Automata and Group Theory

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Universitat Politècnica de Catalunya

AutoMatha ABCD Workshop, Bratislava,

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Outline

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Definitions and notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
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The universal property

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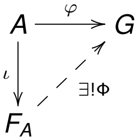

commutes (where ι is the inclusion map).

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$$G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle = F_A / \ll r_1, \ldots, r_m \gg .$$

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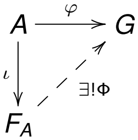

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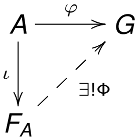

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Comparison with linear algebra

vector spaces	free groups
• K^n f.d. K-vector space	• F_n f.g. free group
• Every f.d. K-vector space is like this,	• Every group G is a quotient of a free group,
• $K^n \simeq K^m \Leftrightarrow n = m$,	• $F_n \simeq F_m \Leftrightarrow n = m$,
• –	• (Nielsen-Schreier) Every subgroup of a free group is free,
• Steinitz Lemma,	• Not true,
• $F \leqslant E \Rightarrow \dim F \leqslant \dim E$,	• Very false: $F_{\aleph_0} \leqslant F_2$.
• A basis	• The A-Stallings automata

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A Stallings automata is a finite A-labeled oriented graph with a distinguished vertex, (X, v), such that:

1- X *is connected,*

2- *no vertex of degree 1 except possibly* v *(*X *is a core-graph),*

3- *no two edges with the same label go out of (or in to) the same vertex.*

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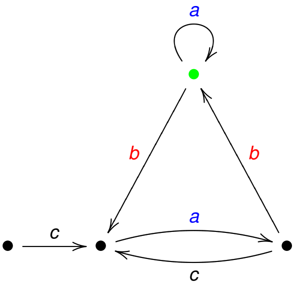

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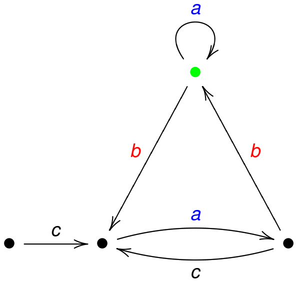

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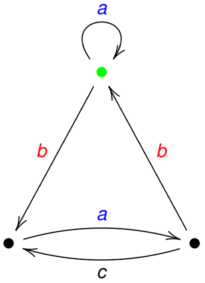

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Stallings (building on previous works) gave a bijection between finitely generated subgroups of F_A and Stallings automata:

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To any given (Stallings) automaton (X, v), we associate its fundamental group:

$$\pi(X, v) = \{ \text{ labels of closed paths at } v\} \leqslant F_A,$$

clearly, a subgroup of F_A.

$X=$

$$\pi(X, \bullet) = \{1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \ldots\}$$

$$\pi(X, \bullet) \not\ni bc^{-1}bcaa$$

Membership problem in $\pi(X, \bullet)$ is solvable.

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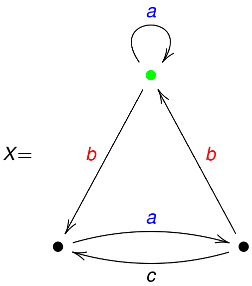

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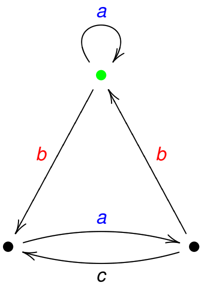

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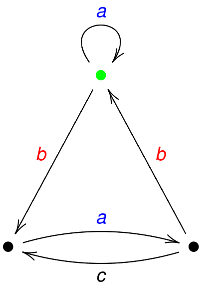

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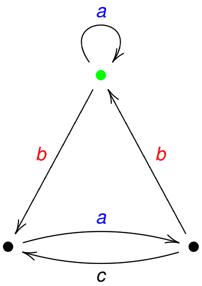
$X=$ 

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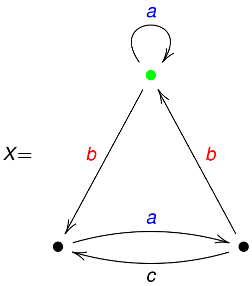

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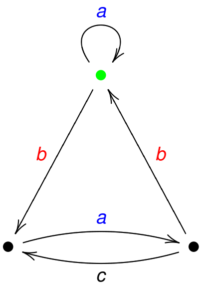

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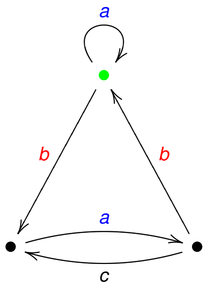
$X=$ 

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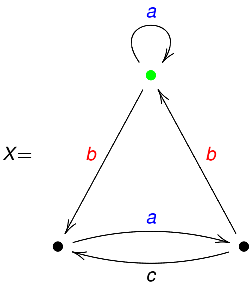

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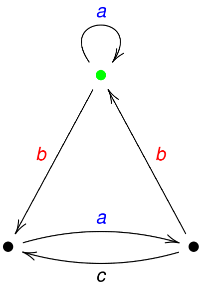
$X=$ 

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A basis for $\pi(X, v)$

Proposition

For every Stallings automaton (X, v), the group $\pi(X, v)$ is free of rank $rk(\pi(X, v)) = 1 - |VX| + |EX|$.

