On the difficulty of inverting automorphisms of free groups

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Groups in Galway, 2011

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Outline

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Motivation

(Joint work with P. Silva and M. Ladra.)

Find a group G where $\cdot$ is "easy" but $(\,)^{-1}$ is "difficult".

Natural candidate: $\mathrm{Aut}\,(F_n)$, where $F_r = \langle a_1, \ldots, a_r \mid \rangle$.

$F_3 = \langle a, b, c \mid \rangle$.

$$\begin{array}{rcl} \phi\colon F_3 & \to & F_3 \\ a & \mapsto & ab \\ b & \mapsto & ab^2c \\ c & \mapsto & bc^2 \end{array} \qquad \begin{array}{rcl} \psi\colon F_3 & \to & F_3 \\ a & \mapsto & bc^{-1} \\ b & \mapsto & a^{-1}bc \\ c & \mapsto & c^{-1}. \end{array}$$

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$F_5 = \langle a, b, c, d, e \mid \rangle.$

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- We have formalized the situation.
- We have seen that inverting in $\mathrm{Aut}(F_r)$ is not that bad.
- We now want to look for worse groups G.

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Main definition

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Let $A = \{a_1, \dots, a_r\}$ be a finite alphabet, and $G = \langle A \mid R \rangle$ be a finite presentation for a group G. We have the word metric:

$$\text{for } g \in G, \quad |g| = \min\{n \mid g = a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n}\}.$$

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For $\theta \in Aut(G)$, note θ is determined by $a_1\theta, \dots, a_r\theta$ and define

$$||\theta||_1 = |a_1\theta| + \cdots + |a_r\theta|,$$

$$||\theta||_\infty = \max\{|a_1\theta|, \dots, |a_r\theta|\}.$$

Observation

For every $\theta \in Aut(F_r)$, $||\theta||_\infty \leqslant ||\theta||_1 \leqslant r||\theta||_\infty$

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$$\alpha_A(n) = \max\{||\theta^{-1}||_1 \mid \theta \in Aut(G),\ ||\theta||_1 \leqslant n\}.$$

Clearly, $\alpha_A(n) \leqslant \alpha_A(n+1)$.

The bigger is α_A, the more "difficult" will be to invert automorphisms of G (with respect to the given set of generators A).

Question

Determine the asymptotic growth of the function α_A.

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Free group case

For the rest of the talk, $G = F_r = \langle a_1, \ldots, a_r \mid \rangle$.

Definition

Every $w \in F_r$ has its length, $|w|$, and its cyclic length, $\cdot|w|\cdot$:

$|a_1 a_1^{-1} a_2| = |a_2| = \cdot|a_2|\cdot = 1,$

$|a_1 a_2 a_1^{-2}| = 4,$

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Observation

i) $|w^n| \leqslant |n||w|$ and $\cdot|w^n|\cdot = |n|\cdot|w|\cdot$;

ii) $|vw| \leqslant |v| + |w|$, but $\cdot|vw|\cdot \leqslant \cdot|v|\cdot + \cdot|w|\cdot$ is not true in general.

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$|\cdot\theta\cdot|_1 \leqslant |||\theta|||_1 \leqslant ||\theta||_1$, *but not equal in general.*

Example

Consider $\theta\colon F_4 \to F_4$, $a \mapsto a$, $b \mapsto a^{-1}ba$, $c \mapsto a^{-1}ca$, $d \mapsto d$. *We have* $|\cdot\theta\cdot|_1 = 4$, $|||\theta|||_1 = 6$ *and* $||\theta||_1 = 8$.

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Are these functions equal up to multiplicative constants ?

α_r *and* γ_r *are not;*
β_r *is not clear.*

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Main results

Theorem

For rank $r = 2$ we have

(i) *for* $n \geqslant 4$, $\alpha_2(n) \leqslant \frac{(n-1)^2}{2}$,

(ii) *for* $n \geqslant n_0$, $\alpha_2(n) \geqslant \frac{n^2}{16}$,

(iii) *for* $n \geqslant 1$, $\beta_2(n) = n$,

(iv) *for* $n \geqslant 1$, $\gamma_2(n) = n$.

Theorem

For $r \geqslant 3$ *there exist* $K = K(r)$ *and* $M = M(r)$ *such that, for* $n \geqslant 1$,

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For $r \geqslant 3$ *there exist* $K = K(r)$ *and* $M = M(r)$ *such that, for* $n \geqslant 1$,

(i) $\alpha_r(n) \geqslant Kn^r$,

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Main results

Theorem

For rank $r = 2$ we have

(i) *for* $n \geqslant 4$, $\alpha_2(n) \leqslant \frac{(n-1)^2}{2}$,

(ii) *for* $n \geqslant n_0$, $\alpha_2(n) \geqslant \frac{n^2}{16}$,

(iii) *for* $n \geqslant 1$, $\beta_2(n) = n$,

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Outline

A lower bound for γ_r

Theorem

For $r \geqslant 2$, and $n \geqslant n_0$, we have $\gamma_r(n) \geqslant \frac{1}{2r^{r-1}} n^{r-1}$.

