Unsolvability of the CP and IP for automaton groups

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Outline

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Main results

Consider the family of automaton groups.

Observation

The word problem is solvable for all automaton groups.

Theorem (Sunic-V.)

There exist automaton groups with unsolvable conjugacy problem.

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The isomorphism problem is unsolvable within the family of automaton groups.

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Reduction to matrices

Both results come from...

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Let $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$ be f.g. Then, $\mathbb{Z}^d \rtimes \Gamma$ is an automaton group.

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Theorem (Bogopolski-Martino-V.)

There exists $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$ f.g. such that $\mathbb{Z}^d \rtimes \Gamma$ has unsolvable conjugacy problem.

Theorem (Sunic-V.)

Given $\Gamma, \Delta \leqslant \mathrm{GL}_d(\mathbb{Z})$ f.g., it is undecidable whether $\mathbb{Z}^d \rtimes \Gamma \simeq \mathbb{Z}^d \rtimes \Delta$.

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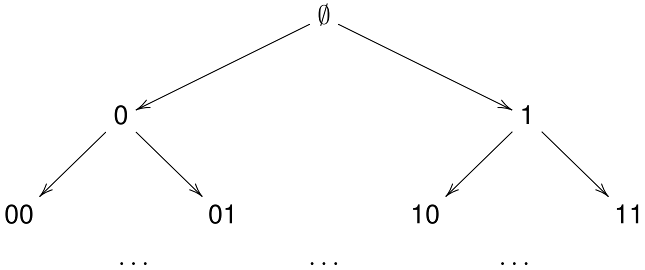
Tree automorphisms

Let X be an alphabet on k letters, and let X^* be the free monoid on X, thought as a rooted k-ary tree:



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- Every *tree automorphism* g decomposes as a *root permutation* $\pi_g : X \to X$, and k *sections* $g|_x$, for $x \in X$:

$$g(xw) = \pi_g(x) g|_x(w).$$

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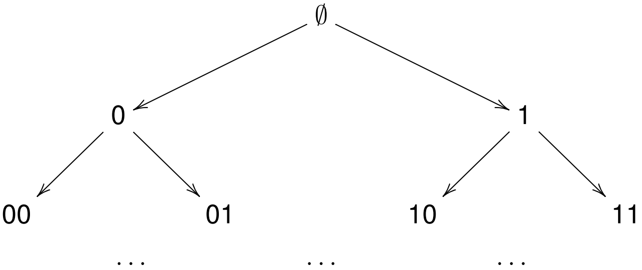

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Automaton groups

Definition

- *A set of tree automorphisms is self-similar if it contains all sections of all of its elements.*
- *A finite automaton is a finite self-similar set (elements are called states).*
- *The group $G(\mathcal{A})$ of tree automorphisms generated by an automaton $\mathcal{A}$ is called an automaton group.*

The Grigorchuk group: $G = \langle 1, \alpha, \beta, \gamma, \delta \rangle$, where

$$\alpha = \sigma(1,1), \quad \beta = 1(\alpha,\gamma), \quad \gamma = 1(\alpha,\delta), \quad \delta = 1(1,\beta).$$

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Affinities of n-adic integers

Definition

Let $\mathcal{M} = \{M_1, \ldots, M_m\}$ be integral $d \times d$ matrices with non-zero determinants. Let $n \geqslant 2$ be relatively prime to all these determinants (thus, M_i is invertible over the ring $\mathbb{Z}_n$ of n-adic integers).

For an integral $d \times d$ matrix M and $\mathbf{v} \in \mathbb{Z}^d$, consider the invertible affine transformation ${}_{\mathbf{v}}M \colon \mathbb{Z}_n^d \to \mathbb{Z}_n^d$, ${}_{\mathbf{v}}M(\mathbf{u}) = \mathbf{v} + M\mathbf{u}$.

Let

$$G_{\mathcal{M},n} = \langle \{ {}_{\mathbf{v}}M \mid M \in \mathcal{M},\ \mathbf{v} \in \mathbb{Z}^d \} \rangle \leqslant Aff_d(\mathbb{Z}_n).$$

Lemma

If, in addition, $\det M_i = \pm 1$, then $G_{\mathcal{M},n} \cong \mathbb{Z}^d \rtimes \Gamma$, where $\Gamma = \langle M_1, \ldots, M_m \rangle \leqslant GL_d(\mathbb{Z})$.

