

Stallings graphs for (free-abelian)-by-free groups

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Outline

- 1 Free groups
- 2 Stallings' graphs
- 3 Applications to free groups
- 4 Applications to (free-abelian)-by-free groups

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- 4 Applications to (free-abelian)-by-free groups

Free group: the construction

Definition

- Let $A = \{a_1, \dots, a_r\}$ be a finite **alphabet**, and consider (formally) $\tilde{A} = \{a_1, \dots, a_r, a_1^{-1}, \dots, a_r^{-1}\}$.
- A **word** on A is a finite sequence of symbols $w = a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n}$, where $a_{i_j} \in A$ and $\epsilon_j = \pm 1$. The **length** of w is $\ell(w) = n$.
- The **empty** word is the only one with zero letters, denoted 1 ; $\ell(1) = 0$.
- The collection of all words on A is denoted \tilde{A}^* .
- Operation of **concatenation** in \tilde{A}^* : $u \cdot v = uv$; $\ell(uv) = \ell(u) + \ell(v)$.

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- Two consecutive letters in $w \in \tilde{A}^*$ of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$ are called a **cancellation**. A word w is called **reduced** if it has no cancellations. Denote $R(A) \subseteq \tilde{A}^*$ the set of reduced words.
- The **reduction** is the equivalence relation \sim generated by

$$ua_i^\epsilon a_i^{-\epsilon} v \sim uv.$$

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Definition

- The *free group* on A is $F(A) = \tilde{A}^* / \sim$ with the operation of concatenation (i.e. *concatenation + reduction*).
- The *neutral element* is 1, and the *inverse* of $w = a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n}$ is $w^{-1} = (a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n})^{-1} = a_{i_n}^{-\epsilon_n} \cdots a_{i_1}^{-\epsilon_1}$.
- Of course, $(a_i^{-1})^{-1} = a_i$.

Lemma

For every $w \in \tilde{A}^*$, there is a unique $\bar{w} \in R(A)$, s.t. $w = \bar{w}$ in $F(A)$.

This allows us to “forget” the \sim , and work in $F(A)$ by just manipulating words (and reducing every time it is possible).

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Membership problem

Definition

Let G be a group. The *membership problem* in G consists on finding an algorithm which, on

input: $g_0, g_1, \dots, g_n \in G$;

decides whether $g_0 \in \langle g_1, \dots, g_n \rangle \leq G$, or not.

Proposition

- (i) *Finite groups have solvable membership problem.*
- (ii) *\mathbb{Z}^n and \mathbb{Q}^n have solvable membership problem.*
- (iii) *There are groups G with **UNSOLVABLE** membership problem.*
- (iv) *What about F_r ?*

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Intersection problem

Definition

A group G has the *Howson property* if the intersection of any two finitely generate subgroups is again finitely generated.

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Let G be a group. The *intersection problem* in G consists on finding an algorithm which, on

input: $u_1, \dots, u_n, v_1, \dots, v_m \in G$;

decides whether $\langle u_1, \dots, u_n \rangle \cap \langle v_1, \dots, v_m \rangle$ is finitely generated or not and, if yes, computes a set of generators w_1, \dots, w_p for it.

Proposition

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A first example

Theorem (Howson, 50's)

Free groups are Howson.

Example

Consider F_2 and the subgroups $H = \langle a, b^2, bab \rangle$ and $K = \langle b^2, ba^2 \rangle$.
Can you find generators for $H \cap K$?

- Clearly, $b^2 \in H \cap K$...
- Less obvious but still easy, $a^{-2}b^2a^2 \in H \cap K$ because

$$a^{-2}b^2a^2 = (a)^{-2}(b^2)(a)^2 \in H,$$

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- *Something else?* $H \cap K = \langle b^2, a^{-2}b^2a^2, \dots (?) \dots \rangle$
- *How to be sure you found everything ?*

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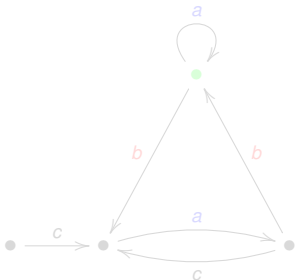
Stallings automata

Definition

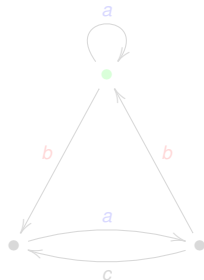
A *Stallings automaton* over A is a finite A -graph (V, E, q_0) , such that:

- 1- it is *connected*,
- 2- it is *trim*, (*no* vertex of degree 1 except possibly q_0),
- 3- it is *deterministic* (*no* two edges with the same label go out of (or into) the same vertex).

