

Automorphisms of free and surface groups and their fixed points

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(Joint work with Q. Zhang and J. Wu.)

Most of this talk is contained in the paper:

Q. Zhang, E. Ventura, J. Wu,
“Fixed subgroups are compressed in surface groups”, *International
Journal of Algebra and Computation* **25** (5) (2015), 865-887.

Outline

- 1 Fixed subgroups in free groups (history)
- 2 New results in free groups
- 3 Fixed subgroups in surface groups (history)
- 4 New results in surface groups
- 5 New results in direct products of free and surface groups

Notation

- Let G be a finitely presented group.
- $\text{Aut}(G) \subseteq \text{Mono}(G) \subseteq \text{End}(G)$.
- Let endomorphisms $\phi: G \rightarrow G$ act on the left, $x \mapsto \phi(x)$.
- $\text{Fix}(\phi) = \{x \in G \mid \phi(x) = x\} \leq G$.
- If $\mathcal{B} \subseteq \text{End}(G)$ then
 $\text{Fix}(\mathcal{B}) = \{x \in G \mid \beta(x) = x \ \forall \beta \in \mathcal{B}\} = \bigcap_{\beta \in \mathcal{B}} \text{Fix}(\beta) \leq G$.
- For $\mathcal{B} \subseteq \text{Hom}(G, H)$,
 $\text{Eq}(\mathcal{B}) = \{x \in G \mid \beta_1(x) = \beta_2(x) \ \forall \beta_1, \beta_2 \in \mathcal{B}\}$.
- Note that if $G \leq H$ and $\mathcal{B} \subseteq \text{Hom}(G, H)$ then $\text{Eq}(\mathcal{B}) = \text{Fix}(\mathcal{B})$.

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What is known about free groups ?

Theorem (Dyer–Scott, 75)

Let $\mathcal{B} \leq \text{Aut}(F_n)$ be a finite group of automorphisms of F_n . Then, $\text{Fix}(\mathcal{B}) \leq_{\text{ff}} F_n$; in particular, $r(\text{Fix}(\mathcal{B})) \leq n$.

Conjecture (Scott)

For every $\phi \in \text{Aut}(F_n)$, $r(\text{Fix}(\phi)) \leq n$.

Theorem (Gersten, 83 (published 87))

Let $\phi \in \text{Aut}(F_n)$. Then $r(\text{Fix}(\phi)) < \infty$.

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Train-tracks

Main result in this story:

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introducing the theory of [train-tracks](#) for graphs.

After Bestvina–Handel, live continues ...

Theorem (Imrich–Turner, 89)

Let $\phi \in \text{End}(F_n)$. Then $r(\text{Fix}(\phi)) \leq n$.

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Inertia

Definition

A subgroup $H \leq G$ is called

- *inert in G* if $r(H \cap K) \leq r(K)$ for every $K \leq G$;
 - *compressed in G* if $r(H) \leq r(K)$ for every $H \leq K \leq G$;
- *Free factors and cyclic subgroups of F_n are inert in F_n ;*
 - *intersection of inert subgroups are inert;*
 - *free subgroups of rank 1 and 2 in F_n are inert in F_n ;*
 - *$A \leq B \leq C$; if A is inert in B , and B is inert in C then A is inert in C .*
 - *$H \leq G$ inert $\Rightarrow H \leq G$ compressed $\Rightarrow r(H) \leq r(G)$;*
 - *not known if all compressed subgroups of F_n are inert in F_n , or not (Compressed-Inert Conjecture, Dicks-V.)*

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Let $\mathcal{B} \subseteq \text{Mon}(F_n)$ be an arbitrary set of monomorphisms of F_n . Then, $\text{Fix}(\mathcal{B})$ is inert in F_n ; in particular, $r(\text{Fix}(\mathcal{B})) \leq n$.

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The inertia conjecture

Inertia Conjecture (Dicks–V.)

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Main result for free groups

Theorem (Zhang–Wu–V., 15)

Let F be a f.g. free group, let $\mathcal{B} \subseteq \text{End}(F)$, and let $\beta_0 \in \langle \mathcal{B} \rangle \leq \text{End}(F)$ be with $r(\beta_0(F))$ minimal. Then, for every subgroup $K \leq F$ such that $\beta_0(K) \cap \text{Fix } \mathcal{B} \leq K$, we have $r(K \cap \text{Fix } \mathcal{B}) \leq r(K)$.

