{array} \right) = \left(\begin{array}{c} * \\ \hline * \\ \hline Q \end{array} \right) - \left(\begin{array}{c} 0 \\ \hline * \\ \hline -I_m + Q \end{array} \right) = \left(\begin{array}{c} * \\ \hline * \\ \hline I_m \end{array} \right)$$

and so, it is clearly onto. Now,

$$\begin{aligned} l = [\mathbb{Z}^m : L_H] = [\mathbb{Z}^m : L_H + L_K] = [\mathbb{Z}^{n_3} : (L_H + L_K)R^{-1}] = \\ = [H\pi \cap K\pi : (L_H + L_K)R^{-1}\rho_3^{-1}] = [H\pi \cap K\pi : (H \cap K)\pi] \end{aligned}$$

and, therefore,

$$\begin{array}{rcl} \frac{\tilde{\mathrm{rk}}(H\cap K)}{\tilde{\mathrm{rk}}(K)} = \frac{\dim(L_H \cap L_K) + \tilde{\mathrm{rk}}((H\cap K)\pi)}{\dim(L_K) + \tilde{\mathrm{rk}}(K\pi)} & = & \frac{l\cdot \tilde{\mathrm{rk}}(H\pi \cap K\pi)}{\tilde{\mathrm{rk}}(K\pi)} \\ & = & l \cdot \frac{\tilde{\mathrm{rk}}(H\pi\cap M) + m' + m}{\tilde{\mathrm{rk}}(M) + m} \\ & \geqslant & l \cdot \frac{\tilde{\mathrm{rk}}(H\pi\cap M) + m}{\tilde{\mathrm{rk}}(M) + m} \\ & \geqslant & l \cdot \frac{\tilde{\mathrm{rk}}(H\pi\cap M)}{\tilde{\mathrm{rk}}(M)} \\ & > & l \cdot (\mathrm{di}(H\pi) - \epsilon), \end{array}$$

which is what we wanted to prove.

4. FUTURE PLANS

In all the results and formulas which I have done during my first year of PhD, the ambient group is the direct product of a free-abelian group and a finitely generated free group. For the immediate future, I plan to do a bit more work in this family of groups (namely, develop a Takahasi-type theorem) and then try to generalize the similar types of results obtained to other bigger and more complicated families of groups.

First, in Section 4.1. below, I will briefly discuss a list of consequences of Takahasi theorem 2.6 for free groups, already developed in the literature. Then, I will discuss some possible reasonable extensions of all those results to $\mathbb{Z}^m \times F_n$.

In Sections 4.2. and 4.3. I give a brief description of several particular candidate families of groups, where I will try to obtain extensions of some of the above results: semidirect products of free-abelian and free groups, and direct products of finitely many factors (not just two) being free-abelian, or free, or surface groups.

4.1. **Takahasi theorem and its applications.** Of particular interest to our discussion is the result given by Takahasi [9] in 1951. The original proof, due to M. Takahasi was combinatorial, using words and their lengths with respect to different sets of generators. And the more geometrical proof was done later independently by Ventura in [10] and by Kapovich–Miasnikov in [5]. Takahasi theorem is an important tool in free groups as there has been several research works where it played a crucial role in proving of them. Here are some of these applications:

- Computation of the *auto-closure* of a subgroup $H \leqslant_{fg} F_n$, namely,

$$a - cl(H) = \bigcap_{\substack{\varphi \in Aut(F_n) \\ H \leqslant Fix(\varphi)}} Fix(\varphi)$$

  as well as the *endo-closure* (the same with endomorphisms). This was done by E. Ventura in [11] where, additionally, an algorithm is given to decide if a given subgroup is the fixed subgroup of a finite family of autos (or endos) or not, and in the affirmative case, computing such a family of autos (or endos).

- E. Ventura [7] also proved that, for every autos (or endos) f, g there is another one h such that $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is a free factor of $\mathrm{Fix}(h)$. And in the same paper [7] Ventura conjectured that the family of fixed subgroups is closed by intersections (i.e., one can always avoid the free complement). In other words, is $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ always equal to $\mathrm{Fix}(h)$ for some h? This is still an open problem even in free groups.

