# Finite automata for Schreier graphs of virtually free groups

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(Joint work with P. Silva, X. Soler-Escrivà)

# Outline

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# Stallings automata

## Definition

*A Stallings automata is a finite A-labeled oriented graph with a distinguished vertex, $(X, v)$, such that:*

1- *$X$ is connected,*

2- *no vertex of degree 1 except possibly $v$ ($X$ is a core-graph),*

3- *no two edges with the same label go out of (or in to) the same vertex.*

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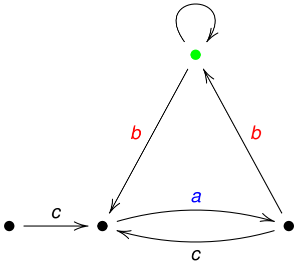


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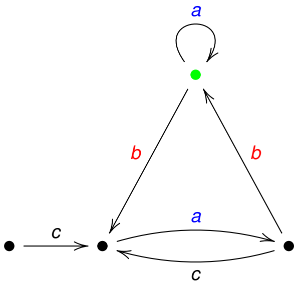


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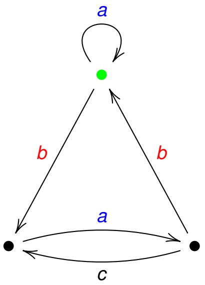

# Stallings automata

In the influent paper

J. R. Stallings, Topology of finite graphs, Inventiones Math. 71 (1983), 551-565,

Stallings (building on previous works) gave a bijection between finitely generated subgroups of $F_A$ and Stallings automata:

$$\{\text{f.g. subgroups of } F_A\} \quad \longleftrightarrow \quad \{\text{Stallings automata}\},$$

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# Reading the subgroup from the automata

## Definition

*To any given (Stallings) automaton $(X, v)$, we associate its fundamental group:*

$$\pi(X, v) = \{ \text{labels of closed paths at } v\} \leqslant F_A,$$

*clearly, a subgroup of $F_A$.*

$X=$ 


$\pi(X, \bullet) = \{1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \ldots\}$

$\pi(X, \bullet) \not\ni bc^{-1}bcaa$

Membership problem in $\pi(X, \bullet)$ is solvable.

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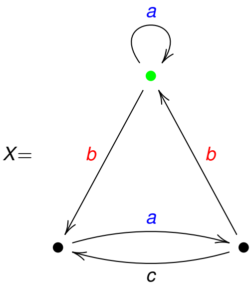


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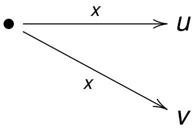


we can fold and identify vertices $u$ and $v$ to obtain

$$\bullet \xrightarrow{\ x\ } u = v \ .$$

This operation, $(X, v) \rightsquigarrow (X', v)$, is called a Stallings folding.

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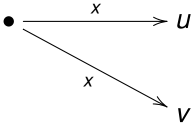


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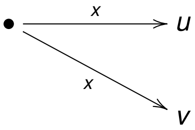


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**Lemma (Stallings)**

*If $(X, v) \rightsquigarrow (X', v')$ is a Stallings folding then $\pi(X, v) = \pi(X', v')$.*

*Given a f.g. subgroup $H = \langle w_1, \ldots, w_m \rangle \leqslant F_A$ (we assume $w_i$ are reduced words), do the following:*

1- *Draw the flower automaton,*

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Well defined?

Need to see that the output does not depend on the process...

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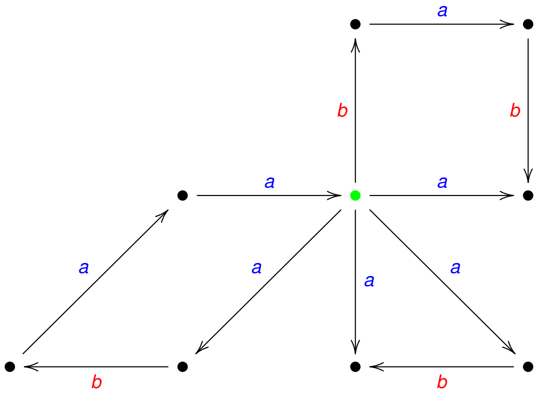


