Algebraic extensions and computations of closures in free groups

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Outline

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1. Algebraic extensions
2. The bijection between subgroups and automata
3. Takahasi's theorem
4. The pro-$\mathcal{V}$ topology

Definitions and notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- Usually, $A = \{a, b, c\}$.
- $(A^{\pm 1})^*$ the free monoid on $A^{\pm 1}$ (words on $A^{\pm 1}$).
- $F_A = (A^{\pm 1})^* / \sim$ is the free group on A (words on $A^{\pm 1}$ modulo reduction).
- Every $w \in A^*$ has a unique reduced form,
- 1 denotes the empty word, and $|\cdot|$ the (shortest) length in F_A:
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- In basic linear algebra:

$$U \leqslant V \leqslant K^n \quad \Rightarrow \quad V = U \oplus L.$$

- In $\mathbb{Z}^n$, the analog is almost true:

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Mimicking field theory...

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Let $H \leqslant F(A)$ and $w \in F(A)$. We say that w is

- *algebraic over H if $\exists\ 1 \neq e_H(x) \in H * \langle x \rangle$ such that $e_H(w) = 1$;*
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Observation

w is transcendental over $H \iff \langle H, w \rangle \simeq H * \langle w \rangle$
$\iff H$ is contained in a proper f.f. of $\langle H, w \rangle$.

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w_1, w_2 algebraic over $H \not\Rightarrow w_1 w_2$ algebraic over H.

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Algebraic and free extensions

A relative notion works better...

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Let $H \leqslant K \leqslant F(A)$ and $w \in K$. We say that w is

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w is algebraic over H if and only if it is $\langle H, w\rangle$-algebraic over H.

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If w_1 and w_2 are K-algebraic over H, then so is $w_1 w_2$.

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Can we compute them all ?

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Takahasi's Theorem

Theorem (Takahasi, 1951)

For every $H \leqslant_{fg} F_A$, the set of algebraic extensions, denoted $\mathcal{AE}(H)$, is finite.

- Original proof by Takahasi was combinatorial and technical,
- Modern proof, using Stallings automata, is much simpler, and due independently to Ventura (1997), Margolis-Sapir-Weil (2001) and Kapovich-Miasnikov (2002).
- Additionally, $\mathcal{AE}(H)$ is computable.

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Outline

Definition

A Stallings automaton is a finite A-labeled oriented graph with a distinguished vertex, (X, v), such that:

1- *X is connected,*

2- *no vertex of degree 1 except possibly v (X is a core-graph),*

3- *no two edges with the same label go out of (or in to) the same vertex.*

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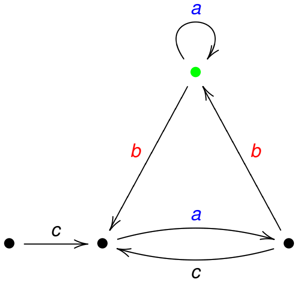

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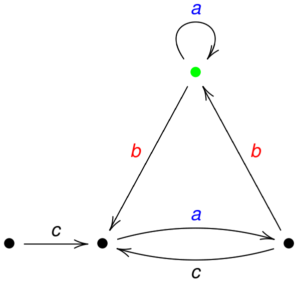

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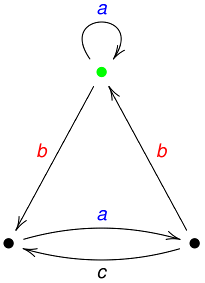

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Stallings (building on previous works) gave a bijection between finitely generated subgroups of F_A and Stallings automata:

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Definition

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$$\pi(X, v) = \{ \text{ labels of closed paths at } v\} \leqslant F_A,$$

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$\pi(X, \bullet) = \{1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \ldots\}$

$\pi(X, \bullet) \not\ni bc^{-1}bcaa$

Membership problem in $\pi(X, \bullet)$ is solvable.