(Proof)

- Since, $\text{Fix } \alpha \cap \text{Fix } \beta \leq \text{Fix } (\alpha\beta)$, we have $\text{Fix } \langle \mathcal{B} \rangle = \text{Fix } \mathcal{B}$ and so, we can assume that $\text{Id} \in \langle \mathcal{B} \rangle = \mathcal{B}$.
- Now choose $\beta_0 \in \mathcal{B}$ with $r(\beta_0(F)) = \min\{r(\gamma(F)) \mid \gamma \in \mathcal{B}\}$. Thus, all elements of \mathcal{B} act *injectively* on $\beta_0(F)$.
- Restricting $\beta_0\mathcal{B} = \{\beta_0\gamma \mid \gamma \in \mathcal{B}\} \subseteq \mathcal{B}$ to $\beta_0(F)$ we get the family of injective endos: $\beta_0\gamma|_{\beta_0(F)}: \beta_0(F) \rightarrow \beta_0(F)$, for $\gamma \in \mathcal{B}$.
- Hence, $\text{Fix}(\beta_0\mathcal{B}) = \text{Fix}(\beta_0\mathcal{B}|_{\beta_0(F)})$ is inert in $\beta_0(F)$ that is, for every $L \leq \beta_0(F)$, we have $r(L \cap \text{Fix}(\beta_0\mathcal{B})) \leq r(L)$.

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Let F be a f.g. free group, let $\mathcal{B} \subseteq \text{End}(F)$, and let $\beta_0 \in \langle \mathcal{B} \rangle \leq \text{End}(F)$ be with $r(\beta_0(F))$ minimal. Then, for every subgroup $K \leq F$ such that $\beta_0(K) \cap \text{Fix } \mathcal{B} \leq K$, we have $r(K \cap \text{Fix } \mathcal{B}) \leq r(K)$.

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As a first corollary, we obtain

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Let $\mathcal{B} \subseteq \text{End}(F_n)$ be an arbitrary set of endomorphisms of F_n . Then, $\text{Fix}(\mathcal{B})$ is compressed in F_n .

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Outline

- 1 Fixed subgroups in free groups (history)
- 2 New results in free groups
- 3 Fixed subgroups in surface groups (history)**
- 4 New results in surface groups
- 5 New results in direct products of free and surface groups

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- Σ_g denotes the orientable surface of genus g , $g \geq 0$;
 - $S_g = \pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$;
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Let G be a surface group. For every $\mathcal{B} \subseteq \text{End}(G)$, $\text{Fix}(\mathcal{B})$ is inert in G .

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- 1 Fixed subgroups in free groups (history)
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Main result for surface groups

The proof of main Theorem for free groups works for surface groups of negative Euler characteristic as well. For non-negative Euler characteristic one can prove the inertia conjecture directly.

Proposition

Let G be either $F_0 = S_0 = 1$, or $S_1 = \mathbb{Z}^2$, or $NS_1 = \mathbb{Z}/2\mathbb{Z}$, or NS_2 , and let $\mathcal{B} \subseteq \text{End}(G)$. Then, $\text{Fix } \mathcal{B}$ is inert in G .

Theorem (Zhang–Wu–V., 15)

Let G be a surface group, let $\mathcal{B} \subseteq \text{End}(G)$, and let $\beta_0 \in \langle \mathcal{B} \rangle \leq \text{End}(G)$ be with $r(\beta_0(G))$ minimal. Then, for every subgroup $K \leq G$ such that $\beta_0(K) \cap \text{Fix } \mathcal{B} \leq K$, we have $r(K \cap \text{Fix } \mathcal{B}) \leq r(K)$.

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Theorem (Zhang–Wu–V., 15)

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Product groups

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A **product** group is a group of the form $G = G_1 \times \cdots \times G_n$, where $n \geq 1$, and each G_i is either F_r , $r \geq 1$, or S_g , $g \geq 1$, or NS_k , $k \geq 1$.
Block notation: $G = G_1^{n_1} \times \cdots \times G_m^{n_m}$, $n_i \geq 1$, and $G_i \not\cong G_j$ for $i \neq j$; of course, $n = n_1 + \cdots + n_m$.

Definition

A product group $G = G_1 \times \cdots \times G_n$ is of

- **hyperbolic type** if G_i is hyperbolic for every i ;
- **Euclidean type** if G_i is Euclidean for every i ;
- **mixed type** if G_i is hyperbolic and G_j is Euclidean, for some i, j ;

Initial properties

In general, $r(A \times B) \leq r(A) + r(B)$, but...

Lemma

For a product group, $r(G_1 \times \cdots \times G_n) = r(G_1) + \cdots + r(G_n)$.

Lemma

Let G be a product group. Then, $Z(G) = 1 \Leftrightarrow G$ is of hyperbolic type.