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Proof. *Denote the translation by* $\tau_{\mathbf{v}}: \mathbb{Z}_n^d \to \mathbb{Z}_n^d$, $\mathbf{u} \mapsto \mathbf{u} + \mathbf{v}$.
Since ${}_{\mathbf{v}}M = \tau_{\mathbf{v}}\ {}_{\mathbf{0}}M$, *we have* $G_{\mathcal{M},n}$ *generated by* ${}_{\mathbf{0}}M$ *for* $M \in \mathcal{M}$, *and* $\tau_{\mathbf{e}_i}$, *where the* $\mathbf{e}_i$*'s are the canonical vectors.*

If $M \in \mathrm{GL}_d(\mathbb{Z})$, *then* ${}_{\mathbf{v}}M \in \mathit{Aff}_d(\mathbb{Z}_n)$ *restricts to an integral bijective affine transformation* ${}_{\mathbf{v}}M \in \mathit{Aff}_d(\mathbb{Z})$; *hence, we can view* $G_{\mathcal{M},n} \leqslant \mathit{Aff}_d(\mathbb{Z})$ *(and is independent from n; let's denote it by* $G_{\mathcal{M}}$*).*

They get multiplied as

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{}_{\mathbf{v}}M_{\mathbf{v}'}M' : \mathbf{u} \longrightarrow \mathbf{v}' + M'\mathbf{u} \longrightarrow \begin{aligned} &\mathbf{v} + M(\mathbf{v}' + M'\mathbf{u}) = \\ &(\mathbf{v} + M\mathbf{v}') + MM'\mathbf{u} = \\ &{}_{\mathbf{v}+M\mathbf{v}'}(MM')(\mathbf{u}). \end{aligned}
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So, $G_{\mathcal{M}} \cong \mathbb{Z}^d \rtimes \Gamma$, *where* $\Gamma = \langle M_1, \dots, M_m \rangle \leqslant \mathrm{GL}_d(\mathbb{Z})$.

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So, $G_{\mathcal{M}} \cong \mathbb{Z}^d \rtimes \Gamma$, *where* $\Gamma = \langle M_1, \ldots, M_m \rangle \leqslant \mathrm{GL}_d(\mathbb{Z})$.

$G_{\mathcal{M}}$ is an automaton group

So, we have the groups $G_{\mathcal{M},n}$ (with $\mathcal{M} = \{M_1, \ldots, M_m\}$ as before) and

$$\det M_i = \pm 1 \quad \Rightarrow \quad G_{\mathcal{M},n} \cong \mathbb{Z}^d \rtimes \Gamma,$$

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Definition

Elements in $\mathbb{Z}_n$ may be (uniquely) represented as right infinite words over $Y_n = \{0, \ldots, n-1\}$:

$$y_1 y_2 y_3 \cdots \quad \longleftrightarrow \quad y_1 + n \cdot y_2 + n^2 \cdot y_3 + \cdots .$$

Similarly, elements of $\mathbb{Z}_n^d$ (the free d-dimensional module, viewed as column vectors), may be (uniquely) represented as right infinite words over $X_n = Y_n^d = \{(y_1, \ldots, y_d)^T \mid y_i \in Y_n\}$:

$$\mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_3 \cdots \quad \longleftrightarrow \quad \mathbf{x}_1 + n \cdot \mathbf{x}_2 + n^2 \cdot \mathbf{x}_3 + \cdots .$$

Note that $|Y_n| = n$ and $|X_n| = n^d$.

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For every $\mathbf{v} \in \mathbb{Z}^d$, *and every* $\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3 \ldots \in \mathbb{Z}_n^d$, *we have*

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Definition

For $M \in \mathcal{M}$, let V_M be the set of integral vectors with coordinates between $-\|M\|$ and $\|M\| - 1$ (note that $|V_M| = (2\|M\|)^d$).

Definition

Construct the automaton $\mathcal{A}_{M,n}$:

- *Alphabet: X_n.*
- *States: $m_{\mathbf{v}}$ for $\mathbf{v} \in V_M$, with root permutation and sections*

$$m_{\mathbf{v}}(\mathbf{x}) = \mathrm{Mod}(\mathbf{v} + M\mathbf{x}), \quad \text{and} \quad m_{\mathbf{v}}|_{\mathbf{x}} = m_{\mathrm{Div}(\mathbf{v}+M\mathbf{x})}.$$

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$G_{\mathcal{M}}$ is an automaton group

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For $M \in \mathcal{M}$, let V_M be the set of integral vectors with coordinates between $-\|M\|$ and $\|M\| - 1$ (note that $|V_M| = (2\|M\|)^d$).

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Observation

The state $m_{\mathbf{v}} \in \mathcal{A}_{M,n}$ acts on a vector $\mathbf{u} = \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3 \cdots \in \mathbb{Z}_n^d$ as $m_{\mathbf{v}}(\mathbf{u}) = {}_{\mathbf{v}}M(\mathbf{u})$.

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Construct the automaton $\mathcal{A}_{\mathcal{M},n}$ as the disjoint union of the automata $\mathcal{A}_{M_1,n}, \ldots, \mathcal{A}_{M_m,n}$.