Proof: For $r \geqslant 2$ and $n \geqslant 1$, consider

$$\begin{array}{rcl} \psi_{r,n}: F_r & \to & F_r \\ a_1 & \mapsto & a_1 \\ a_2 & \mapsto & a_1^n a_2 \\ a_3 & \mapsto & a_2^n a_3 \\ & \vdots & \\ a_r & \mapsto & a_{r-1}^n a_r \end{array} \qquad \begin{array}{rcl} \psi_{r,n}^{-1}: F_r & \to & F_r \\ a_1 & \mapsto & a_1 \\ a_2 & \mapsto & a_1^{-n} a_2 \\ & \vdots & \\ a_i & \mapsto & (a_{i-1}^{-n})\psi_{r,n}^{-1} \cdot a_i \\ & (2 \leqslant i \leqslant r) & \end{array}$$

A straightforward calculation shows that
$|\!|\psi_{r,n}|\!|_1 = ||\psi_{r,n}||_1 = (r-1)n + r$, and
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Hence, for $n \geqslant r$,

$$\gamma_r(rn) \geqslant \gamma_r((r-1)n+r) \geqslant n^{r-1}.$$

Now, for n big enough, take the closest multiple of r below,

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Outline

Outer space

To prove the upper bound

(ii) $\beta_r(n) \leqslant Kn^M$,

we'll need to use the recently discovered metric in the outer space $\mathcal{X}_r$.

Definition

- By *graf* Γ we mean a finite, connected graph of rank r, with no vertices of degree 1 or 2.
- A *metric* on Γ is a map $\ell \colon E\Gamma \to [0,1]$ such that $\sum_{e \in E\Gamma} \ell(e) = 1$, and $\{e \in E\Gamma \mid \ell(e) = 0\}$ is a forest.
- For a graph Γ, $\Sigma_\Gamma = \{\text{metrics on } \Gamma\}$ = a simplex with missing faces.
- If $\Gamma' = \Gamma/\text{forest}$, then we identify points in $\Sigma_{\Gamma'}$ with the corresponding points in Σ_Γ by assigning length 0 to the collapsed edges.
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The outer space $\mathcal{X}_r$ is

$$\mathcal{X}_r = \{\, (\Gamma, f, \ell)\,\}/\sim$$

(where $\sim$ is an equivalence relation).

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There is a natural action of $Aut(F_r)$ on $\mathcal{X}_r$, given by

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Metric on $\mathcal{X}_r$

Definition

Let $x, x' \in \mathcal{X}_r$, $x = (\Gamma, f, \ell)$, $x' = (\Gamma', f', \ell')$. A difference of markings is a map $\alpha\colon \Gamma \to \Gamma'$, which is linear over edges and $f\alpha \simeq f'$.

For such an α, define $\sigma(\alpha)$ to be its maximum slope over edges.

Definition

$\mathcal{X}_r$ admits the following "metric":

$$d(x, x') = \min\{\log(\sigma(\alpha)) \mid \alpha \text{ diff. markings}\}.$$

This minimum is achieved by Arzela-Ascoli's theorem.

This is Bestvina-AlgomKfir version of Martino-Francaviglia's original metric.

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Proposition

(i) $d(x, y) \geqslant 0$, *and* $= 0 \Leftrightarrow x = y$.

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For $\epsilon > 0$, *the* ϵ*-thick part of* $\mathcal{X}_r$ *is*

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Bestvina-AlgomKfir theorem

Theorem (Bestvina-AlgomKfir)

For any $\epsilon > 0$ there is constant $M = M(r, \epsilon)$ such that for all $x, y \in \mathcal{X}_r(\epsilon)$,

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Corollary

For $r \geqslant 2$, there exists $M = M(r)$ such that

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Remind $\beta_r(n) = \max\{|||\theta^{-1}|||_1 \mid \theta \in Aut\, F_r,\ |||\theta|||_1 \leqslant n\}$.

Proof. Given $\phi \in \mathrm{Aut}\,(F_r)$, consider $x = (R_r, id, \ell_0) \in \mathcal{X}_r$, and $\phi \cdot x = (R_r, \phi, \ell_0) \in \mathcal{X}_r$, where ℓ_0 is the uniform metric.

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Remind $\beta_r(n) = \max\{|||\theta^{-1}|||_1 \mid \theta \in Aut\, F_r,\ |||\theta|||_1 \leqslant n\}$.

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Outline

The rank 2 case

These functions for $\mathrm{Aut}\,(F_2)$ are much easier to understand due to the following technical lemmas.

Lemma

Let $\varphi \in Aut(F_2)$ be positive. Then φ^{-1} is cyclically reduced and $||\varphi^{-1}||_1 = ||\varphi||_1$.

Lemma

For every $\theta \in Aut(F_2)$, there exist two letter permuting autos $\psi_1, \psi_2 \in Aut(F_2)$, a positive one $\varphi \in Aut^+(F_2)$, and an element $g \in F_2$, such that $\theta = \psi_1 \varphi \psi_2 \lambda_g$ and $||\varphi||_1 + 2|g| \leqslant ||\theta||_1$.

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The rank 2 case: γ_2

Theorem

For every $\theta \in Aut(F_2)$, $|\!\cdot\!|\theta^{-1}|\!\cdot\!|_1 = |\!\cdot\!|\theta|\!\cdot\!|_1$. Hence, $\gamma_2(n) = n$.

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For $n \geqslant 4$ we have $\alpha_2(n) \leqslant \frac{(n-1)^2}{2}$.

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For $n \geqslant n_0$ we have $\alpha_2(n) \geqslant \frac{n^2}{16}$.

So, the global known picture is

(i) $\frac{n^2}{16} \leqslant \alpha_2(n) \leqslant \frac{(n-1)^2}{2}$,
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(iv) $Kn^r \leqslant \alpha_r(n)$,
(v) $\beta_r(n) \leqslant Kn^M$,
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for some constants $K = K(r)$, $M = M(r)$, and for $n \geqslant n_0$.

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THANKS