Fixed subgroups are compressed in surface groups

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(Joint work with Q. Zhang and J. Wu.)

Most of this talk is contained in the paper:

Q. Zhang, E. Ventura, J. Wu,
"Fixed subgroups are compressed in surface groups", *International Journal of Algebra and Computation* **25** (5) (2015), 865-887.

Outline

1. Fixed subgroups in free groups (history)
2. New results in free groups
3. Fixed subgroups in surface groups (history)
4. New results in surface groups
5. New results in direct products of free and surface groups

Notation

- Let G be a finitely presented group.
- $\mathrm{Aut}(G) \subseteq \mathrm{Mono}(G) \subseteq \mathrm{End}(G)$.
- I let endomorphisms $\phi\colon G \to G$ act on the left, $x \mapsto \phi(x)$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid \phi(x) = x\} \leqslant G$.
- If $\mathcal{B} \subseteq \mathrm{End}(G)$ then
  $\mathrm{Fix}(\mathcal{B}) = \{x \in G \mid \beta(x) = x \ \forall \beta \in \mathcal{B}\} = \cap_{\beta \in \mathcal{B}} \mathrm{Fix}(\beta) \leqslant G$.
- For $\mathcal{B} \subseteq \mathrm{Hom}(G, H)$,
  $\mathrm{Eq}(\mathcal{B}) = \{x \in G \mid \beta_1(x) = \beta_2(x) \ \forall \beta_1, \beta_2 \in \mathcal{B}\}$.
- Note that if $G \leqslant H$ and $\iota \in \mathcal{B}$ then $\mathrm{Eq}(\mathcal{B}) = \mathrm{Fix}(\mathcal{B})$.

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What is known about free groups ?

Theorem (Dyer–Scott, 75)

Let $\mathcal{B} \leqslant Aut(F_n)$ be a finite group of automorphisms of F_n. Then, $Fix(\mathcal{B}) \leqslant_{\mathrm{ff}} F_n$; in particular, $r(Fix(\mathcal{B})) \leqslant n$.

Conjecture (Scott)

For every $\phi \in Aut(F_n)$, $r(Fix(\phi)) \leqslant n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) < \infty$.

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Train-tracks

Main result in this story:

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Let $\phi \in Aut(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

introducing the theory of train-tracks for graphs.

After Bestvina–Handel, live continues ...

Theorem (Imrich–Turner, 89)

Let $\phi \in End(F_n)$. Then $r(Fix(\phi)) \leqslant n$.

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Inertia

Definition

A subgroup $H \leqslant G$ is called

- *inert in G if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant G$;*
- *compressed in G if $r(H) \leqslant r(K)$ for every $H \leqslant K \leqslant G$;*

- *Free factors and cyclic subgroups of F_n are inert in F_n;*
- *intersections of inert subgroups are inert;*
- *subgroups of rank 1 and 2 in F_n are inert in F_n;*
- *$A \leqslant B \leqslant C$; if A is inert in B, and B is inert in C then A is inert in C.*
- *$H \leqslant G$ inert $\Rightarrow H \leqslant G$ compressed $\Rightarrow r(H) \leqslant r(G)$;*
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Let $\mathcal{B} \subseteq Mon(F_n)$ be an arbitrary set of monomorphisms of F_n. Then, $Fix(\mathcal{B})$ is inert in F_n; in particular, $r(Fix(\mathcal{B})) \leqslant n$.

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The inertia conjecture

Inertia Conjecture (Dicks–V.)

For every $\mathcal{B} \subseteq End(F_n)$, $Fix(\mathcal{B})$ is inert in F_n.

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Let $\phi\colon G \twoheadrightarrow H$ be an epimorphism of free groups, with H f.g. Then, the equalizer of any family of sections of ϕ is a free factor of H.

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Main result for free groups

Theorem (Zhang–Wu–V., 15)

Let F be a f.g. free group, let $\mathcal{B} \subseteq End(F)$, and let $\beta_0 \in \langle \mathcal{B} \rangle \leqslant End(F)$ be with $r(\beta_0(F))$ minimal. Then, for every subgroup $K \leqslant F$ such that $\beta_0(K) \cap Fix\,\mathcal{B} \leqslant K$, we have $r(K \cap Fix\,\mathcal{B}) \leqslant r(K)$.