NO :



YES :



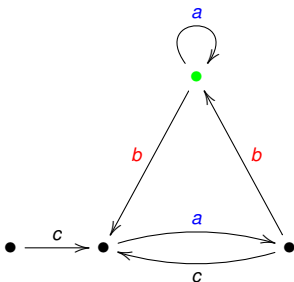
Stallings automata

Definition

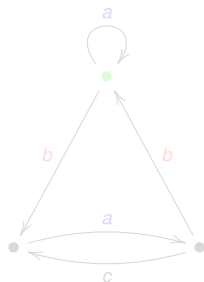
A *Stallings automaton* over A is a finite A -graph (V, E, q_0) , such that:

- 1- it is *connected*,
- 2- it is *trim*, (no vertex of degree 1 except possibly q_0),
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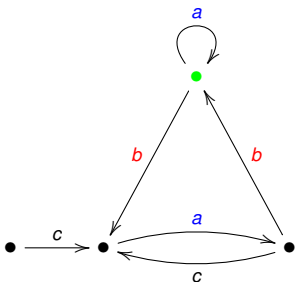
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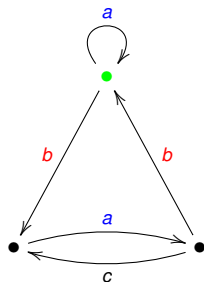
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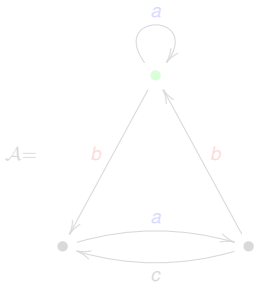
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Reading the subgroup from the automata

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To any given Stallings automaton $\mathcal{A} = (V, E, q_0)$, we associate its language:

$$L(\mathcal{A}) = \{ \text{labels of closed paths at } q_0 \} \leq F(A).$$



$$L(\mathcal{A}) = \{ 1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \dots \}$$

$$L(\mathcal{A}) \not\ni bc^{-1}bcaa$$

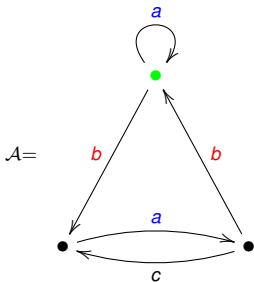
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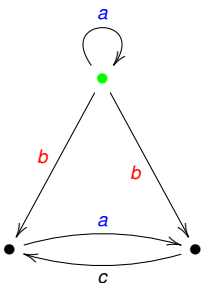
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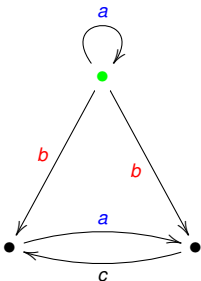


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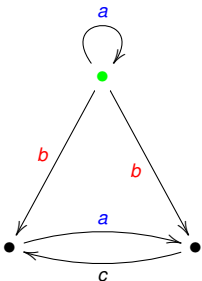
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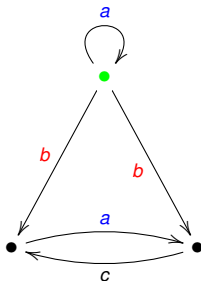
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A basis for $L(\mathcal{A})$

Proposition

For every Stallings automaton $\mathcal{A} = (V, E, q_0)$, and every maximal tree T , the group $L(\mathcal{A})$ is free with free basis

$$\{x_e = \text{label}(T[q_0, \iota e]) \cdot e \cdot T[\tau e, q_0] \in L(\mathcal{A}) \mid e \in EX - ET\},$$

where $T[p, q]$ denotes the geodesic in T from p to q . In particular, $\text{rk}(L(\mathcal{A})) = 1 - |V| + |E|$.

Constructing the automaton from the subgroup

Given $H = \langle w_1, \dots, w_n \rangle \in F(A)$, construct the *flower automaton*, denoted $\mathcal{F}(H)$.

Clearly, $L(\mathcal{F}(H)) = H$.

