The conjugacy and isomorphism problems for automaton groups

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Outline

1. Algorithmic problems
2. Automaton groups
3. Mihailova's construction and orbit undecidability
4. Unsolvability of IP

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The three Dehn problems

Let $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ be a finite presentation.

- *Word Problem (WP): Given a word $w(x_1, \ldots, x_n)$ decide whether $w =_G 1$.*
- *Conjugacy Problem (CP): Given two words $u(x_1, \ldots, x_n)$ and $v(x_1, \ldots, x_n)$, decide whether $u \sim_G v$.*
- *Isomorphism Problem (IP): Given two finite presentations like above, G_1 and G_2, decide whether $G_1 \simeq G_2$.*

The three of them are known to be unsolvable in general.

Theorem (Novikov 1955, Boone 1957)

There exists a finitely presented group with unsolvable WP.

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More algorithmic problems

- *Membership problem (MP):* Given G and $h, h_1, \ldots, h_n \in G$, decide whether $h \in H = \langle h_1, \ldots, h_n \rangle \leqslant G$.
- *Generation problem (GP):* Given G and $g_1, \ldots, g_n \in G$, decide whether $\langle g_1, \ldots, g_n \rangle = G$.

Theorem (Mihailova 1958)

The membership problem in $F_2 \times F_2$ is unsolvable.

Theorem (Miller 1971)

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Orbit decidability

Definition

Let G be a f.g. group. A subgroup $\Gamma \leqslant \mathrm{Aut}(G)$ is said to be orbit decidable (O.D.) if there is an algorithm s.t., given $u, v \in G$, it decides whether v and $\alpha(u)$ are conjugate, for some $\alpha \in \Gamma$.

First examples: $G = \mathbb{Z}^d$

Observation (folklore)

The full group $\mathrm{Aut}(\mathbb{Z}^d) = \mathrm{GL}_d(\mathbb{Z})$ is orbit decidable.

Proof. *For $u, v \in \mathbb{Z}^d$, there exists $A \in \mathrm{GL}_d(\mathbb{Z})$ such that $v = Au$ if and only if $gcd(u_1, \ldots, u_d) = gcd(v_1, \ldots, v_d)$.*

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OD subgroups in $GL_d(\mathbb{Z})$...

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Proposition (Bogopolski-Martino-V., 08)

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Main results

Theorem (Sunic-V.)

There exist automaton groups with unsolvable conjugacy problem.

Easy: *the word problem is solvable for all such groups.*

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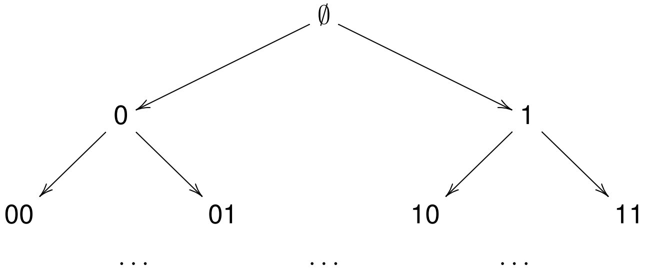
Tree automorphisms

Let X be an alphabet on k letters, and let X^* be the free monoid on X, thought as a rooted k-ary tree:



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- Every *tree automorphism* g decomposes as a *root permutation* $\pi_g : X \to X$, and k *sections* $g|_x$, for $x \in X$:

$$g(xw) = \pi_g(x) g|_x(w).$$

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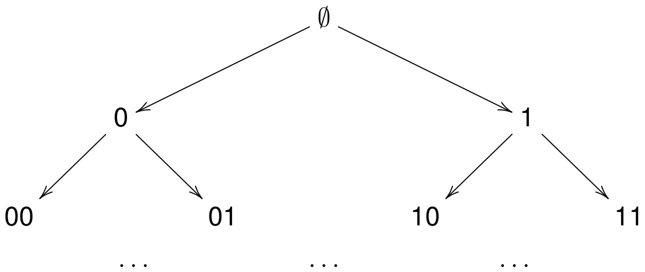

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Automaton groups

Definition

- *A set of tree automorphisms is self-similar if it contains all sections of all of its elements.*
- *A finite automaton is a finite self-similar set (elements are called states).*
- *The group $G(\mathcal{A})$ of tree automorphisms generated by an automaton $\mathcal{A}$ is called an automaton group.*

The Grigorchuk group: $G = \langle \alpha, \beta, \gamma, \delta \rangle$, where

$$\alpha = \sigma(1, 1), \quad \beta = 1(\alpha, \gamma), \quad \gamma = 1(\alpha, \delta), \quad \delta = 1(1, \beta).$$

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Reduction to matrices

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Both results come from...

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Given $\Gamma, \Delta \leqslant \mathrm{GL}_d(\mathbb{Z})$ f.g., it is undecidable whether $\mathbb{Z}^d \rtimes \Gamma \simeq \mathbb{Z}^d \rtimes \Delta$.