Corollary

Let G be Euclidean, $G = NS_2^\ell \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$, for $\ell, p, q \geq 0$. Then any subgroup $H \leq G$ satisfies $r(H) \leq r(G) = 2\ell + p + q$. In particular, $r(\text{Fix}(\phi)) \leq r(G)$ for every $\phi \in \text{End}(G)$.

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So, in the hyperbolic case, $\text{Cen}_G(g_1, \dots, g_n) \simeq \widehat{G}_1 \times \dots \times \widehat{G}_n$, where $\widehat{G}_i = G_i$ if $g_i = 1$, or $\widehat{G}_i = \mathbb{Z}$ if $g_i \neq 1$.

In general $\mathbb{Z} \times A \simeq \mathbb{Z} \times B \not\Rightarrow A \simeq B$, but...

Proposition

Let $G = G_1 \times \dots \times G_n$ and $H = H_1 \times \dots \times H_m$ be two product groups of hyperbolic type. Then, $G \simeq H \Leftrightarrow n = m$ and $G_i \simeq H_i$ up to reordering.

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Main result for product groups

Theorem (Zhang–Wu–V., 15)

*Let $G = G_1 \times \cdots \times G_n$ be a product group. Then, $r(\text{Fix } \phi) \leq r(G)$
 $\forall \phi \in \text{Aut}(G) \Leftrightarrow G$ is either of Euclidean or of hyperbolic type.*

(Proof)

- Step 1: If G Euclidean then ok. Done.
- Step 2: If G hyperbolic the ok ...
- Step 3: For any mixed type G , $\exists \phi \in \text{Aut}(G)$ s.t. $r(\text{Fix } \phi) > r(G)$...

Proposition (Zhang–Wu–V., 15)

Let $G = G_1^{n_1} \times \cdots \times G_m^{n_m}$ be a product group in block notation. If G is of hyperbolic type then, $\forall \phi \in \text{Aut}(G)$, $\exists \phi_{i,j} \in \text{Aut}(G_i)$ and $\sigma_i \in S_{n_i}$, such that

$$\phi = \sigma_1 \circ \cdots \circ \sigma_m \circ \left(\prod_{i=1}^m \prod_{j=1}^{n_i} \phi_{i,j} \right) = \prod_{i=1}^m \left(\sigma_i \circ \prod_{j=1}^{n_i} \phi_{i,j} \right).$$

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Main result for free products

• Assume G of hyperbolic type, let $\phi \in \text{Aut}(G)$, and let us prove that $r(\text{Fix } \phi) \leq r(G)$.

• By previous result, $\phi = \prod_{i=1}^m (\sigma_i \circ \prod_{j=1}^{n_i} \phi_{i,j})$. So,

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we are reduced to the case $m = 1$, i.e., $G = G_1^n = G_{1,1} \times \cdots \times G_{1,n}$ ($G_{1,j} = G_1$) and $\phi = \sigma \circ (\phi_1 \times \cdots \times \phi_n)$, for $\sigma \in S_n$, $\phi_j \in \text{Aut}(G_{1,j})$.

• If $\sigma = \text{Id}$ then $\text{Fix } \phi = \text{Fix } \phi_1 \times \cdots \times \text{Fix } \phi_n$ and so,

$$r(\text{Fix } \phi) \leq r(\text{Fix } \phi_1) + \cdots + r(\text{Fix } \phi_n) \leq n r(G_1) = r(G_1^n) = r(G).$$

• If $\sigma \neq \text{Id}$, considering its decomposition as a product of cycles, we can reduce to the case of a cycle, $\sigma = (n, n-1, \dots, 1)$.

• In this situation, $\phi = \sigma \circ (\phi_1 \times \cdots \times \phi_n)$ has the form

$$\begin{aligned} \phi: G_{1,1} \times \cdots \times G_{1,n} &\rightarrow G_{1,1} \times \cdots \times G_{1,n} \\ (g_1, \dots, g_n) &\mapsto \sigma(\phi_1(g_1), \phi_2(g_2), \dots, \phi_n(g_n)) = \\ &= (\phi_n(g_n), \phi_1(g_1), \dots, \phi_{n-1}(g_{n-1})). \end{aligned}$$

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Finally, for step 3 let us prove that ...

Proposition (Zhang–Wu–V., 15)

Let G be a product group of mixed type. Then, $\exists \phi \in \text{Aut}(G)$ such that $r(\text{Fix } \phi) > r(G)$.

(Proof)

- We can reduce to the case $G = G_1 \times G_2$ with G_1 Euclidean and G_2 hyperbolic. Take $1 \neq t \in Z(G_1)$, and $Z(G_2) = 1$.