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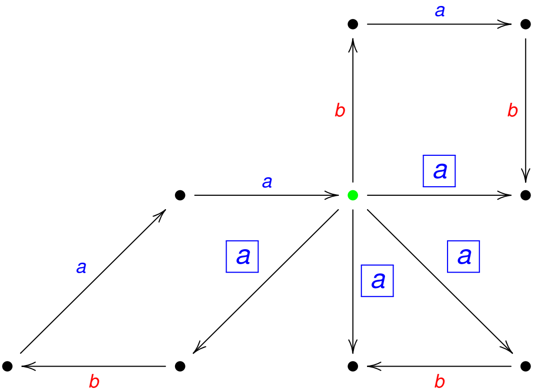


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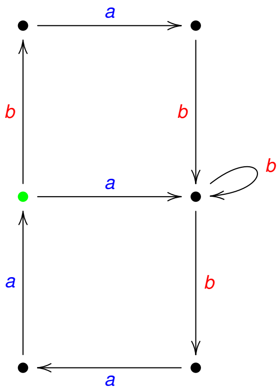


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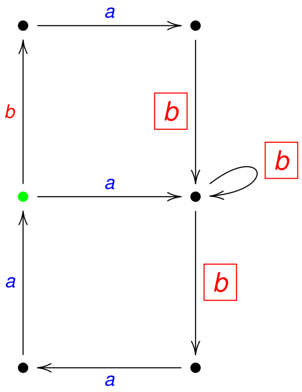


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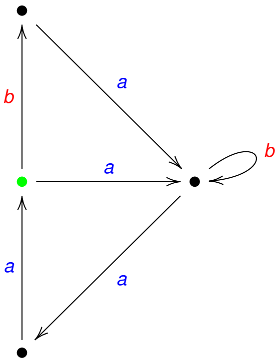


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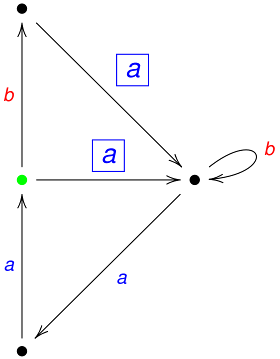


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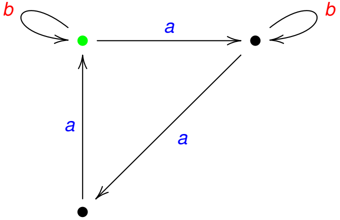


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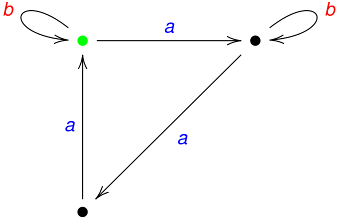


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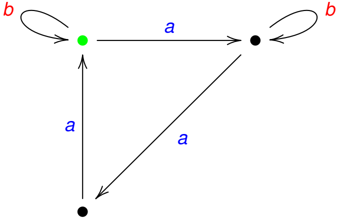


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## Theorem

*The following is a bijection between f.g subgroups and Stallings automata:*

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**Corollary (Nielsen-Schreier)**

*Every subgroup of $F_A$ is free.*

- We have proved the finitely generated case, but everything extends easily to the general case.
- The original proof (1920's) is combinatorial and much more technical.
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*Does $w$ belong to $H = \langle w_1, \ldots, w_m \rangle$ ?*

- Construct $\Gamma(H)$,
- Check whether $w$ is readable as a closed path in $\Gamma(H)$ (at the basepoint).

**(Containment)**

*Given $H = \langle w_1, \ldots, w_m \rangle$ and $K = \langle v_1, \ldots, v_n \rangle$, is $H \leqslant K$ ?*

- Construct $\Gamma(K)$,
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- Check whether $w$ is readable as a closed path in $\Gamma(H)$ (at the basepoint).

**(Containment)**

*Given $H = \langle w_1, \ldots, w_m \rangle$ and $K = \langle v_1, \ldots, v_n \rangle$, is $H \leqslant K$ ?*

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# Membership & containment

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# Basis & conjugacy

**(Computing a basis)**

*Given $H = \langle w_1, \ldots, w_m \rangle$, find a basis for $H$.*

- Construct $\Gamma(H)$,
- Choose a maximal tree,
- Read the corresponding basis.