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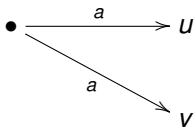
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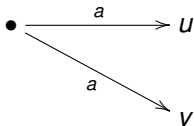
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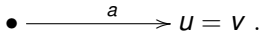
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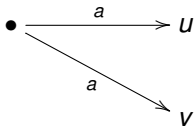
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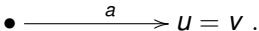
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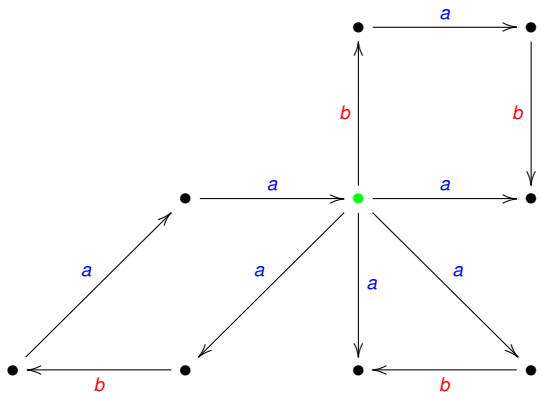
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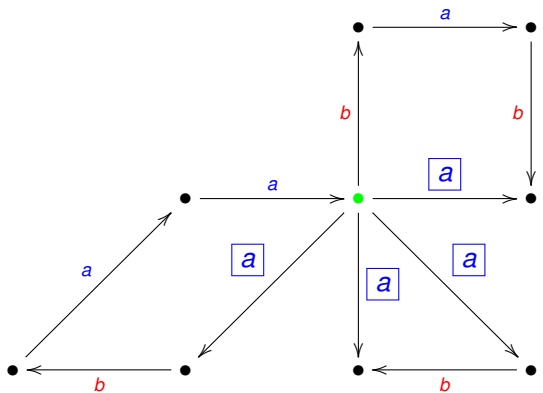
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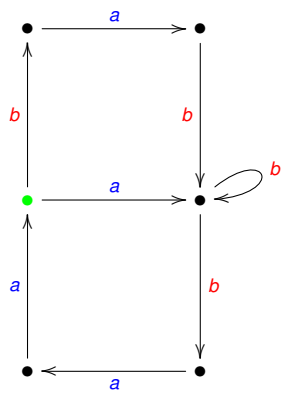
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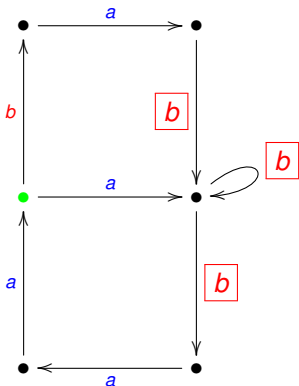
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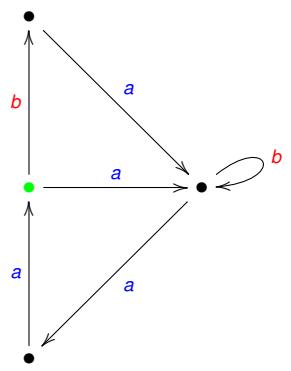
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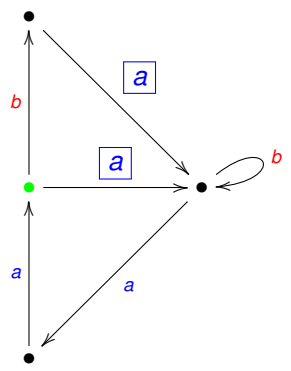
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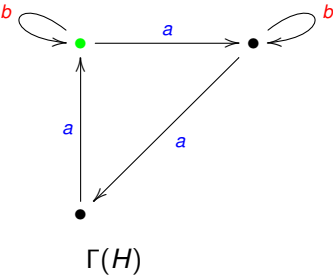
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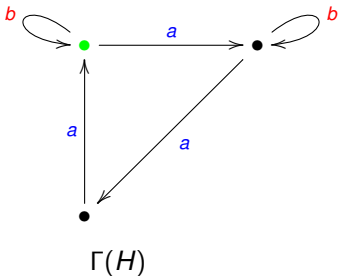
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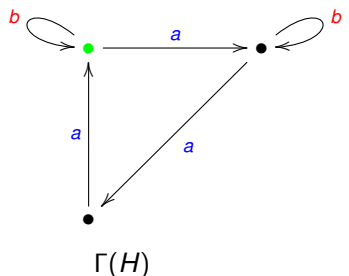
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Local confluence

It can be shown that

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The automaton $\Gamma(H)$ does not depend on the sequence of foldings.

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The automaton $\Gamma(H)$ does not depend on the generators of H .

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The following is a well defined bijection:

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Outline

- 1 Free groups
- 2 Stallings' graphs
- 3 Applications to free groups**
- 4 Applications to (free-abelian)-by-free groups

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Corollary (Nielsen-Schreier)

Every subgroup of F_A is free.

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Free groups have solvable membership problem.