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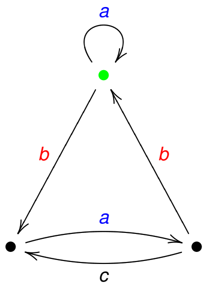
$X=$ 

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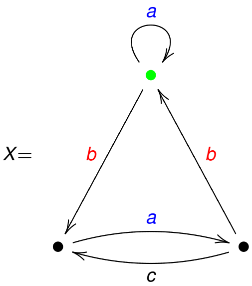

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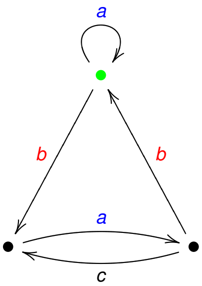

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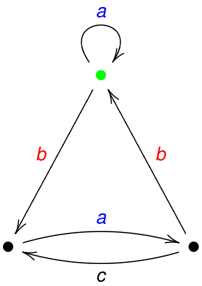
$X=$ 

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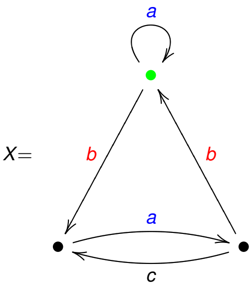

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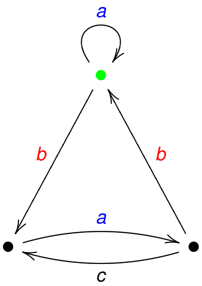

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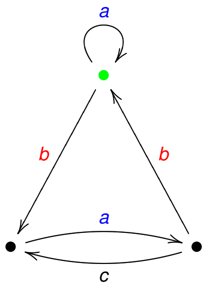
$X=$ 

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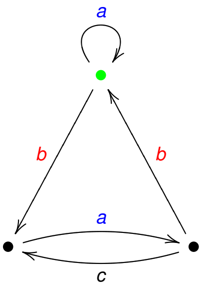

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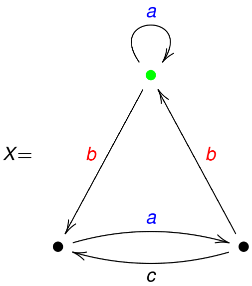

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A basis for $\pi(X, v)$

Proposition

For every Stallings automaton (X, v), the group $\pi(X, v)$ is free of rank $rk(\pi(X, v)) = 1 - |VX| + |EX|$.

Proof:

- Take a maximal tree T in X.
- Write $T[p, q]$ for the geodesic (i.e. the unique reduced path) in T from p to q.
- For every $e \in EX - ET$, $x_e = label(T[v, \iota e] \cdot e \cdot T[\tau e, v])$ belongs to $\pi(X, v)$.
- Not difficult to see that $\{x_e \mid e \in EX - ET\}$ is a basis for $\pi(X, v)$.
- And, $$\begin{aligned} |EX - ET| &= |EX| - |ET| \\ &= |EX| - (|VT| - 1) = 1 - |VX| + |EX|. \; \square \end{aligned}$$

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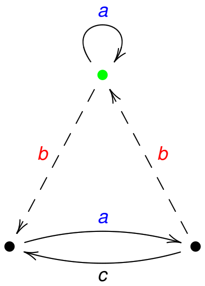

$H = \langle \quad \rangle$

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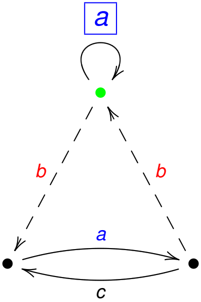

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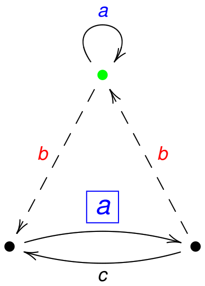
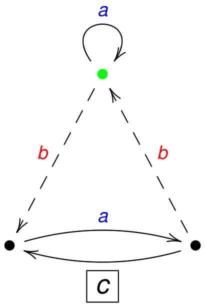


$H = \langle a,\ bab,\ \quad \rangle$

Example



$H = \langle a,\ bab,\ b^{-1}cb^{-1} \rangle$

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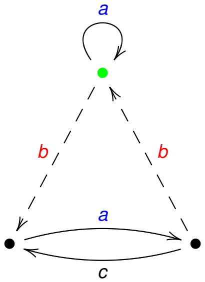
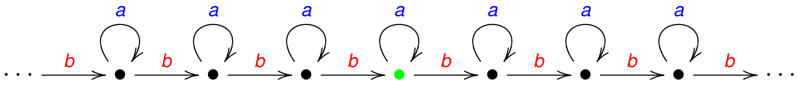


