How are the majority of groups? It depends on the glasses in use...

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Outline

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Claim (Gromov '87)

Most finite presentations of groups, present an hyperbolic infinite group.

- Stated in his influential paper on hyperbolic groups: "Essays in group theory", 75-263, Springer, 1987,
- no proof, only the idea,
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- statement made precise and proved, later by other authors.

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The meaning of "most"

Let X be an infinite set. What is the meaning of sentences like "most elements in X have property $\mathcal{P}$" ?

- Define a notion of size, $|\cdot|: X \to \mathbb{N}$, with finite preimages.
- Define the balls: $B(n) = \{x \in X \mid |x| \leqslant n\}$ (which are finite).
- Count the proportion $\rho_n = \frac{|\{x \in B(n) \mid x \text{ satisfies } \mathcal{P}\}|}{|B(n)|} = \frac{|\mathcal{P} \cap B(n)|}{|B(n)|}$.
- Define the density of $\mathcal{P}$ as $\rho = \lim_{n\to\infty} \rho_n$ $\quad(\in [0, 1]$ if it exists).
- $\mathcal{P}$ is generic (or generically many elements satisfy $\mathcal{P}$) if $\rho = 1$.
- $\mathcal{P}$ is negligible if $\rho = 0$.

Of course, everything depends on the chosen size function, i.e. on the direction to infinity inside X.

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Classical example: visible points

Definition

A point $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ *is visible if* $\gcd(x_1, \ldots, x_k) = 1$.

Theorem (Mertens, 1874 (case $k = 2$))

The density of visible points in $\mathbb{Z}^k$ *is* $1/\zeta(k)$, *where* $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$ *is the Riemann zeta-function (with respect to* $||\cdot||_\infty$*).*

In particular, visible points in the plane have density $\frac{6}{\pi^2}$.

With artificial definitions of size, one can force it to be any $\alpha \in [0, 1]$.

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Arzhantseva-Ol'shanskii's proof

- Fix $r \geqslant 2$ and $k \geqslant 1$.
- Consider the free group $F_A = \langle a_1, \ldots, a_r \mid - \rangle$.
- In F_A we have the natural notion of size and balls.
- For $w_1, \ldots, w_k \in F_A$, let $G_{w_1,\ldots,w_k} = \langle a_1, \ldots, a_r \mid w_1, \ldots, w_k \rangle$.

Theorem (Arzhantseva-Ol'shanskii, '96)

$$\exists \quad \lim_{n\to\infty} \frac{|\{(w_1, \ldots, w_k) \in B(n)^k \mid G_{w_1,\ldots,w_k} \text{ is infinite hyperbolic } \}|}{|B(n)|^k} = 1.$$

- Hence, generically many presentations present an infinite hyperbolic group.
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Comments

- This fits the algebraic intuition: the longer the relations are, the closest will the group be to a free group.
- Problem-1: this counts r-generated, k-related groups, with r and k fixed.
- Problem-2: this counts presentations, not really groups !
- maybe different k-tuples $(w_1, \ldots, w_k) \neq (w'_1, \ldots, w'_k)$ generate the same subgroup $\langle w_1, \ldots, w_k \rangle = \langle w'_1, \ldots, w'_k \rangle$.
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A new point of view

Observation

Let $N = \langle w_1, \ldots, w_k \rangle \leqslant F_A$. *Then,*

$$\langle a_1, \ldots, a_r \mid w_1, \ldots, w_k \rangle \simeq \langle a_1, \ldots, a_r \mid N \rangle.$$

and let us count f.g. subgroups N of F_A, instead of counting k-tuples of words.

Advantages:

- r still fixed, but not k.
- less redundancy.
- it will be an equally natural way of counting.

... but with very different results... this is a very different direction to infinity.

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$$\langle a_1, \ldots, a_r \mid w_1, \ldots, w_k \rangle \simeq \langle a_1, \ldots, a_r \mid N \rangle.$$

and let us count f.g. subgroups N of F_A, instead of counting k-tuples of words.

Advantages:

- r still fixed, but not k.
- less redundancy.
- it will be an equally natural way of counting.