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Mihailova's subgroup

Definition

Let $U = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ be a finite presentation. The Mihailova group corresponding to U is

$$M(U) = \{(v, w) \in F_n \times F_n \mid v =_U w\} =$$

$$= \langle (x_1, x_1), \ldots, (x_n, x_n), (1, r_1), \ldots, (1, r_m) \rangle \leqslant F_n \times F_n.$$

Theorem (Mihailova 1958)

The membership problem in $F_2 \times F_2$ is unsolvable.

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Connection to orbit decidability

Proposition (Bogopolski-Martino-V. 2008)

Let G be a group, and let $A \leqslant B \leqslant \mathrm{Aut}(G)$ and $v \in G$ be such that $B \cap Stab^*(v) = 1$. Then,

$$OD(A) \text{ solvable} \quad \Rightarrow \quad MP(A, B) \text{ solvable.}$$

Proof. *Given $\varphi \in B \leq \mathrm{Aut}(G)$, let $w = v\varphi$ and*

$$\{\phi \in B \mid v\phi = w\} = B \cap (Stab(v) \cdot \varphi) = (B \cap Stab(v)) \cdot \varphi = \{\varphi\}.$$

$$\{\phi \in B \mid v\phi \sim w\} = B \cap (Stab^*(v) \cdot \varphi) = (B \cap Stab^*(v)) \cdot \varphi = \{\varphi\}.$$

So, deciding whether v can be mapped to w, up to conjugacy, by somebody in A, is the same as deciding whether φ belongs to A. Hence,

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Orbit undecidable subgroups

Proposition (Bogopolski-Martino-V., 08)

For $d \geqslant 4$, there exist f.g., orbit undecidable, subgroups $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$.

Proof. *Consider* $F_2 \simeq \langle P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \rangle \leq_{24} GL_2(\mathbb{Z})$.

- $Stab(1,0) = \{M \mid (1,0)M = (1,0)\} = \{ \begin{pmatrix} 1 & 0 \\ n & \pm 1 \end{pmatrix} \mid n \in \mathbb{Z}\}$.
- $\langle P, Q \rangle \cap Stab(1,0) = \langle \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix} \rangle$.
- *Choose a free subgroup* $F_2 \simeq \langle P', Q' \rangle \leq \langle P, Q \rangle$ *such that* $\langle P', Q' \rangle \cap Stab(1,0) = \{I\}$ *and consider*

$$B = \langle \left(\begin{array}{c|c} P' & 0 \\ \hline 0 & I \end{array}\right), \left(\begin{array}{c|c} Q' & 0 \\ \hline 0 & I \end{array}\right), \left(\begin{array}{c|c} I & 0 \\ \hline 0 & P' \end{array}\right), \left(\begin{array}{c|c} I & 0 \\ \hline 0 & Q' \end{array}\right) \rangle \leq GL_4(\mathbb{Z}).$$

- *Note that* $B \simeq F_2 \times F_2$.

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- *Write* $v = (1, 0, 1, 0)$. *By construction,* $B \cap Stab(v) = \{I\}$.
- *Take* $A \leq B \simeq F_2 \times F_2$ *with unsolvable membership problem.*
- *By previous Proposition,* $A \leqslant \mathrm{GL}_4(\mathbb{Z})$ *is orbit undecidable.*
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Connection to semidirect products

Observation (B-M-V)

Let H be f.g., and $\Gamma \leqslant \mathrm{Aut}(H)$ f.g. If $H \rtimes \Gamma$ has solvable CP, then $\Gamma \leqslant \mathrm{Aut}(H)$ is orbit decidable.

Proof. *$G = H \rtimes \Gamma$ contains elements $(h, \gamma) \in H \times \Gamma$ operated like*

$$(h_1, \gamma_1) \cdot (h_2, \gamma_2) = (h_1 \gamma_1(h_2), \gamma_1 \gamma_2)$$

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For $h_1, h_2 \in H \leqslant G$, we have $h_1 \sim_G h_2 \iff \exists (h, \gamma) \in H \rtimes \Gamma$ s.t.

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Outline

1. Algorithmic problems
2. Automaton groups
3. Mihailova's construction and orbit undecidability
4. Unsolvability of IP

A construction due to Gordon

Let $X = \{x_1, \ldots, x_n\}$ *and* $U = \langle X \mid R \rangle$ *be a finite presentation. For* $w = w(x_1, \ldots, x_n)$ *consider*

$$H_w = \left\langle X, a, b, c \;\middle|\; \begin{array}{l} R \\ a^{-1}ba = c^{-1}b^{-1}cbc \\ a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}cbc^2 \\ a^{-3}[w,b]a^3 = c^{-3}bc^3 \\ a^{-(3+i)}x_iba^{3+i} = c^{-(3+i)}bc^{3+i}, \;\; i \geqslant 1 \end{array} \right\rangle$$

Lemma

1) *If* $w \neq_U 1$ *then* U *embeds in* H_w.
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The isomorphism problem, the triviality problem, the finite problem are all unsolvable.

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The generation problem

Take U with unsolvable WP (in particular $|U| = \infty$), consider the presentations H_w as above, and consider the Mihailova group corresponding to H_w:

$$L_w = M(H_w) = \{(u, v) \in F_{n+3} \times F_{n+3} \mid u =_{H_w} v\} \leqslant F_{n+3} \times F_{n+3}.$$

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