- Computation of pro-$\mathcal{V}$ closures (like pro-p, pro-solvable, pro-nilpotent, etc) of finitely generated subgroups of a free group F_n. Consider an extension closed variety $\mathcal{V}$ of finite groups (i.e., a family of finite groups closed under taking subgroups, quotients, and direct products; in other words, for any short exact sequence $1 \to A \to B \to C \to 1$ of finite groups, if $B \in \mathcal{V}$ then $A, C, A \times C \in \mathcal{V}$). Given such a variety $\mathcal{V}$ and an arbitrary group G, one can put the pro-$\mathcal{V}$ topology in G defined (metrically) in the following way: given two elements $g, g' \in G$ define the distance between them as $d(g, g') = 2^{-v(g,g')}$, where $v(g, g')$ is the smallest cardinal of a group $H \in \mathcal{V}$ for which there is a homomorphism $\varphi\colon G \to H$ separating g and g', i.e., $g\varphi \neq g'\varphi$ (take $d(g, g') = 0$ if $v(g, g') = \infty$ meaning that there is no such finite group H). This is a metric in G which induces a topology called the pro-$\mathcal{V}$ topology; typical examples are the pro-finite topology (take $\mathcal{V}$ to be all finite groups), the pro-p topology (take $\mathcal{V}$ to be all finite p-groups), the pro-nilpotent topology (take $\mathcal{V}$ to be all finite nilpotent groups), the pro-solvable topology (take $\mathcal{V}$ to be all finite solvable groups), etc.

  Let us particularize the situation in the free group, $G = F_n$. In [6], the authors proved among other results that, when the variety $\mathcal{V}$ is extension-closed (if $A, C \in \mathcal{V}$ then $B \in \mathcal{V}$) then free factors of closed subgroups are again closed subgroups. This automatically connects with Takahasi theorem because it implies that, for any subgroup $H \leqslant_{fg} F_n$, its pro-$\mathcal{V}$ closure must be one of the H_i's, $\overline{H} \in \mathcal{AE}(H)$. Using this idea the authors of [6] give algorithms to compute the pro-finite, pro-p, and pro-nilpotent closures of finitely generated subgroups of F_n (the computation of the pro-solvable closure is still an open problem).

Adapting appropriately the notions of "free factor" and "algebraic extension" from free groups to $Z^m \times F_n$, it seems reasonable to be able to:

Project 4.1. Give a version of Takahasi's theorem for free-abelian times free groups.

If I succeed in this, a natural plan is to try to generalize the above applications of Takahasi theorem for free groups to my family of groups:

Project 4.2. Give an algorithm to compute auto-closures and endo-closures of finitely generated subgroups of $Z^m \times F_n$. This seems plausible, specially having in mind the explicit description of all automorphisms and endomorphisms of $Z^m \times F_n$ given by Delgado–Ventura in [2].

Project 4.3. Is the family of fixed subgroups $Z^m \times F_n$ in some sense closed under intersections ? Maybe up to "free factors" ? Can such an intersection be not finitely generated ? (remind that $Z^m \times F_n$ is not Howson and so, all questions related to intersections tend in general to be more tricky).

Project 4.4. Consider the pro-$\mathcal{V}$ topology in $Z^m \times F_n$. Reprove here the fact that, in the extension closed case, "free factors" of closed subgroups are closed again (with the appropriate notion of "free factor"), and then extend the algorithms for computing finite, p-, and nilpotent closures, from the free group to $F_n \times Z^m$.

4.2. Semidirect products of the form $\mathbb{Z}^m \rtimes F_n$.

Definition 4.5. Consider $\{t^a \mid a \in \mathbb{Z}^m\}$ same as before, i.e., $\mathbb{Z}^m$ in multiplicative notation, let $A_1, \ldots, A_n \in GL(m, \mathbb{Z})$ acting as $A_i \colon t^a \mapsto t^{aA_i}$, and consider the semidirect product

$$G = \mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n = \langle u_1, \ldots, u_n, t_1, \ldots, t_m \mid [t_i, t_j] = 1,\, u_i^{-1} t^a u_i = t^{aA_i} \rangle$$

The following is an easy observation for this group G.