**(Conjugacy)**

*Given $H = \langle w_1, \ldots, w_m \rangle$ and $K = \langle v_1, \ldots, v_n \rangle$, are they conjugate (i.e. $H^x = K$ for some $x \in F_A$) ?*

- Construct $\Gamma(H)$ and $\Gamma(K)$,
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- Every path between the two basepoints spells a valid $x$.

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# Finite index subgroups

**(Finite index)**

*Given $H = \langle w_1, \ldots, w_m \rangle$, we can decide whether $H \leqslant_{f.i.} F_A$; and, if yes, compute a set of coset representatives.*

**(Schreier index formula)**

*If $H \leqslant_{f.i.} F_A$ is of index $[F : H]$, then $r(H) = 1 + [F : H] \cdot (r(F_A) - 1)$.*

**Theorem (M. Hall)**

*Every f.g. subgroup $H \leqslant_{fg} F_A$ is a free factor of a finite index one, $H \leqslant_{ff} H * L \leqslant_{f.i.} F_A$.*

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**Theorem (Howson)**

*The intersection of finitely generated subgroups of $F_A$ is again finitely generated.*

**Theorem**

*We can effectively compute a basis for $H \cap K$ from a set of generators for $H$ and from $K$.*

**Theorem (H. Neumann)**

$\tilde{r}(H \cap K) \leqslant 2\tilde{r}(H)\tilde{r}(K)$, *where* $\tilde{r}(H) = \max\{0, r(H) - 1\}$.

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# Outline

# Our goal

*Can we extend this to other families of groups $G = \langle A \mid R \rangle$ ?*

- *f.g. subgroups $H \leqslant G$ are not free in general,*
- *there exist subgroups $H \leqslant F_2 \times F_2$ with unsolvable membership problem,*
- *... for general G this is asking too much.*

**(Goal 1)**

*Put conditions to the presentation $G = \langle A \mid R \rangle$ to recreate the bijection with f.g. subgroups and the membership problem, algorithmically.*

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*Identify which are the groups admitting such a presentation.*

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# The Schreier graph

## Definition

*The* Schreier graph $\Gamma(G, H, A)$ *of a subgroup* $H \leqslant G = \langle A \mid R \rangle$ *w.r.t.* $A$ *is:*

- *vertices: left cosets of* $G$ *modulo* $H$, $V = \{Hg \mid g \in G\}$,
- *edges:* $Hg \xrightarrow{a} Hga$, *for* $g \in G$ *and* $a \in A$,
- *basepoint:* $H \cdot 1$.

*Note that* $\Gamma(G, H, A)$ *is finite if and only if* $H \leqslant_{f.i.} G$.

## Definition

*The* core *of a graph* $(\Gamma, v)$ *is the* smallest *subgraph containing* $v$ *and having the same fundamental group; i.e.* $c(\Gamma)$ *is* $\Gamma$ *after deletion of all "pending trees".*

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## Observation

*$\Gamma(H)$ is the core of the Schreier graph $\Gamma(F_A, H, A)$, for $H \leqslant F_A$.*

## (Key observation)

*In the free case, $\Gamma(H)$ is the "central" part of $\Gamma(F_A, H, A)$, i.e. it is a part of $\Gamma(F_A, H, A)$ such that*

- *it is finite,*
- *it is computable from a set of generators for $H$,*
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## (Finite groups)

*If $G = \langle A \mid R \rangle$ is finite and $H \leqslant G$, then we can take $\Gamma(H)$ to be the whole $\Gamma(G, H, A)$...*

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*For all the talk,* $G = \langle A \mid R \rangle$ *and* $\pi : \widetilde{A}^* \twoheadrightarrow G$.

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*A section of* $\pi$ *is a subset* $S \subseteq \widetilde{A}^*$ *such that* $S\pi = G$ *and* $S^{-1} = S$.