(Proof)

- Since, $Fix\,\alpha \cap Fix\,\beta \leqslant Fix(\alpha\beta)$, we have $Fix\,\langle \mathcal{B} \rangle = Fix\,\mathcal{B}$ and so, we can assume that $Id \in \langle \mathcal{B} \rangle = \mathcal{B}$.
- Now choose $\beta_0 \in \mathcal{B}$ with $r(\beta_0(F)) = \min\{r(\gamma(F)) \mid \gamma \in \mathcal{B}\}$. Thus, all elements of $\mathcal{B}$ act injectively on $\beta_0(F)$.
- Restricting $\beta_0\mathcal{B} = \{\beta_0\gamma \mid \gamma \in \mathcal{B}\} \subseteq \mathcal{B}$ to $\beta_0(F)$ we get the family of injective endos: $\beta_0\gamma|_{\beta_0(F)} \colon \beta_0(F) \to \beta_0(F)$, for $\gamma \in \mathcal{B}$.
- Hence, $Fix(\beta_0\mathcal{B}) = Fix(\beta_0\mathcal{B}|_{\beta_0(F)})$ is inert in $\beta_0(F)$ that is, for every $L \leqslant \beta_0(F)$, we have $r(L \cap Fix(\beta_0\mathcal{B})) \leqslant r(L)$.

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Let F be a f.g. free group, let $\mathcal{B} \subseteq End(F)$, and let $\beta_0 \in \langle \mathcal{B} \rangle \leqslant End(F)$ be with $r(\beta_0(F))$ minimal. Then, for every subgroup $K \leqslant F$ such that $\beta_0(K) \cap Fix\,\mathcal{B} \leqslant K$, we have $r(K \cap Fix\,\mathcal{B}) \leqslant r(K)$.

(Proof)

- *Since, $Fix\,\alpha \cap Fix\,\beta \leqslant Fix\,(\alpha\beta)$, we have $Fix\,\langle \mathcal{B} \rangle = Fix\,\mathcal{B}$ and so, we can assume that $Id \in \langle \mathcal{B} \rangle = \mathcal{B}$.*
- *Now choose $\beta_0 \in \mathcal{B}$ with $r(\beta_0(F)) = \min\{r(\gamma(F)) \mid \gamma \in \mathcal{B}\}$. Thus, all elements of $\mathcal{B}$ act injectively on $\beta_0(F)$.*
- *Restricting $\beta_0\mathcal{B} = \{\beta_0\gamma \mid \gamma \in \mathcal{B}\} \subseteq \mathcal{B}$ to $\beta_0(F)$ we get the family of injective endos: $\beta_0\gamma|_{\beta_0(F)} : \beta_0(F) \to \beta_0(F)$, for $\gamma \in \mathcal{B}$.*
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As a first corollary, we obtain

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Let F be a f.g. free group, let $\mathcal{B} \subseteq End(F)$, and let $\beta_0 \in \langle \mathcal{B} \rangle \leqslant End(F)$ be with $r(\beta_0(F))$ minimal. Then, $Fix\,\mathcal{B}$ is inert in $\beta_0(F)$. Moreover, if $\beta_0(F)$ is inert in F then $Fix\,\mathcal{B}$ is inert in F as well.

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Let F be a f.g. free group, and $\mathcal{B} \subseteq End(F)$. If some composition of endos from $\mathcal{B}$ has image of rank 1 or 2, then $Fix\,\mathcal{B}$ is inert in F.

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Theorem (Martino–V., 04)

Let $\mathcal{B} \subseteq End(F_n)$ be an arbitrary set of endomorphisms of F_n. Then, $Fix(\mathcal{B})$ is compressed in F_n.

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Clearly, $Fix\,\mathcal{B} \leqslant K \leqslant F \Rightarrow \beta_0(K) \cap Fix\,\mathcal{B} \leqslant K$. So, main theorem applies to those K, and $r(Fix\,\mathcal{B}) = r(K \cap Fix\,\mathcal{B}) \leqslant r(K)$.

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Outline

Notation

- Σ_g denotes the orientable surface of genus g, $g \geqslant 0$;
- $S_g = \pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$;
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- $N\Sigma_k$ denotes the connected sum of k projective planes, $k \geqslant 1$;
- $NS_k = \pi_1(N\Sigma_k) = \langle a_1, \ldots, a_k \mid a_1^2 \cdots a_k^2 \rangle$;
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- Euler characteristic: $\chi(\Sigma_g) = 2 - 2g$, $\chi(N\Sigma_k) = 2 - k$;
- Euclidean type ($\chi \geqslant 0$): S_0, S_1, NS_1, NS_2, (and $F_1 = \mathbb{Z}$);
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Main result for surface groups

The proof of main Theorem for free groups works for surface groups of negative Euler characteristic as well. For non-negative Euler characteristic one can prove the inertia conjecture directly.