- *Alphabet: X_n,*
- *It has $2^d \sum_{i=1}^{m} ||M_i||^d$ states.*

Proposition

$G_{\mathcal{M},n}$ is an automaton group generated by the automaton $\mathcal{A}_{\mathcal{M},n}$ (over an alphabet of size n^d, and having $2^d \sum_{i=1}^{m} ||M_i||^d$ states).

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Orbit decidability

Definition

Let G be a f.g. group. A subgroup $\Gamma \leqslant \mathrm{Aut}(G)$ is said to be orbit decidable (O.D.) if there is an algorithm s.t., given $u, v \in G$, it decides whether there exists $\alpha \in \Gamma$ such that $\alpha(u)$ is conjugate to v.

First examples: $G = \mathbb{Z}^d$

Observation (folklore)

The full group $\mathrm{Aut}(\mathbb{Z}^d) = \mathrm{GL}_d(\mathbb{Z})$ is orbit decidable.

Proof. *For $u, v \in \mathbb{Z}^d$, there exists $A \in \mathrm{GL}_d(\mathbb{Z})$ such that $v = Au$ if and only if $gcd(u_1, \ldots, u_d) = gcd(v_1, \ldots, v_d)$.*

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Does there exist an orbit undecidable subgroup of $GL_3(\mathbb{Z})$?

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Definition

Let $U = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ be a finite presentation. The Mihailova group corresponding to U is

$$M(U) = \{(v, w) \in F_n \times F_n \mid v =_U w\} =$$

$$= \langle (x_1, x_1), \ldots, (x_n, x_n), (1, r_1), \ldots, (1, r_m) \rangle \leqslant F_n \times F_n.$$

Theorem (Mihailova 1958)

The membership problem in $F_2 \times F_2$ is unsolvable.

Theorem (Grunewald 1978)

If $m \geqslant 1$ (i.e. at least one relation) then:
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Connection with orbit decidability

Proposition (Bogopolski-Martino-V. 2008)

Let G be a group, and let $A \leqslant B \leqslant \mathrm{Aut}(G)$ and $v \in G$ be such that $B \cap Stab([v]) = 1$. Then,

$$OD(A) \text{ solvable} \quad \Rightarrow \quad MP(A, B) \text{ solvable.}$$

Proof. *Given $\varphi \in B \leq \mathrm{Aut}(G)$, let $w = v\varphi$ and*

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So, deciding whether v can be mapped to w, up to conjugacy, by somebody in A, is the same as deciding whether φ belongs to A. Hence,

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Proposition (Bogopolski-Martino-V., 08)

For $d \geqslant 4$, there exist f.g., orbit undecidable, subgroups $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$.

Proof.

- *Take a copy of $F_2 = \langle P, Q \rangle$ inside $GL_2(\mathbb{Z})$.*
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- *The technical condition can be satisfied.*
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Connection to semidirect products

Observation (Bogopolski-Martino-V.)

Let H be f.g., and $\Gamma \leqslant \mathrm{Aut}(H)$ f.g. If $H \rtimes \Gamma$ has solvable CP, then $\Gamma \leqslant \mathrm{Aut}(H)$ is orbit decidable.

Proof. *$OD(\Gamma)$ is exactly the CP in G applied to $u, v \in H$.* $\square$

Corollary (Bogopolski-Martino-V.)

There exists $\Gamma \leqslant GL_d(\mathbb{Z})$ f.g. such that $\mathbb{Z}^d \rtimes \Gamma$ has unsolvable conjugacy problem.

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A construction due to Gordon

Let $U = \langle x_1, \dots, x_n \mid R \rangle$ *be fin. pres. For* $w = w(x_1, \dots, x_n)$*, consider*

$$H_w = \left\langle X,\ a,\ b,\ c \;\middle|\; \begin{array}{l} R \\ a^{-1}ba = c^{-1}b^{-1}cbc \\ a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}cbc^2 \\ a^{-3}[w,b]a^3 = c^{-3}bc^3 \\ a^{-(3+i)}x_iba^{3+i} = c^{-(3+i)}bc^{3+i},\ \ i \geqslant 1 \end{array} \right\rangle$$

Lemma

1) *If* $w \neq_U 1$ *then* U *embeds in* H_w.
2) *If* $w =_U 1$ *then* $H_w = \{1\}$.
3) H_w *is two generated (by* b *and* ca^{-1}*).*

Theorem (Adian-Rabin)

The isomorphism problem, the triviality problem, the finite problem are all unsolvable.

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The isomorphism problem, the triviality problem, the finite problem are all unsolvable.

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Let $U = \langle x_1, \ldots, x_n \mid R \rangle$ *be fin. pres. For* $w = w(x_1, \ldots, x_n)$*, consider*

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Take U with unsolvable WP (in particular $|U| = \infty$), consider the presentations H_w as above, and consider the Mihailova group corresponding to H_w:

$$L_w = M(H_w) = \{(u, v) \in F_2 \times F_2 \mid u =_{H_w} v\} \leqslant F_2 \times F_2.$$

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