## Definition

*Given a section* $S \subseteq \widetilde{A}^*$ *and* $H \leqslant_{f.g.} G$*, define* $\Gamma(G, H, A) \sqcap S$ *to be the smallest subgraph of* $\Gamma(G, H, A)$ *where you can read all* $w \in S$ *as closed paths at the basepoint.*

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# Outline

# Stallings sections

## Definition

*A section $S \subseteq \widetilde{A}^*$ is a Stallings section if*

(S0) *S is a regular language and effectively computable,*

(S1) $\forall\ g \in G, \quad S_g = g\pi^{-1} \cap S$ *is rational and effectively computable,*

(S2) $\forall\ g, h \in G, \quad S_{gh} \subseteq \overline{S_g S_h}$.

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*If $\mathcal{A}$ is an automaton and $L \subseteq \widetilde{A}^*$ is regular and effectively computable then $\mathcal{A} \sqcap L$ is regular and effectively computable.*

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**Proposition**

*For the free group $F_A = \langle A \mid - \rangle$, $S = R_A$ is a Stallings section.*

***Proof.*** $R_A\pi = F_A$ and $R_A^{-1} = R_A$.

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**Theorem (Benois)**

$L \subseteq \widetilde{A}^*$ *rational* $\Rightarrow \overline{L} \subseteq \widetilde{A}^*$ *is rational and effectively computable.*

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***Proof.*** $R_A\pi = F_A$ and $R_A^{-1} = R_A$.

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*For a finite group $G = \langle A \mid R \rangle$, $S = R_A$ is a Stallings section.*

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# Independence of the presentation

## Proposition

*Suppose* $\langle A \mid R\rangle \simeq G \simeq \langle A' \mid R'\rangle$. *Then, there exists a Stallings section for* $\pi\colon \widetilde{A}^* \twoheadrightarrow G$ *if and only if there exists a Stallings section for* $\pi'\colon \widetilde{A'}^* \twoheadrightarrow G$.

**Proof.** *Take a monoid morphism* $\varphi\colon \widetilde{A}^* \to \widetilde{A'}^*$ *such that* $\varphi\pi' = \pi$.
*If S is a Stallings section for* $\pi\colon \widetilde{A}^* \twoheadrightarrow G$, *then* $\overline{S\varphi}$ *will be a Stallings section for* $\pi'\colon \widetilde{A'}^* \twoheadrightarrow G$, *and viceversa.* □

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# Constructing $\Gamma(G, H, A) \sqcap S$

**Lemma**

*Let $S$ be a Stallings section for $\pi \colon \widetilde{A}^* \twoheadrightarrow G$, let $H \leqslant_{f.g.} G$, and let $\mathcal{A}$ be an inverse automata such that*

- $S_H \subseteq L(\mathcal{A}) \subseteq H\pi^{-1}$,
- *there is no path* $p \xrightarrow{w} q$ *with* $p \neq q$ *and* $w\pi = 1$.

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**Theorem**

*Let $S$ be a Stallings section for $\pi \colon \widetilde{A}^* \twoheadrightarrow G$.*
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# Membership

## Corollary

*Let $S$ be a Stallings section for $\pi\colon \widetilde{A}^* \twoheadrightarrow G$, $H \leqslant_{f.g.} G$, and $g \in G$. TFAE:*

(a) $g \in H$,

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(c) $S_g \cap L(\Gamma(G,H,A) \sqcap S) \neq \emptyset$.

*Hence, the membership problem is solvable in $G$.*

***Proof.***

(a) $\Rightarrow$ (b). *If* $g \in H$ *then* $S_g \subseteq S_H \subseteq L(\Gamma(G,H,A) \sqcap S)$.

(b) $\Rightarrow$ (c). $S_g \neq \emptyset$ *because* $S$ *is a section.*

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# Outline

# Amalgamation and HNN

*After several quite technical computations...*

**Theorem**

*If $G_1$ and $G_2$ are groups with Stallings sections, and $H$ is a finite subgroup of both, then the amalgamated product $G_1 *_H G_2$ also admits a Stallings section.*

**Theorem**

*If $G$ is a group with a Stallings section and $K$ is a finite subgroup, then the HNN extension $G*_K$ also admits a Stallings section.*

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*Virtually free groups admit Stallings sections.*

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THANKS