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A new point of view

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Outline

Definition

A Stallings automaton is a finite A-labeled oriented graph with a distinguished vertex, (X, v), such that:

1- *X is connected,*

2- *no vertex of degree 1 except possibly v (X is a core-graph),*

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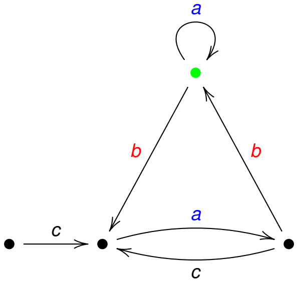

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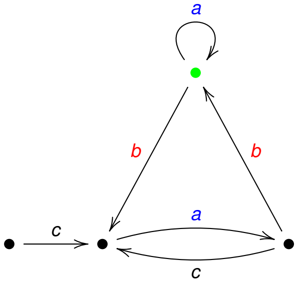

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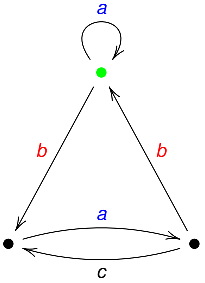

Stallings automata

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Stallings (building on previous works) gave a bijection between finitely generated subgroups of F_A and Stallings automata:

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Reading the subgroup from the automata

Definition

To any given (Stallings) automaton (X, v), we associate its fundamental group:

$$\pi(X, v) = \{ \text{ labels of closed paths at } v \} \leqslant F_A,$$

clearly, a subgroup of F_A.

$$\pi(X, \bullet) = \{1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \ldots\}$$

$$\pi(X, \bullet) \not\ni bc^{-1}bcaa$$

Membership problem in $\pi(X, \bullet)$ is solvable.

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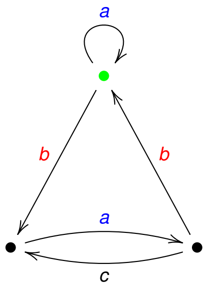
$X=$ 

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A basis for $\pi(X, v)$

Proposition

For every Stallings automaton (X, v), the group $\pi(X, v)$ is free of rank $rk(\pi(X, v)) = 1 - |VX| + |EX|$.

Proof:

- Take a maximal tree T in X.
- Write $T[p, q]$ for the geodesic (i.e. the unique reduced path) in T from p to q.
- For every $e \in EX - ET$, $x_e = label(T[v, \iota e] \cdot e \cdot T[\tau e, v])$ belongs to $\pi(X, v)$.
- Not difficult to see that $\{x_e \mid e \in EX - ET\}$ is a basis for $\pi(X, v)$.
- And, $$\begin{aligned} |EX - ET| &= |EX| - |ET| \\ &= |EX| - (|VT| - 1) = 1 - |VX| + |EX|. \; \square \end{aligned}$$

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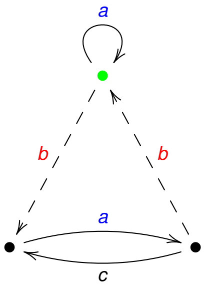
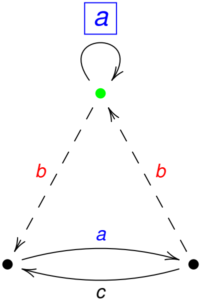


$H = \langle \quad \rangle$

Example



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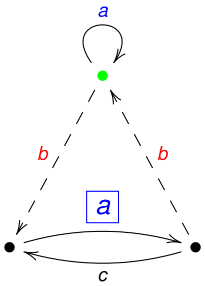

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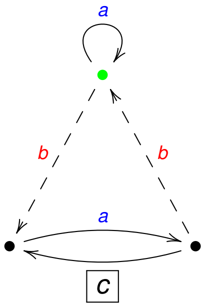

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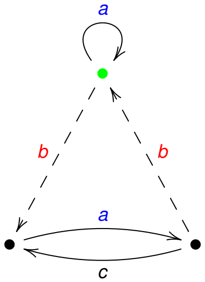

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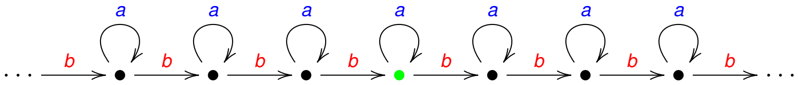
$rk(H) = 1 - 3 + 5 = 3.$

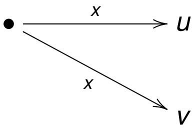
Example-2



$F_{\aleph_0} \simeq H = \langle \ldots, b^{-2}ab^{2}, b^{-1}ab, a, bab^{-1}, b^{2}ab^{-2}, \ldots \rangle \leqslant F_2.$

Constructing the automata from the subgroup

In any automaton containing the following situation, for $x \in A^{\pm 1}$,



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$$\bullet \xrightarrow{\ x\ } u = v \ .$$

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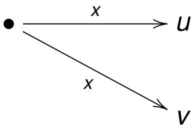

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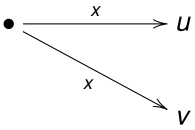

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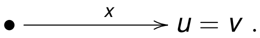

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Lemma (Stallings)

If $(X, v) \rightsquigarrow (X', v')$ is a Stallings folding then $\pi(X, v) = \pi(X', v')$.