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- Given w_0 and $H = \langle w_1, \dots, w_n \rangle$ in F_m ,
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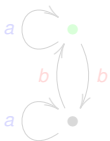
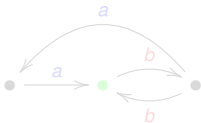
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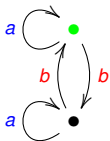
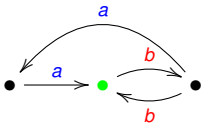
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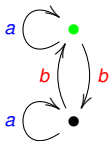
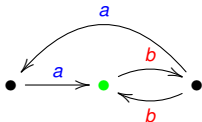
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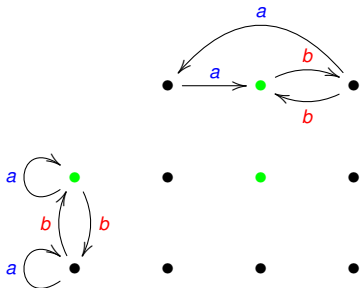
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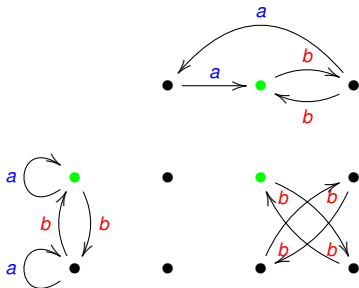
Let $H = \langle a, b^2, bab \rangle$ and $K = \langle b^2, ba^2 \rangle$ be subgroups of F_2 .
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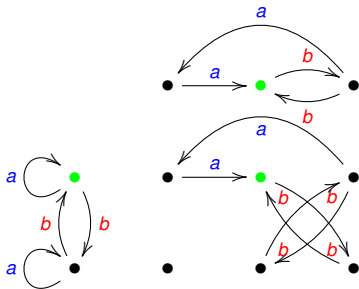
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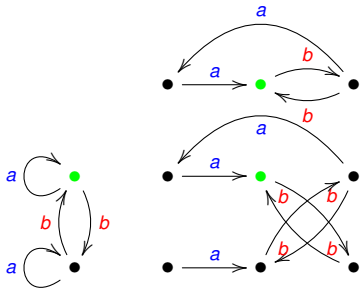
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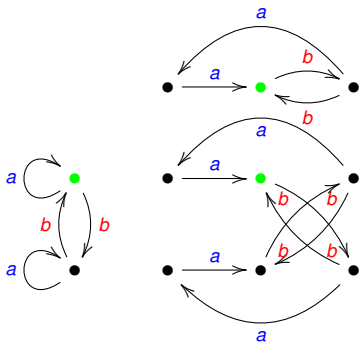
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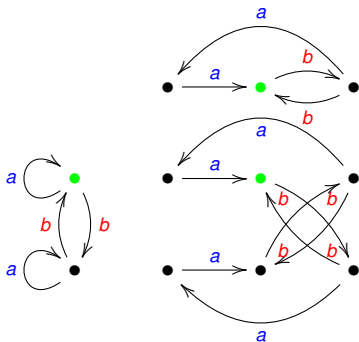
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Outline

- 1 Free groups
- 2 Stallings' graphs
- 3 Applications to free groups
- 4 Applications to (free-abelian)-by-free groups**

(Free-abelian)-by-free groups

Definition

Consider $\{t^v \mid v \in \mathbb{Z}^m\}$ (i.e., \mathbb{Z}^m in multiplicative notation), let $A_1, \dots, A_n \in GL_m(\mathbb{Z})$ acting as $A_i: t^v \mapsto t^{vA_i}$, and consider the group

$$G = F_n \rtimes_{A_1, \dots, A_n} \mathbb{Z}^m = \langle a_1, \dots, a_n, t_1, \dots, t_m \mid [t_i, t_j] = 1, a_i^{-1} t^v a_i = t^{vA_i} \rangle$$

Observation

We have the split short exact sequence

$$1 \rightarrow \mathbb{Z}^m \rightarrow G \rightarrow F_n \rightarrow 1,$$

and normal forms $w(\vec{a})t^v$ for the elements of G (where $v \in \mathbb{Z}^m$ and $w \in F(\{a_1, \dots, a_n\})$), computable using $t^v a_i = a_i t^{vA_i}$. Furthermore,

$$t^v w(\vec{a}) = w(\vec{a}) t^{vW},$$

where $W = W(A_1, \dots, A_n) \in GL_m(\mathbb{Z})$.

