Algebraic extensions

in free groups

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Everybody knows the

Steinitz Lemma: In a vector space K^n, every set of L.I. vectors can be extended to a basis of K^n. Or, equivalently, every subspace $U \leqslant K^n$ is a direct summand, $K^n = U \oplus V$ for some complement $V \leqslant K^n$.

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In free abelian groups, the same is almost true:

Proposition: Let $U \leqslant \mathbb{Z}^n$. If U is pure $(\lambda u \in U \Rightarrow u \in U)$ then $\mathbb{Z}^n = U \oplus V$, for some complement $V \leqslant \mathbb{Z}^n$.
More precisely: for every $U \leqslant \mathbb{Z}^n$ there exists $U' \leqslant \mathbb{Z}^n$ such that $U \leqslant_{\mathsf{fi}} U'$ and $\mathbb{Z}^n = U' \oplus V$ for some $V \leqslant \mathbb{Z}^n$.

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$$\begin{array}{rcl} \langle (2,0,4),\ (3,3,6)\rangle & \leqslant_6 & \langle (1,0,2),\ (1,1,2)\rangle \\ & \leqslant & \langle (1,0,2),\ (1,1,2)\rangle \oplus \langle (0,0,1)\rangle \\ & = & \mathbb{Z}^3. \end{array}$$

Takahasi's Theorem: Let $F(A)$ be the free group on A and $H \leq F(A)$ a f.g. subgroup. Then, there exists a finite computable collection of f.g. extensions of H, say $H \leqslant H_1, \ldots, H_n \leqslant F(A)$, s.t. every $H \leqslant K \leqslant F(A)$ is a free multiple of one of the H_i's,

$$H \leqslant H_i \leqslant_{\mathsf{ff}} H_i * L = K \leqslant F(A).$$

The set $\mathcal{O}(H) = \{H_1, \ldots, H_n\}$ is called the fringe or the set of overgroups of H.

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Example: The fringe of $H = \langle ab, acba \rangle$ as a subgroup of $F = \langle a, b, c \rangle$ is

$$\mathcal{O}(H) = \left\{ \begin{array}{ll} H_1 = \langle ab, acba \rangle, & H_2 = \langle ab, ac, ba \rangle, \\ H_3 = \langle ab, a^2, acba \rangle, & H_4 = \langle ab, aca, acba \rangle, \\ H_5 = \langle ab, a^2, ab^{-1}, ac \rangle, & H_6 = \langle a, b, c \rangle. \end{array} \right\}$$

Definition: Take $H \leqslant K \leqslant F(A)$ and $x \in K$. We say that x is K-algebraic over H if every free factor of K containing H, $H \leqslant L \leqslant_{\mathsf{ff}} K$, satisfies $x \in L$. Otherwise, we say that x is K-transcendental over H.

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• The notion of algebraicity over H is relative to K:

$$\langle a^2b^2 \rangle \leqslant \langle a^2, b^2 \rangle \leqslant \langle a, b \rangle$$

→ in $\langle a^2, b^2 \rangle$, a^2 is transcendental over $\langle a^2b^2 \rangle$ (in fact, primitive),

→ in $\langle a, b \rangle$, a^2 is algebraic over $\langle a^2b^2 \rangle$.

Observation: Let $H \leqslant K \leqslant F(A)$. TFAE:

(a) H is contained in no proper free factor of K;

(b) every $x \in K$ is K-algebraic over H;

(c) $K = \langle H, k_1, \ldots, k_t \rangle$ for some k_i's K-algebraic over H.

In this case, we say that the extension $H \leqslant K$ is algebraic, denoted $H \leqslant_{\mathsf{alg}} K$.

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Theorem: For every extension $H \leqslant K$ of f.g. subgroups of $F(A)$ there exists a unique L such that $H \leqslant_{\mathsf{alg}} L \leqslant_{\mathsf{ff}} K$. This is called the algebraic K-closure of H.

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Theorem: Let $H \leqslant_{\mathsf{fg}} F(A)$. Then,

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(b) $\mathcal{AE}(H)$ is finite (i.e., H admits only finitely many algebraic extensions);

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Proof: (a) Let $H \leqslant_{\mathsf{alg}} K \leqslant F(A)$. Then, $H \leqslant H_i \leqslant_{\mathsf{ff}} K$ for some $H_i \in \mathcal{O}(H)$. But K is algebraic over H so, $H_i = K$.

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(b) Immediate.

(c) Given H, $\rightarrow$ compute $\Gamma(H)$,
$\rightarrow$ compute all quotients, $\mathcal{O}(H) = \{H_1, \ldots, H_n\}$,
$\rightarrow$ clean H_i if $H \leqslant_{\text{ff}} H_i$ (using Roig-V.-Weil in polynomial time).

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Theorem (Ribes-Zalesskii, Margolis-Sapir-Weil) For every prime p, there is an algorithm to compute the pro-p closure of f.g. subgroups of $F(A)$.

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So, there are uncountably many $\mathcal{V}$'s which are indistinguishable by means of closures of f.g. subgroups (!)

Thank you