Given a f.g. subgroup $H = \langle w_1, \ldots w_m \rangle \leqslant F_A$ (we assume w_i are reduced words), do the following:

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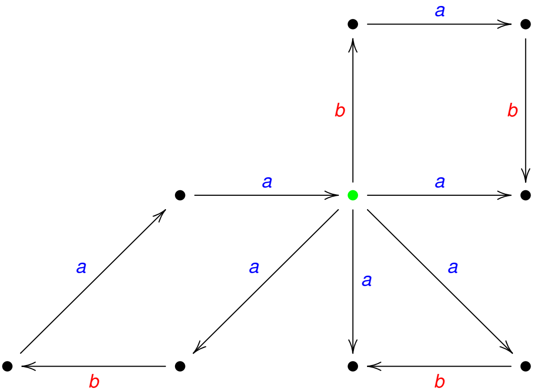

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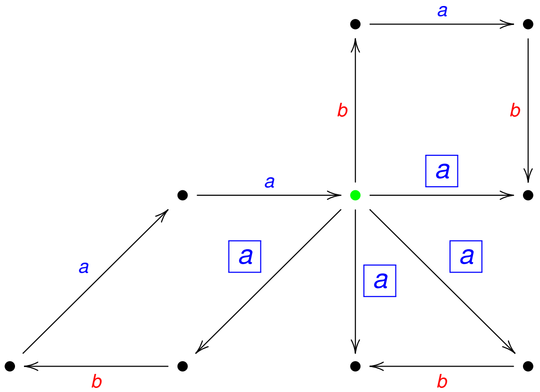

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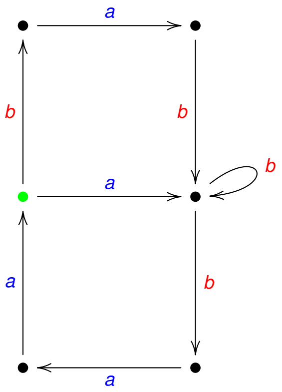

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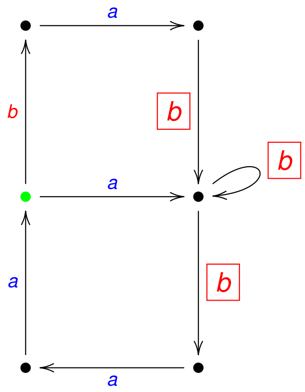

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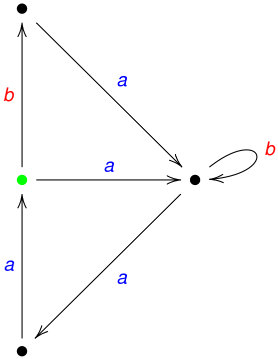

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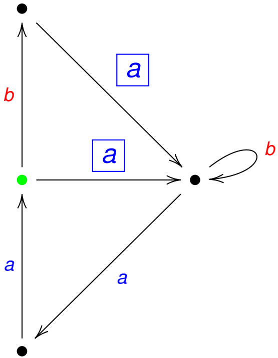

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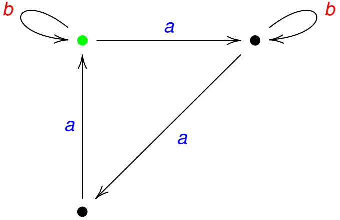

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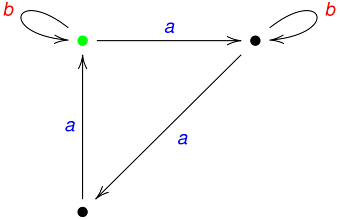

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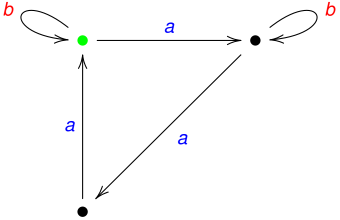

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Local confluence

It can be shown that

Proposition

The automaton $\Gamma(H)$ does not depend on the sequence of foldings.

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The automaton $\Gamma(H)$ does not depend on the generators of H.

Theorem

The following is a bijection:

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Corollary (Nielsen-Schreier)

Every subgroup of F_A is free.

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Outline

Counting Stallings graphs

From now on, let us think presentations as

$$\langle a_1, \ldots, a_r \mid \Gamma \rangle,$$

where Γ *is a Stallings graph.*

The natural size function to consider is the number of vertices:

$$\begin{array}{rcl} |\cdot|: \{\textit{Stallings graphs}\} & \to & \mathbb{N}, \\ \Gamma & \mapsto & \# V\Gamma. \end{array}$$

Goal: Count (estimate) the number of Stallings graphs with $\leqslant n$ *vertices, satisfying a certain property* $\mathcal{P}$.

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Stallings' graphs as partial injections

Definition

Let Γ be a Stallings graph. Every letter in A determines a partial injection of the set of vertices VΓ: $a(i) = j \quad \text{iff} \quad i \xrightarrow{a} j$.