$H = \langle a,\ bab,\ b^{-1}cb^{-1} \rangle$
$rk(H) = 1 - 3 + 5 = 3.$

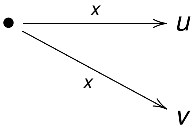
Example-2



$$F_{\aleph_0} \simeq H = \langle \ldots, b^{-2}ab^2, b^{-1}ab, a, bab^{-1}, b^2ab^{-2}, \ldots \rangle \leqslant F_2.$$

Constructing the automata from the subgroup

In any automaton containing the following situation, for $x \in A^{\pm 1}$,



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This operation, $(X, v) \rightsquigarrow (X', v)$, is called a Stallings folding.

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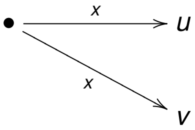

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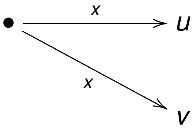

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Lemma (Stallings)

If $(X, v) \rightsquigarrow (X', v')$ is a Stallings folding then $\pi(X, v) = \pi(X', v')$.

Given a f.g. subgroup $H = \langle w_1, \ldots w_m \rangle \leqslant F_A$ (we assume w_i are reduced words), do the following:

1- *Draw the flower automaton,*

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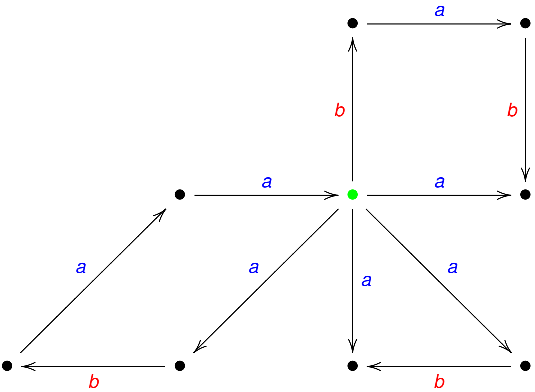

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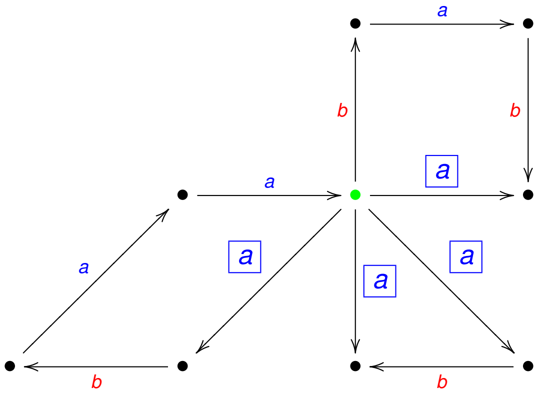

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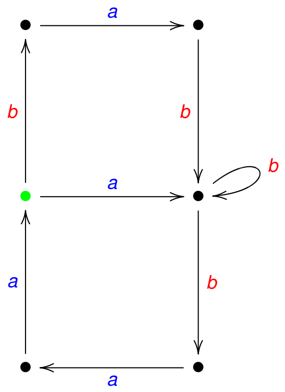

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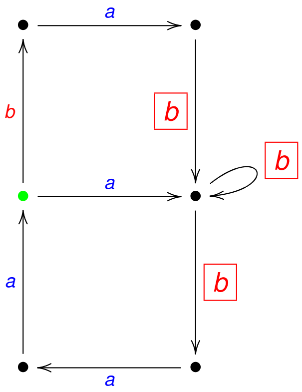

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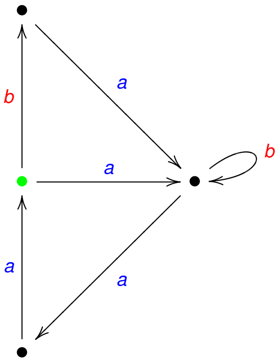

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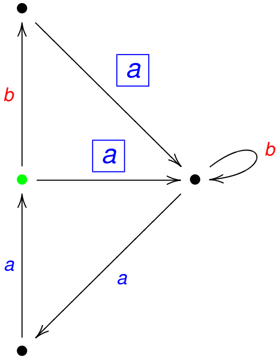

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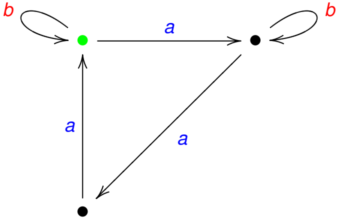

Folding #3.