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(Free-abelian)-by-free groups

Proposition

For every subgroup $H \leq G = F_n \rtimes_{A_1, \dots, A_n} \mathbb{Z}^m$, the sub-short exact sequence

$$\begin{array}{ccccccc}
 1 & \rightarrow & \mathbb{Z}^m & \rightarrow & G & \xrightarrow{\pi} & F_n \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \rightarrow & L = H \cap \mathbb{Z}^m & \rightarrow & H & \xrightarrow{\pi} & H\pi \rightarrow 1
 \end{array}$$

also splits and so, $H \simeq H\pi \rtimes_{\mathcal{A}} L$, where \mathcal{A} is the restriction of the defining action $F_n \rightarrow \text{Aut}(\mathbb{Z}^m)$ to $\mathcal{A}: H\pi \rightarrow \text{Aut}(L)$.

In particular, every $H \leq F_n \rtimes_{A_1, \dots, A_n} \mathbb{Z}^m$, $n \geq 2$, is of the form $H \simeq F_{n'} \rtimes_{A'_1, \dots, A'_{n'}} \mathbb{Z}^{m'}$ for some $n' \in \mathbb{N} \cup \{\infty\}$ and $m' \leq m$.

Stallings graphs with vectors

Definition

Let us consider now, *vectored A -automata*, i.e., A -graphs with *vectors* assigned at the heads and tails of the edges,

$$\bullet \xrightarrow{u_1 \quad a \quad u_2} \bullet ,$$

reading $t^{-u_1} a t^{u_2} = a t^{u_2 - u_1} A$ (and the inverse if traversed backwards).
... plus a *subspace* $L \leq \mathbb{Z}^m$ attached to the basepoint (corresponding to the purely abelian elements).

Example

For a f. g. subgroup $H = \langle w_1 t^{u_1}, \dots, w_r t^{u_r}, t^{v_1}, \dots, t^{v_s} \rangle$ of $G = F_n \times_{A_1, \dots, A_n} \mathbb{Z}^m$, we can also construct the *flower automaton*.

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Abelian moves

Definition

We need now some extra operations to allow moving abelian mass around:

- *edge moves,*
- *vertex moves,*
- *vertex moves at the basepoint,*
- *open foldings,*
- *closed foldings,*
- *increase L to its closure by the labels of all closed paths at \bullet .*

Definition

A *vectored Stallings A -automata* is a connected and trim vectored A -automata satisfying:

- \mathcal{A}' is deterministic,
- L' is invariant by the labels of all closed paths at \bullet ,
- vectors are zero everywhere except, maybe, at the heads of edges outside a maximal tree T .

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Main results

Proposition

With repeated use of the above operations, any vectored A -automata A can be converted into a Stallings vectored A -automata A' .

Theorem (Delgado–V., 2016)

- (i) A' with the above conditions is uniquely determined by the subgroup H (modulo the choice of the maximal tree, and with all vectors around being viewed 'modulo' L).*
- (ii) The membership problem is solvable in (free-abelian)-by-free groups.*
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Example of intersection

Observation

$F_2 \times \mathbb{Z}$ is **NOT** Howson.

Example

In $F_2 \times \mathbb{Z}^2$, consider the subgroups

$$H = \langle at^{(1,0)}, b^2, bab^{-1}t^{(1,2)}, t^{(1,1)} \rangle,$$

$$K = \langle b^2, ba^2t^{(2,1)}, t^{(1,5)}, t^{(2,6)} \rangle.$$

After some computations...

$$(4, 4) - (4, 2) = (0, 2) \notin L_H + L_K = \langle (1, 1), (1, 5), (2, 6) \rangle = \langle (1, 1), (0, 4) \rangle,$$

But... $H \cap K$ is f.g. because

$$2((4, 4) - (4, 2)) = 2(0, 2) = (0, 4) = -(1, 1) + (1, 5) \in L_H + L_K,$$

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$$(ba^2)^2 b^2 (ba^2)^{-2} = (bab^{-1} t^{(1,2)})^2 b^2 (at^{(1,0)})^2 b^2 (at^{(1,0)})^{-2} b^{-2} \cdot (bab^{-1} t^{(1,2)})^{-2} \in H$$

$$(ba^2)^2 b^2 (ba^2)^{-2} = (ba^2 t^{(2,1)})^2 b^2 (ba^2 t^{(2,1)})^{-2} \in K$$

$$(ba^2)^4 t^{(9,9)} = (bab^{-1} t^{(1,2)})^2 b^2 (at^{(1,0)})^2 (bab^{-1} t^{(1,2)})^2 b^2 (at^{(1,0)})^2 t^{(1,1)} \in H$$

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$$L_H \cap L_K = \langle (1, 1) \rangle \cap \langle (1, 5), (2, 6) \rangle = \langle (1, 1) \rangle.$$

Example of intersection

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THANKS