Observation 4.6. *We have the split short exact sequence*

$$1 \to \mathbb{Z}^m \to G \to F_n \to 1,$$

and normal forms $w(\vec{x})t^a$ for the elements of G, where $a \in \mathbb{Z}^m$ and $w \in F_n = F(\{x_1, \ldots, x_n\})$, computable using $t^a x_i = x_i t^{aA_i}$ (or, in general, $t^a w(\vec{x}) = w(\vec{x}) t^{aW}$, where $W = W(A_1, \ldots, A_n) \in GL(m, \mathbb{Z})$).

Proposition 4.7. *For every subgroup $H \leqslant G = \mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$, the sub- short exact sequence*

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z}^m & \to & G & \stackrel{\pi}{\to} & F_n & \to & 1 \\ & & \vee & & \vee & & \vee & & \\ 1 & \to & L_H = H \cap \mathbb{Z}^m & \to & H & \stackrel{\pi}{\to} & H\pi & \to & 1 \end{array}$$

also splits and so, $H \simeq L_H \rtimes_{\mathcal{A}} H\pi$, where $\mathcal{A}$ is the restriction of the defining action $F_n \to Aut(\mathbb{Z}^m)$ to $\mathcal{A} \colon H\pi \to Aut(L)$.

In particular, every $H \leqslant \mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$, $n \geqslant 2$, is of the form $H \simeq \mathbb{Z}^{m'} \rtimes_{A'_1,\ldots,A'_{n'}} F_{n'}$ for some $n' \in \mathbb{N} \cup \{\infty\}$ and $m' \leqslant m$.

The first reasonable step in this family is to study the degree of compression. The arguments involved in the study and computability of the degree of compression for a subgroup of $Z^m \times F_n$ are purely about the free group (Stallings graphs, fringe, algebraic extensions, etc) or about linear algebra (PAQ-reduction of integral matrices, linear systems of equations, manipulation of direct summands, etc). It seems reasonable to think that these arguments will extend and work in a semidirect product $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$, just with the matrices making the calculations more complicated and the arguments more involved. An interesting point here is the fact that, while the rank of $\mathbb{Z}^m \times F_n$ (i.e., the minimal number of generators) is $m+n$, the rank of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$ could easily be less than this because of the effect of the action matrices. I do not see yet a clear way to compute/understand ranks of (free-abelian)-by-free groups; maybe the notion of degree of compression will have to be considered with respect to the invariant $dim(\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n) = m+n$ instead of the usual rank. In whatever sense it needs to be considered, the project here is to

Project 4.8. Find formulas and algorithms to compute the degree of compression of finitely generated subgroups of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$.

Other possibilities for future investigation are the search of a Takahasi theorem for semidirect products (subject to finding a good enough notion of "free factor", which is not clear at the moment, and needs more detailed thinking), and study of particular properties of fixed subgroups of automorphisms (subject to being able to obtain a more or less explicit description of all automorphisms of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$):

Project 4.9. Find a god enough notion of "free factor" for subgroups of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$ and prove a Takahasi-like theorem for this family of groups. Obtain similar applications as those done for the free case (see above).

Project 4.10. Find a good enough description of the automorphisms of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$ and, from it, analyze special properties of fixed point subgroups of automorphisms in this family of groups (bounding the rank, compression, inertia, etc, following again the guide of what happens in free groups).

About degree of inertia I am much more skeptical: our understanding of the degree of inertia for subgroups of $\mathbb{Z}^m \times F_n$ strongly relies on the diagram of the equation (12) invented by Delgado–Ventura in [2] to understand arbitrary intersections of finitely generated subgroups. As far as I know, these arguments do not extend to semidirect products, where the control of intersections seems to be much more involved, and unknown at the present time. Without a way of understanding well intersections, it does not seem plausible to try to understand the degree of inertia in semidirect products.

Delgado-Ventura [1] have built an adaptation of the Stallings automata theory very useful to work with subgroups of $\mathbb{Z}^m \rtimes_{A_1,\ldots,A_n} F_n$; they are essentially classical Stallings graphs decorated with vectors in a clever enough way to keep all the information of the subgroup in a finite geometric object. It is very possible that this nice construction helps us in our goals within this family of groups.