Proposition

Let G be either $F_0 = S_0 = 1$, or $S_1 = \mathbb{Z}^2$, or $NS_1 = \mathbb{Z}/2\mathbb{Z}$, or NS_2, and let $\mathcal{B} \subseteq End(G)$. Then, $Fix\,\mathcal{B}$ is inert in G.

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Let G be a surface group, let $\mathcal{B} \subseteq End(G)$, and let $\beta_0 \in \langle \mathcal{B} \rangle \leqslant End(G)$ be with $r(\beta_0(G))$ minimal. Then, for every subgroup $K \leqslant G$ such that $\beta_0(K) \cap Fix\,\mathcal{B} \leqslant K$, we have $r(K \cap Fix\,\mathcal{B}) \leqslant r(K)$.

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Let G be Euclidean, $G = NS_2^\ell \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$, for $\ell, p, q \geqslant 0$. Then any subgroup $H \leqslant G$ satisfies $r(H) \leqslant r(G) = 2\ell + p + q$. In particular, $r(Fix(\phi)) \leqslant r(G)$ for every $\phi \in End(G)$.

Initial properties

Lemma

Let G be a product group. Then, $Z(G) = 1 \Leftrightarrow G$ is of hyperbolic type.

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For $g_i \in G_i$, $Cen_G(g_1, \ldots, g_n) \simeq Cen_{G_1}(g_1) \times \cdots \times Cen_{G_n}(g_n)$.
So, in the hyperbolic case, $Cen_G(g_1, \ldots, g_n) \simeq \widehat{G_1} \times \cdots \times \widehat{G_n}$, where $\widehat{G_i} = G_i$ if $g_i = 1$, or $\widehat{G_i} = \mathbb{Z}$ if $g_i \neq 1$.

In general $\mathbb{Z} \times A \simeq \mathbb{Z} \times B \not\Rightarrow A \simeq B$, but...

Proposition

Let $G = G_1 \times \cdots \times G_n$ and $H = H_1 \times \cdots \times H_m$ be two product groups of hyperbolic type. Then, $G \simeq H \Leftrightarrow n = m$ and $G_i \simeq H_i$ up to reordering.
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Theorem (Zhang–Wu–V., 15)

Let $G = G_1 \times \cdots \times G_n$ be a product group. Then, $r(Fix\,\phi) \leqslant r(G)$ $\forall \phi \in Aut(G) \Leftrightarrow G$ is either of Euclidean or of hyperbolic type.

(Proof)

- *Step 1: If G Euclidean then ok. Done.*
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Let $G = G_1^{n_1} \times \cdots \times G_m^{n_m}$ be a product group in block notation. If G is of hyperbolic type then, $\forall \phi \in Aut(G)$, $\exists \phi_{i,j} \in Aut(G_i)$ and $\sigma_i \in S_{n_i}$, such that

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• *Since $Z(G_1)$ is $Z(F_1)=\mathbb{Z}$, or $Z(S_1)=\mathbb{Z}^2$, or $Z(NS_1)=\mathbb{Z}/2\mathbb{Z}$, or $Z(NS_2)=\mathbb{Z}$, we deduce $o(t)=2,\infty$.*

• *Let us distinguish the 3 cases: $G_2=F_r$, $G=S_g$, or $G=NS_k$.*

→ *Case 1:* $G_2=F_r=\langle a_1,\ldots,a_r \mid \rangle$, $r\geqslant 2$.

• *Consider $\phi\in Aut(G)$ fixing G_1 pointwise and mapping $a_1\mapsto ta_1,\ a_2\mapsto a_2,\ \ldots,\ a_r\mapsto a_r$. This is well defined because t commutes with all of G_1.*

• *Now, ϕ maps $w(a_1,\ldots,a_r)\mapsto w(ta_1,a_2,\ldots,a_r)=t^{|w|_1}w(a_1,\ldots,a_r)$, where $|w|_1\in\mathbb{Z}$ is the total a_1-exponent of $w\in G_2$.*

• *Hence, $Fix\,\phi=G_1\times\{w\in G_2 \mid |w|_1\equiv 0\}=G_1\times\ker\pi$, where $\pi\colon G_2\twoheadrightarrow\mathbb{Z}/o(t)\mathbb{Z}$, $w\mapsto|w|_1$, and $\equiv$ means equality of integers modulo $o(t)$.*