$\Gamma(H)$

By Stallings Lemma, $\pi(\Gamma(H), \bullet) = \langle baba^{-1}, aba^{-1}, aba^2 \rangle$

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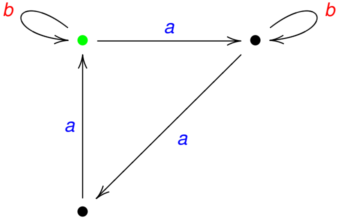

Folding $\#3$. $\Gamma(H)$

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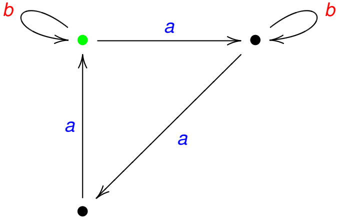

Folding #3. $\Gamma(H)$

By Stallings Lemma, $\pi(\Gamma(H), \bullet) = \langle baba^{-1}, aba^{-1}, aba^2 \rangle$
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Local confluence

It can be shown that

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The automaton $\Gamma(H)$ does not depend on the sequence of foldings

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The automaton $\Gamma(H)$ does not depend on the generators of H.

Theorem

The following is a bijection:

$$\begin{array}{rcl} \{\text{f.g. subgroups of } F_A\} & \longleftrightarrow & \{\text{Stallings automata}\} \\ H & \to & \Gamma(H) \\ \pi(X, v) & \leftarrow & (X, v) \end{array}$$

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Corollary (Nielsen-Schreier)

Every subgroup of F_A is free.

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Outline

Takahasi's theorem

Definition

Let $H \leqslant K \leqslant F(A)$. Then, $H \leqslant K$ is algebraic if and only if H is not contained in any proper free factor of K.

Theorem (Takahasi, 1951)

For every $H \leqslant_{fg} F_A$, the set of algebraic extensions, $\mathcal{AE}(H)$, is finite.

Proof (Ventura; Margolis-Sapir-Weil; Kapovich-Miasnikov):

- Consider $\tilde{\Gamma}(H)$, the result of attaching all possible (infinite) "hairs" to $\Gamma(H)$ (i.e. the covering of the bouquet corresponding to H).
- Given $H \leqslant K$ (both f.g.), we can obtain $\tilde{\Gamma}(K)$ from $\tilde{\Gamma}(H)$ by performing the appropriate identifications of vertices (plus subsequent foldings).

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- Hence, if $H \leqslant K$ (both f.g.) then $\Gamma(K)$ contains as a subgraph either $\Gamma(H)$ or some quotient of it (i.e. $\Gamma(H)$ after some identifications of vertices, $\Gamma(H)/\sim$).
- The overgroups of H: $\mathcal{O}(H) = \{\pi(\Gamma(H)/\sim, \bullet) \mid \sim \text{ is a partition of } V\Gamma(H)\}$.
- Hence, for every $H \leqslant K$, there exists $L \in \mathcal{O}(H)$ such that $H \leqslant L \leqslant_{ff} K$.
- Thus, $\mathcal{AE}(H) \subseteq \mathcal{O}(H)$ and so, it is finite. □

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Computing $\mathcal{AE}(H)$

Corollary

$\mathcal{AE}(H)$ is computable.

Proof:

- Compute $\Gamma(H)$,
- Compute $\Gamma(H)/\sim$ for all partitions $\sim$ of $V\Gamma(H)$,
- Compute $\mathcal{O}(H)$,
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Deciding free-factorness

Proposition

Given $H, K \leqslant F_A$, it is algorithmically decidable whether $H \leqslant_{ff} K$ or not.