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and so,

$$\text{Fix } \phi = \{(g, \phi_1(g), \phi_2\phi_1(g), \dots, (\phi_{n-1} \cdots \phi_1)(g)) \mid g \in \text{Fix}(\phi_n \cdots \phi_1)\}.$$

- Hence, $r(\text{Fix } \phi) = r(\text{Fix}(\phi_n \cdots \phi_1)) \leq r(G_1) \leq r(G_1^n) = r(G)$.
- This finishes step 2.

Finally, for step 3 let us prove that ...

Proposition (Zhang–Wu–V., 15)

Let G be a product group of mixed type. Then, $\exists \phi \in \text{Aut}(G)$ such that $r(\text{Fix } \phi) > r(G)$.

(Proof)

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• Since $Z(G_1)$ is $Z(F_1) = \mathbb{Z}$, or $Z(S_1) = \mathbb{Z}^2$, or $Z(NS_1) = \mathbb{Z}/2\mathbb{Z}$, or $Z(NS_2) = \mathbb{Z}$, we deduce $o(t) = 2, \infty$.

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• Now, ϕ maps $w(a_1, \dots, a_r) \mapsto w(ta_1, a_2, \dots, a_r) = t^{|w|_1} w(a_1, \dots, a_r)$, where $|w|_1 \in \mathbb{Z}$ is the total a_1 -exponent of $w \in G_2$.

• Hence, $\text{Fix } \phi = G_1 \times \{w \in G_2 \mid |w|_1 \equiv 0\} = G_1 \times \ker \pi$, where $\pi: G_2 \rightarrow \mathbb{Z}/o(t)\mathbb{Z}$, $w \mapsto |w|_1$, and \equiv means equality of integers modulo $o(t)$.

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Main result for product groups

→ Case 3: $G_2 = NS_k = \langle a_1, a_2, \dots, a_k \mid a_1^2 \cdots a_k^2 \rangle$, $k \geq 3$.

- Consider $\phi \in \text{Aut}(G)$ fixing G_1 pointwise and mapping $a_1 \mapsto ta_1$, $a_2 \mapsto t^{-1}a_2$, $a_3 \mapsto a_3$, \dots , $a_k \mapsto a_k$. It is well defined because t commutes with a_1, a_2 and all of G_1 .
- Observe now that, due to the form of the def. rel. in G_2 , the “total a_i -exponent” of an element of $w \in G$ is not well defined; however, the difference of two of them, say $|w|_1 - |w|_2 \in \mathbb{Z}$, it really is.
- Hence, the projection $\pi: G_2 \rightarrow \mathbb{Z}/o(t)\mathbb{Z}$, $w \mapsto |w|_1 - |w|_2$ is well defined, ϕ maps $w(a_1, \dots, a_k)$ to $w(ta_1, t^{-1}a_2, a_3, \dots, a_k) = t^{|w|_1 - |w|_2} w(a_1, \dots, a_k)$, and we proceed and conclude as in case 2.

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Characterizing compression

It is natural to ask for similar characterizations of full compression and full inertia.

Theorem (Zhang–Wu–V., 15)

Let $G = G_1 \times \cdots \times G_n$ be a product group. If $\text{Fix } \phi$ is compressed in G for every $\phi \in \text{Aut}(G)$, then G must be of one of the following forms:

(euc1) $G = \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $p, q \geq 0$; or

(euc2) $G = NS_2 \times (\mathbb{Z}/2\mathbb{Z})^q$ for some $q \geq 0$; or

(euc3) $G = NS_2 \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})$ for some $p \geq 1$; or

(euc4) $G = NS_2^\ell \times \mathbb{Z}^p$ for some $\ell \geq 1, p \geq 0$; or

(hyp1) $G = F_r \times NS_3^\ell$ for some $r \geq 2, \ell \geq 0$; or

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Characterizing inertia

Theorem (Zhang–Wu–V., 15)

Let $G = G_1 \times \cdots \times G_n$ be a product group. If $\text{Fix } \phi$ is inert in G for every $\phi \in \text{Aut}(G)$, then G is of one of the forms: (euc1), or (euc2), or (euc3), or (euc4), or

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Conjecture (Zhang–Wu–V., 15)

Let $G = G_1 \times \cdots \times G_n$ be a product group. Then, the following are equivalent:

- every $\phi \in \text{End}(G)$ satisfies that $\text{Fix } \phi$ is inert in G ,
- every $\phi \in \text{Aut}(G)$ satisfies that $\text{Fix } \phi$ is inert in G ,
- G is of the form (euc1), or (euc2), or (euc3), or (euc4), or (hyp1'), or (hyp2'), or (hyp3').

Characterizing inertia

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