Deciding endo-fixedness

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- $H \leqslant \mathbb{Z}^n$ is 1-auto-fixed $\quad \Leftrightarrow \quad H$ is a direct summand of $\mathbb{Z}^n$,
- All is easy from algorithmic point of view.

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Dually,

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- $H \leqslant F_n$ is *1-endo-fixed* $\Leftrightarrow H = \mathrm{Fix}(\varphi)$ for some $\varphi \in \mathrm{End}(F_n)$,
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Example. (Stallings automorphism) *Let*

$$\begin{array}{rcl} \varphi \colon F_4 & \longrightarrow & F_4 \\ a & \mapsto & dac \\ b & \mapsto & c^{-1}a^{-1}d^{-1}ac \\ c & \mapsto & c^{-1}a^{-1}b^{-1}ac \\ d & \mapsto & c^{-1}a^{-1}bc \end{array}$$

Then $\text{Fix}(\varphi) = \langle w \rangle$, where

$$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdadacdcdbcda^{-1}a^{-1}d^{-1}a^{-1} \\ d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}.$$

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- [Martino-V., 2004] Explicit description of 1-auto-fixed subgroups of F_n.

• [Martino-V.] If $\varphi \in \mathrm{End}(F_3)$ fixes $[a,b] = a^{-1}b^{-1}ab$ and $[a,c] = a^{-1}c^{-1}ac$, then it must also fix a.
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Theorem. (V., 2008) *Given $H \leqslant F_n$, it is algorithmically decidable whether*

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- Clearly, H is auto-fixed $\Leftrightarrow \overline{H} = H$.

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Definition. *The auto-closure and endo-closure of H in F_n are*

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Theorem. (Turner) *Given $H \leqslant F_n$ finitely generated, it is algorithmically decidable whether H is a retract of F_n.*

Technical Lemma. *Let* $H \leqslant F_n$, *and* $\mathcal{AE}_{ret}(H) = \{H_0 = H, \ldots, H_s\}$ *be the set of algebraic extensions of* H *where* H *is a retract. Then,*

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• By comp. of auto-closures, compute *a-Cl* $_{H_0}(H), \ldots,$ *a-Cl* $_{H_s}(H)$,

Technical Lemma. *Let* $H \leqslant F_n$*, and* $\mathcal{AE}_{ret}(H) = \{H_0 = H, \ldots, H_s\}$ *be the set of algebraic extensions of* H *where* H *is a retract. Then,*

$$\textit{e-Cl}_{F_n}(H) = \textit{a-Cl}_{H_0}(H) \cap \cdots \cap \textit{a-Cl}_{H_s}(H)$$

Algorithm for computing the endo-cloure of H**:**

- By Takahasi, compute $\mathcal{AE}(H) = \{H_0 = H, \ldots, H_r\}$,

- By Turner, choose those H_i where H is a retract, $\mathcal{AE}_{ret}(H) = \{H_0 = H, \ldots, H_s\}$, $s \leqslant r$,

- By comp. of auto-closures, compute $\textit{a-Cl}_{H_0}(H), \ldots, \textit{a-Cl}_{H_s}(H)$,

- By Stallings, compute $\textit{a-Cl}_{H_0}(H) \cap \cdots \cap \textit{a-Cl}_{H_s}(H)$,

Technical Lemma. *Let $H \leqslant F_n$, and $\mathcal{AE}_{ret}(H) = \{H_0 = H, \ldots, H_s\}$ be the set of algebraic extensions of H where H is a retract. Then,*

$$e\text{-}Cl_{F_n}(H) = a\text{-}Cl_{H_0}(H) \cap \cdots \cap a\text{-}Cl_{H_s}(H)$$

Algorithm for computing the endo-cloure of H:

- By Takahasi, compute $\mathcal{AE}(H) = \{H_0 = H, \ldots, H_r\}$,

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- By comp. of auto-closures, compute $a\text{-}Cl_{H_0}(H), \ldots, a\text{-}Cl_{H_s}(H)$,

- By Stallings, compute $a\text{-}Cl_{H_0}(H) \cap \cdots \cap a\text{-}Cl_{H_s}(H)$,

- By the Technical Lemma, this equals $e\text{-}Cl_{F_n}(H)$.

THANKS