Proved by:

- Whitehead 1930's (classical and exponential),
- Silva-Weil 2006 (graphical algorithm, faster but still exponential),
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The algebraic closure

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If $H \leqslant_{alg} K_1$ and $H \leqslant_{alg} K_2$ then $H \leqslant_{alg} \langle K_1 \cup K_2 \rangle$.

Corollary

For every $H \leqslant K \leqslant F_A$ (all f.g.), $\mathcal{AE}_K(H)$ has a unique maximal element, called the K-algebraic closure of H, and denoted $Cl_K(H)$.

Corollary

Every extension $H \leqslant K$ of f.g. subgroups of F_A splits, in a unique way, in an algebraic part and a free part, $H \leqslant_{alg} Cl_K(H) \leqslant_{ff} K$.

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Outline

Pseudo-varieties

Definition

A pseudo-variety of groups $\mathcal{V}$ is a class of finite groups closed under taking subgroups, quotients and finite direct products.

- $\mathcal{G}$ = all finite groups,
- $\mathcal{G}_p$ = all finite p-groups,
- $\mathcal{G}_{nil}$ = all finite nilpotent groups,
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- for a finite group V, $[V]$ = all quotients of subgroups of V^k, $k \geqslant 1$.
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The pro-$\mathcal{V}$ topology

Definition

Let G be a group, and $\mathcal{V}$ be a pseudo-variety of finite groups. The pro-$\mathcal{V}$ topology on G can be defined in several equivalent ways:

- *it is the smallest topology making all the morphisms from G into all $V \in \mathcal{V}$ (with the discrete topology) continuous,*
- *a basis of open sets is given by $\varphi^{-1}(x)$, for all morphism $\varphi \colon G \to V \in \mathcal{V}$,*
- *the normal (finite index) subgroups $K \trianglelefteq G$ such that $G/K \in \mathcal{V}$ form a basis of neighborhoods of 1,*
- *it is the topology given by the pseudo-ultra-metric $d(x, y) = 2^{-r(x,y)}$, where $r(x, y) = \min\{|V| \mid V \in \mathcal{V}$ and separates x and y $\}$.*

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This topology is Hausdorf $\iff$ d is an ultra-metric $\iff$ G is residually-$\mathcal{V}$.

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The $\mathcal{V}$-closure

Proposition

Let G be a group equipped with the pro-$\mathcal{V}$ topology, and let $H \leq G$. Then, TFAE:

- *H is open*
- *H is clopen (i.e. open and closed)*
- *$H \leq_{fi} G$ and $G/H_G \in \mathcal{V}$.*

Furthermore,

$$cl_{\mathcal{V}}(H) = \bigcap_{H \leqslant K,\ open} K = \bigcap_{\varphi:\ G \to V \in \mathcal{V}} \varphi^{-1}(\varphi(H)).$$

Corollary

Assume that $\mathcal{V}$ has a finite free object over A, say $F_A(\mathcal{V})$, and let $\sigma \colon F_A \to F_A(\mathcal{V})$ be the natural projection. Then, for every subset $X \subseteq F_A$, $cl_{\mathcal{V}}(X) = \sigma^{-1}(\sigma(X))$.

The $\mathcal{V}$-closure

Proposition

Let G be a group equipped with the pro-$\mathcal{V}$ topology, and let $H \leq G$. Then, TFAE:

- *H is open*
- *H is clopen (i.e. open and closed)*
- *$H \leq_{fi} G$ and $G/H_G \in \mathcal{V}$.*

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The extension-closed case

Proposition (Ribes, Zaleskiĭ)

Let $\mathcal{V}$ be an extension-closed pseudo-variety, and consider F_A the free group on A with the pro-$\mathcal{V}$ topology. For a given $H \leq_{fg} F_A$,

$$H \text{ is closed} \iff H \text{ is a free factor of a clopen subgroup.}$$

Corollary

For an extension-closed $\mathcal{V}$ and a $H \leq_{fg} F_A$, we have $H \leq_{alg} cl_{\mathcal{V}}(H)$.

Furthermore, it can also be proven that

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Containment of pseudo-varieties

Proposition

For an extension-closed $\mathcal{V}$ and a $H \leq_{fg} F_A$, the pro-$\mathcal{V}$ topology in H coincides with the restriction to H of the pro-$\mathcal{V}$ topology in F_A.