4.3. **Product groups involving surface groups.** I want to give some similar results for the product groups which are formed not only using $\mathbb{Z}^m$ and one free group F_n, but also several of them, and also including several copies of surface groups S_g and NS_k at the same time. Surface groups have similar properties to free groups, and have interesting connections to them so, it seems reasonable to be able to extend some of our results to this more general family of groups. Some initial steps into this direction are already done in the literature, specifically investigating automorphisms and their fixed points (see [4], [12] and [13]).

Definition 4.11. A surface group is the fundamental group, $G = \pi_1(X)$, of a connected closed (possibly non-orientable) surface X. To fix the notation, we shall denote by Σ_g the closed orientable surface of genus $g \geqslant 0$, and by

$$S_g = \pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$$

its fundamental group (by convention, $S_0 = \langle \mid \rangle$ stands for the trivial group, the fundamental group of the sphere Σ_0); here we use the notation $[x, y] = xyx^{-1}y^{-1}$. And for the non-orientable case, we shall denote by $N\Sigma_k$ the connected sum of $k \geqslant 1$ projective planes, and by

$$NS_k = \pi(N\Sigma_k) = \langle a_1, a_2 \ldots, a_k, \mid {a_1}^2 \cdots {a_k}^2 \rangle$$

its fundamental group. Note that, among surface groups, the only abelian ones are $S_0 = 1$ (for the sphere), $S_1 = \mathbb{Z}^2$ (for the torus), and $NS_1 = \mathbb{Z}/2\mathbb{Z}$ (for the projective plane).

It is well known that the Euler characteristic of an orientable surface is $\chi(\Sigma_g) = 2 - 2g$, and of the non-orientable ones is $\chi(\Sigma_k) = 2 - k$. Hence, all surfaces have negative Euler characteristic (these are said to be of hyperbolic type) except for the sphere Σ_0, the torus Σ_1, the projective plain $N\Sigma_1$, and the Klein bottle $N\Sigma_2$, homeomorphic to the connected sum of two projective plains (these exceptional ones are said to be of Euclidean type).

These surface groups have some interesting properties:

- Any subgroup H of a surface group G either has finite index in G or it is free; and if H has index d in G, then it is again a surface group with $\chi(H) = d \cdot \chi(G)$.
- The fundamental group of a non-compact surface is free.
- Let G be a surface group with $\chi(G) < 0$. Then its center is trivial, $Z(G) = 1$, and the centralizer of any non-trivial element $1 \neq g \in G$ is infinite cyclic, $Cen_G(g) \simeq \mathbb{Z}$.

Some results about automorphisms and endomorphisms for free groups (specially those concerning compression or inertia) will work in a similar way for surface groups with negative Euler characteristic; S_0, S_1, NS_1, and NS_2 will usually present special and exceptional behaviour (in part, due to the structure of the center and centralizers in these cases).

In this direction the first results were given by Jiang-Wang-Zhang [4] in 2011.

Theorem 4.12 (Jiang-Wang-Zhang, [4])**.** *Let G be a surface group with $\chi(G) < 0$. Then* $\mathrm{rk}(\mathrm{Fix}(\phi)) \leqslant \mathrm{rk}(G)$, *$\forall \phi \in \mathrm{End}(G)$.*

And the result was extended to the following results :

Theorem 4.13 (Wu-Zhang, [12])**.** *Let G be a surface group with $\chi(G) < 0$, and $B \subseteq \mathrm{End}(G)$. Then*

- $\mathrm{rk}(\mathrm{Fix}\, B) \leqslant \mathrm{rk}(G)$*, with equality if and only if $B = \{id\}$.*
- $\mathrm{rk}(\mathrm{Fix}\, B) \leqslant \frac{1}{2}\,\mathrm{rk}(G)$*, if B contains a non-epimorphic endomorphism.*
- *if $B \subseteq \mathrm{Aut}(G)$, then $\mathrm{Fix}\, B$ is inert in G.*

And then, recent results are also given in the inertia direction:

Theorem 4.14 (Zhang–Ventura–Wu, [13]). *(a) Let F_n be a finitely generated free group, let $B \subset \mathrm{End}(F_n)$ and let $\beta_0 \in \langle B \rangle \leqslant \mathrm{End}(F_n)$ be with $\mathrm{rk}(\beta_0(F_n))$ minimal. Then, $\mathrm{Fix}(B)$ is inert in $\beta_0(F_n)$. Moreover, if $\beta_0(F_n)$ is inert in F_n then $\mathrm{Fix}(B)$ is inert in F_n.*

(b) Let G be a surface group, let $B \subseteq \mathrm{End}(G)$ be an arbitrary family of endomorphisms, let $\langle B \rangle \leqslant \mathrm{End}(G)$ be the submonoid generated by B, and let $\beta_0 \in \langle B \rangle \leqslant \mathrm{End}(G)$ with image of minimal rank. Then, for every subgroup $K \leqslant G$ such that $\beta_0(K) \cap \mathrm{Fix}(B) \leqslant K$, we have that $\mathrm{rk}(K \cap \mathrm{Fix}(B)) \leqslant \mathrm{rk}(K)$.

Let us consider the family of groups $\mathcal{P}$ consisting in direct products of finitely many free-abelian groups, free groups, and surface groups, i.e., groups of the form $G = G_1 \times G_2 \times \cdots \times G_n$, where $n \geqslant 1$ and each G_i is either $\mathbb{Z}^m$ with $m \geqslant 1$, or F_n with $n \geqslant 2$, or S_g with $g \geqslant 2$, or NS_k with $k \geqslant 1$. Such a group G is said to be of *hyperbolic* type if all its factors are hyperbolic, of *Euclidean* type if all its factors are Euclidean, and of *mixed* type otherwise.

A first project is to study the degree of compression of subgroups in this family of groups; for similar reasons as those given in the case of semidirect products, the study of the degree of inertia seems to be much more tricky also in the present family of groups:

Project 4.15. Find formulas and algorithms to compute the degree of compression of finitely generated subgroups of a group G in $\mathcal{P}$ (maybe under technical restrictions, if necessary, on the subgroup and/or on the factors of G).

In the paper [13], Zhang–Ventura–Wu show that any automorphism φ of a group $G = G_1 \times G_2 \times \cdots \times G_n \in \mathcal{P}$ of hyperbolic type is always equal to the direct product of automorphisms of each component, $\varphi = \varphi_1, \times \cdots \times \varphi_n$, $\varphi_i \in Aut(G_i)$, just modulo permutations of the possibly repeated factors ($G_i = G_j$), if any; see Proposition 4.4 from [13] for the exact statement. This result allows to connect properties of the automorphisms of G with the corresponding properties about automorphisms of the factors G_i's. In this sense, [13] contains the following nice characterization:

Theorem 4.16 (Zhang–Ventura–Wu, [13]). *Let $G = G_1 \times G_2 \times \cdots \times G_n \in \mathcal{P}$. Then, $rk(Fix(\varphi)) \leqslant rk(G)$ for every $\varphi \in Aut(G)$, if and only if G is either of hyperbolic or of Euclidean type.*

In fact, in the case of mixed type, and copying the idea from $\mathbb{Z} \times F_2$, one can easily construct an automorphism $\varphi \in Aut(G)$ whose fixed subgroup is even not finitely generated.

[13] also contains partial results in the direction of characterizing which $G \in \mathcal{P}$ satisfy that $Fix(\varphi)$ is compressed, or $Fix(\varphi)$ is inert, for every $\varphi \in Aut(G)$. Would be nice to complete this characterization in the spirit of the above theorem:

Project 4.17. Give an explicit characterization of those $G \in \mathcal{P}$ which satisfy: (i) $Fix(\varphi)$ is compressed for every $\varphi \in \mathrm{Aut}(G)$; or (ii) $Fix(\varphi)$ is inert for every $\varphi \in \mathrm{Aut}(G)$. Study the similar questions about endomorphisms.

Project 4.18. If it is possible to compute constants $m, l \in \mathbb{N}$, depending only on the group $G \in \mathcal{P}$ in such a way that, for every $\varphi \in \mathrm{Aut}(G)$, the subgroup $Fix(\varphi)$ is (m, l)-compressed.

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