• *But $\ker\pi$ is a normal subgroup of $G_2=F_r$ of either infinite index (and so, infinitely generated) or of index 2 (and so, $r(\ker\pi)=1+2(r-1)=2r-1$).*

• *In both cases, $r(\ker\pi)>r=r(G_2)$ and so, $r(Fix\,\phi)=r(G_1)+r(\ker\pi)>r(G)$.*

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• *Since* $Z(G_1)$ *is* $Z(F_1) = \mathbb{Z}$, *or* $Z(S_1) = \mathbb{Z}^2$, *or* $Z(NS_1) = \mathbb{Z}/2\mathbb{Z}$, *or* $Z(NS_2) = \mathbb{Z}$, *we deduce* $o(t) = 2, \infty$.

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Main result for product groups

- *Since $Z(G_1)$ is $Z(F_1) = \mathbb{Z}$, or $Z(S_1) = \mathbb{Z}^2$, or $Z(NS_1) = \mathbb{Z}/2\mathbb{Z}$, or $Z(NS_2) = \mathbb{Z}$, we deduce $o(t) = 2, \infty$.*
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Main result for product groups

→ *Case 3:* $G_2 = NS_k = \langle a_1, a_2, \ldots, a_k \mid a_1^2 \cdots a_k^2 \rangle$, $k \geqslant 3$.

- *Consider $\phi \in Aut(G)$ fixing G_1 pointwise and mapping $a_1 \mapsto ta_1,\ a_2 \mapsto t^{-1}a_2,\ a_3 \mapsto a_3,\ \ldots,\ a_k \mapsto a_k$. It is well defined because t commutes with a_1, a_2 and all of G_1.*
- *Observe now that, due to the form of the def. rel. in G_2, the "total a_i-exponent" of an element of $w \in G$ is not well defined; however, the difference of two of them, say $|w|_1 - |w|_2 \in \mathbb{Z}$, it really is.*
- *Hence, the projection $\pi : G_2 \twoheadrightarrow \mathbb{Z}/o(t)\mathbb{Z}$, $w \mapsto |w|_1 - |w|_2$ is well defined, ϕ maps $w(a_1, \ldots, a_k)$ to $w(ta_1, t^{-1}a_2, a_3, \ldots, a_k) = t^{|w|_1 - |w|_2} w(a_1, \ldots, a_k)$, and we proceed and conclude as in case 2.*

Theorem (Zhang–Wu–V., 15)

Let $G = G_1 \times \cdots \times G_n$ be a product group. Then, $r(Fix\,\phi) \leqslant r(G)$ $\forall \phi \in Aut(G) \Leftrightarrow G$ is either of Euclidean or of hyperbolic type.

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Characterizing compression

It is natural to ask for similar characterizations of full compression and full inertia.

Theorem (Zhang–Wu–V., 15)

Let $G = G_1 \times \cdots \times G_n$ be a product group. If $\mathrm{Fix}\,\phi$ is compressed in G for every $\phi \in Aut(G)$, then G must be of one of the following forms:

(euc1) $G = \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $p, q \geqslant 0$; or

(euc2) $G = NS_2 \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $q \geqslant 0$; or

(euc3) $G = NS_2 \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})$ for some $p \geqslant 1$; or

(euc4) $G = NS_2^\ell \times \mathbb{Z}^p$ for some $\ell \geqslant 1$, $p \geqslant 0$; or

(hyp1) $G = F_r \times NS_3^\ell$ for some $r \geqslant 2$, $\ell \geqslant 0$; or

(hyp2) $G = S_g \times NS_3^\ell$ for some $g \geqslant 2$, $\ell \geqslant 0$; or

(hyp3) $G = NS_k \times NS_3^\ell$ for some $k \geqslant 3$, $\ell \geqslant 0$.

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Characterizing inertia

Theorem (Zhang–Wu–V., 15)

Let $G = G_1 \times \cdots \times G_n$ be a product group. If $\mathrm{Fix}\,\phi$ is inert in G for every $\phi \in Aut(G)$, then G is of one of the forms: (euc1), or (euc2), or (euc3), or (euc4), or

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Conjecture (Zhang–Wu–V., 15)

Let $G = G_1 \times \cdots \times G_n$ be a product group. Then, the following are equivalent:

(a) *every $\phi \in End(G)$ satisfies that $\mathrm{Fix}\,\phi$ is inert in G,*

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THANKS