$\Gamma =$

$a: V$	$\to$	V	$b: V$	$\to$	V	$c: V$	$\to$	V
1	$\mapsto$	1	1	$\mapsto$	2	1		
2	$\mapsto$	3	2			2		
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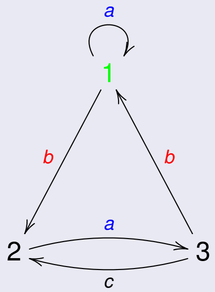

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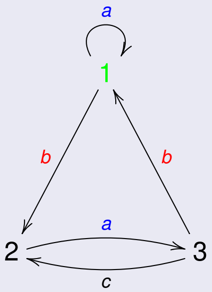

$a\colon V$	$\to$	V
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2	$\mapsto$	3
3		

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2		
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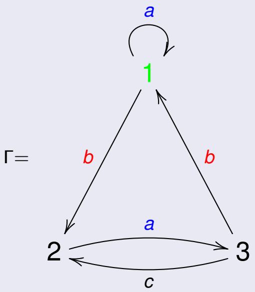
Observation

And the r partial injections $a_1, \ldots, a_r$ determine back the graph Γ.

Stallings' graphs as partial injections

Definition

Let Γ be a Stallings graph. Every letter in A determines a partial injection of the set of vertices $V\Gamma$: $a(i) = j$ *iff* $i \xrightarrow{a} j$.



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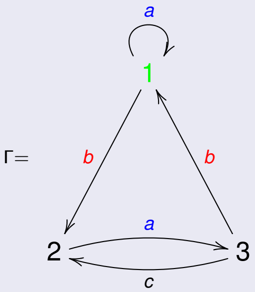

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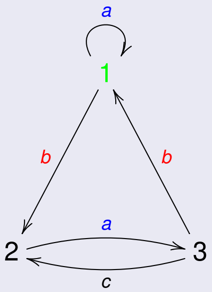
$\Gamma =$ 

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Let I_n be the set of partial injections of $[n] = \{1, 2, \ldots, n\}$ (this is a monoid containing the symmetric group S_n).

A Stallings graph (over A) with n vertices can be thought as a r-tuple of partial injections on the set $[n]$ (taking 1 as the base-point), $\sigma \in I_n^r$, such that

- *the corresponding graph $\Gamma(\sigma)$ is connected,*
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With the word-based distribution malnormality is exponentially generic ...

Proposition

$$\exists \quad \lim_{n\to\infty} \frac{|\{(w_1,\ldots,w_k)\in B(n)^k \mid \langle w_1,\ldots,w_k\rangle \textit{ is malnormal in } F(A)\}|}{|B(n)|^k} = 1$$

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Outline

Permutations and fragmented permutations

Observation

Any partial injection $\sigma \in I_n$ decomposes in orbits of two types: closed and open (i.e. cycles and segments).

Definition

A partial injection $\sigma \in I_n$ is called a

- *permutation if all its orbits are closed,*
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Let S_n and J_n, resp., be the sets of permutations and fragmented permutations in I_n.

Observation

Every partial injection is the disjoint union of a permutation and a fragmented permutation. In particular, $|I_n| = \sum_{k=0}^{n} \binom{n}{k} |S_k||J_{n-k}| = \sum_{k=0}^{n} \frac{n!}{(n-k)!} |J_{n-k}|$.

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Let $\sigma = (\sigma_1, \ldots, \sigma_r) \in I_n{}^r$*, let* $\Gamma(\sigma)$ *be the corresponding (Stallings) graph, and let* $G = \langle a_1, \ldots, a_r \mid \pi(\Gamma(\sigma)) \rangle$*. We have,*

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Let $\sigma = (\sigma_1, \ldots, \sigma_r) \in {I_n}^r$, let $\Gamma(\sigma)$ be the corresponding (Stallings) graph, and let $G = \langle a_1, \ldots, a_r \mid \pi(\Gamma(\sigma)) \rangle$. We have,

- *if* $\mathrm{gcd}(\sigma_i) = 1$ *then $a_i = 1$ in G,*
- *if* $\mathrm{gcd}(\sigma_1) = \cdots = \mathrm{gcd}(\sigma_r) = 1$ *then $G = 1$.*

Most groups are trivial

Definition

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Theorem (Bassino, Martino, Nicaud, V., Weil, 2010)

$$\frac{|\{\sigma \in I_n \mid \mathrm{gcd}(\sigma) > 1\}|}{|I_n|} = \mathcal{O}(\frac{1}{n^{1/6}})$$

Corollary (Bassino, Martino, Nicaud, V., Weil, 2010)

$$\frac{|\{\sigma \in {I_n}^r \mid \Gamma(\sigma) \textit{ St. gr. \& } G \neq 1\}|}{|I_n^r|} = \mathcal{O}(\frac{1}{n^{1/6}}).$$

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Thanks