Proposition

Let $\mathcal{V} \subseteq \mathcal{W}$ be two pseudo-varieties, and let $H \leq_{fg} F_A$. Then,

- *if H is $\mathcal{V}$-closed then H is also $\mathcal{W}$-closed,*
- $cl_{\mathcal{W}}(H) \leqslant cl_{\mathcal{V}}(H)$,
- *if H is $\mathcal{W}$-dense then H is also $\mathcal{V}$-dense.*

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Basic idea (Margolis-Sapir-Weil)

$\mathcal{G}_p$ is extension-closed, so $H \leq_{alg} cl_p(H)$.

Given $H \leqslant F_A$

- *compute* $\Gamma(H)$,
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- *((clean and compute* $\mathcal{AE}(H) = \{H_0, \ldots, H_n\}$, *))*
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Given $H \leqslant F_A$ *we can algorithmically decide whether* H *is p-dense, or otherwise computes an* $H \leq_{alg} H_i \neq F_A$ *which is p-closed.*

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Deciding p-denseness (Margolis-Sapir-Weil)

Key property: In a finite p-group, every maximal proper subgroup is normal of index p.

Lemma

If H is a proper p-clopen subgroup of F_A then $\exists\, \psi\colon F_A \to \mathbb{Z}/p\mathbb{Z}$ which is onto and $H \leqslant \ker \psi$.

Let $\sigma\colon F_A \to (\mathbb{Z}/p\mathbb{Z})^A$ be the natural projection.

Corollary

For $H \leq_{fg} F_A$, TFAE

- *H is p-dense,*
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So, given $H = \langle h_1, \ldots, h_r \rangle \leq_{fg} F_A$,

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- *if $r(M_p(H)) = |A|$ then H is p-dense,*
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Now, we can compute this L ...

- *choose a maximal tree T in $\Gamma(H)$,*
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- *choose a maximal tree T in $\Gamma(H)$,*
- *for every vertex u, let $t_u = T[1, u]$,*
- *for every pair of vertices u, v check whether $t_u t_v^{-1} \in \sigma^{-1}(\sigma(H))$, i.e. whether $\sigma(t_u t_v^{-1}) \in \sigma(H)$,*
- *identify u and v if and only if $\sigma(t_u t_v^{-1}) \in \sigma(H)$,*

- *the resulting graph is* $\Gamma(L)$,
- *choose a maximal tree in* $\Gamma(L)$, *and compute a basis for* L.

Proposition

The complexity is n^5, *where* n *is the sum of lengths of given generators for* H.

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Example

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Let us compute the p-closure of

$$H = \langle a^2,\ ab^2a^{-1},\ aba^2b^{-1}a^{-1},\ ababa^{-1}b^{-1},\ baba^{-1}b^{-1}a^{-1},\ ba^2b^{-1},\ b^2 \rangle$$

in $F_{\{a,b\}}$, for every prime p.

For $p \neq 2$, H is p-dense in $F_{\{a,b\}}$; so, $p - cl(H) = \langle a, b \rangle$.
For $p = 2$,

- $\sigma(H) = \langle a + b \rangle$
- $K = \langle a^2, ab, ab^{-1} \rangle$ is 2-closed and contains H;
- writing H in terms of the generators of K ($x = a^2$, $y = ab$, $z = ab^{-1}$),

$$H = \langle x, yz, yxy^{-1}, y^2x^{-1}z^{-1}, zxzy^{-1}, zxz^{-1}, 1, zy \rangle \leqslant \langle x, y, z \rangle$$

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More consequences (Margolis-Sapir-Weil)

Proposition

Let $H \leqslant K \leqslant F_A$ be f.g. subgroups. Then,

- *the set of primes p for which H is p-dense in K is either empty or co-finite,*
- *the set of primes p for which H is p-closed is either finite or co-finite,*

and both effectively computable.

Proposition

The nil-closure of H is the intersection, over all primes, of the p-closure of H.

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The nil-closure of $H \leq_{fg} F_A$ is effectively computable.

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THANKS