

Deciding endo-fixedness

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- All is easy from algorithmic point of view.

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- $H \leq F_n$ is *auto-fixed* $\Leftrightarrow H = \text{Fix}(S)$ for some $S \subseteq \text{Aut}(F_n)$,
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Example. (Stallings automorphism) Let

$$\begin{aligned}\varphi: F_4 &\rightarrow F_4 \\ a &\mapsto dac \\ b &\mapsto c^{-1}a^{-1}d^{-1}ac \\ c &\mapsto c^{-1}a^{-1}b^{-1}ac \\ d &\mapsto c^{-1}a^{-1}bc\end{aligned}$$

Then $\text{Fix}(\varphi) = \langle w \rangle$, where

$$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdada cdcbcd a^{-1}a^{-1}d^{-1}a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}.$$

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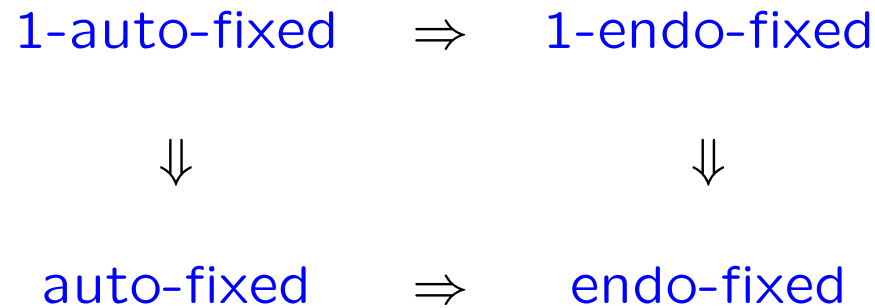
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- [Martino-V., 2004] Explicit description of 1-auto-fixed subgroups of F_n .

- [Martino-V.] If $\varphi \in \text{End}(F_3)$ fixes $[a, b] = a^{-1}b^{-1}ab$ and $[a, c] = a^{-1}c^{-1}ac$, then it must also fix a .
Hence, $H = \langle [a, b], [a, c] \rangle$ is **not endo-fixed**.

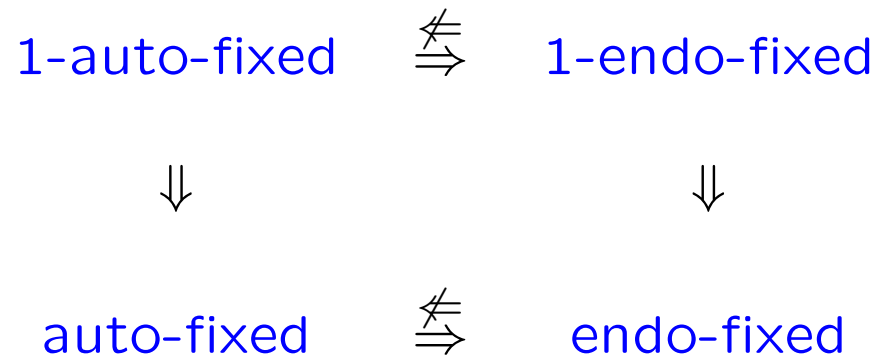
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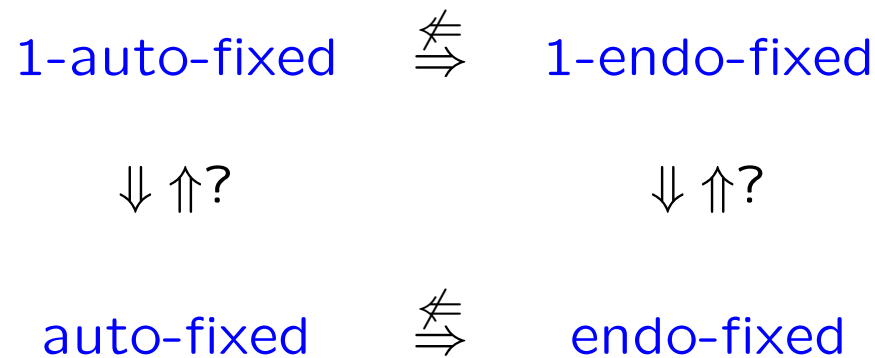
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Theorem. (V., 2008) *Given $H \leq F_n$, it is algorithmically decidable whether*

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and in the affirmative, find $S \subseteq \text{Aut}(F_n)$ or $S \subseteq \text{End}(F_n)$.

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- Clearly, H is auto-fixed $\Leftrightarrow \overline{H} = H$.

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Definition. The *auto-closure* and *endo-closure* of H in F_n are

$$a\text{-Cl}_{F_n}(H) = \text{Fix}(\text{Aut}_H(F_n)) \geq H,$$

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Theorem. (Turner) *Given $H \leq F_n$ finitely generated, it is algorithmically decidable whether H is a retract of F_n .*

Technical Lemma. *Let $H \leq F_n$, and $\mathcal{AE}_{ret}(H) = \{H_0 = H, \dots, H_s\}$ be the set of algebraic extensions of H where H is a retract. Then,*

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- By Turner, choose those H_i where H is a retract, $\mathcal{AE}_{ret}(H) = \{H_0 = H, \dots, H_s\}$, $s \leq r$,
- By comp. of auto-closures, compute $a-Cl_{H_0}(H), \dots, a-Cl_{H_s}(H)$,
- By Stallings, compute $a-Cl_{H_0}(H) \cap \dots \cap a-Cl_{H_s}(H)$,

Technical Lemma. *Let $H \leq F_n$, and $\mathcal{AE}_{ret}(H) = \{H_0 = H, \dots, H_s\}$ be the set of algebraic extensions of H where H is a retract. Then,*

$$e-Cl_{F_n}(H) = a-Cl_{H_0}(H) \cap \dots \cap a-Cl_{H_s}(H)$$

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- By Stallings, compute $a-Cl_{H_0}(H) \cap \dots \cap a-Cl_{H_s}(H)$,
- By the Technical Lemma, this equals $e-Cl_{F_n}(H)$.

THANKS