Proof:

- Take a maximal tree T in X.
- Write $T[p, q]$ for the geodesic (i.e. the unique reduced path) in T from p to q.
- For every $e \in EX - ET$, $x_e = label(T[v, \iota e] \cdot e \cdot T[\tau e, v])$ belongs to $\pi(X, v)$.
- Not difficult to see that $\{x_e \mid e \in EX - ET\}$ is a basis for $\pi(X, v)$.
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$$\begin{aligned} |EX - ET| &= |EX| - |ET| \\ &= |EX| - (|VT| - 1) = 1 - |VX| + |EX|. \; \square \end{aligned}$$

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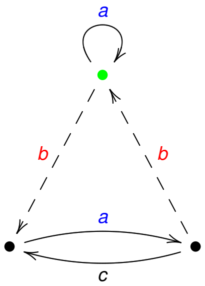
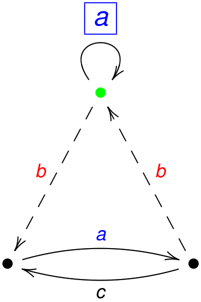


$H = \langle \quad \rangle$

Example



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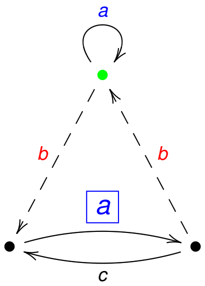

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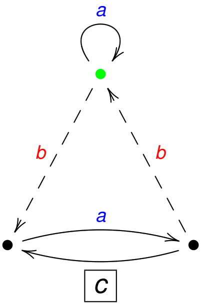

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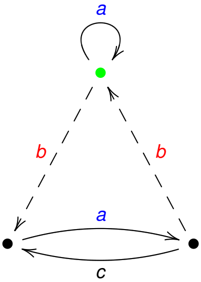
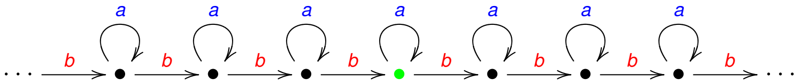


$H = \langle a,\ bab,\ b^{-1}cb^{-1} \rangle$
$rk(H) = 1 - 3 + 5 = 3.$

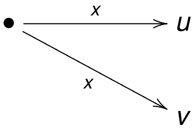
Example-2



$$F_{\aleph_0} \simeq H = \langle \ldots, b^{-2}ab^2, b^{-1}ab, a, bab^{-1}, b^2ab^{-2}, \ldots \rangle \leqslant F_2.$$

Constructing the automata from the subgroup

In any automaton containing the following situation, for $x \in A^{\pm 1}$,



we can fold and identify vertices u and v to obtain

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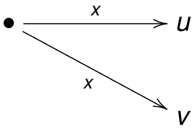

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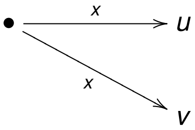

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Lemma (Stallings)

If $(X, v) \rightsquigarrow (X', v')$ is a Stallings folding then $\pi(X, v) = \pi(X', v')$.

Given a f.g. subgroup $H = \langle w_1, \ldots w_m \rangle \leqslant F_A$ (we assume w_i are reduced words), do the following:

1- *Draw the flower automaton,*

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Well defined?

Need to see that the output does not depend on the process...

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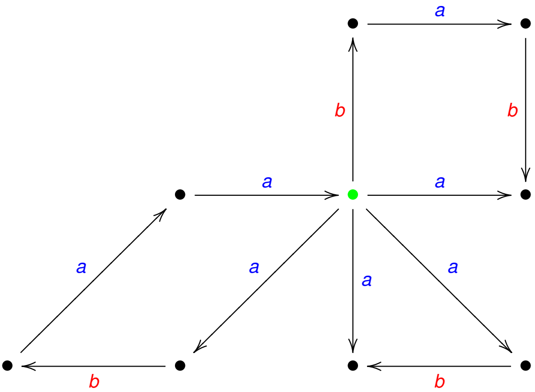

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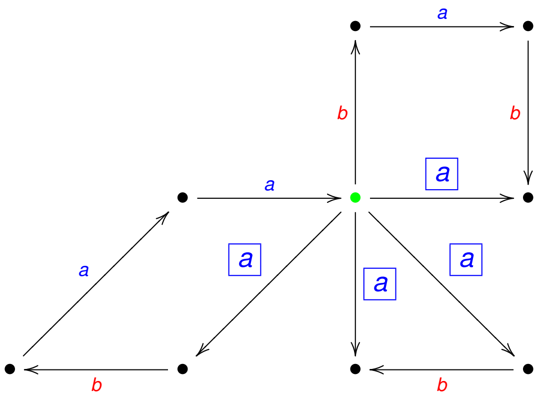

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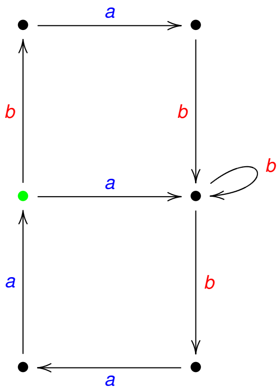

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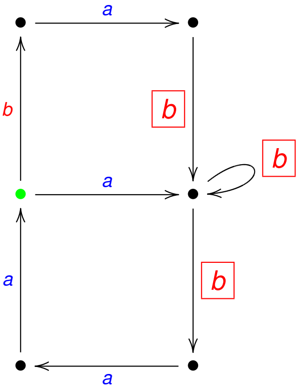

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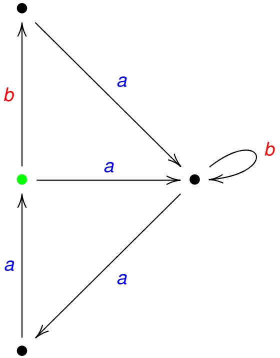

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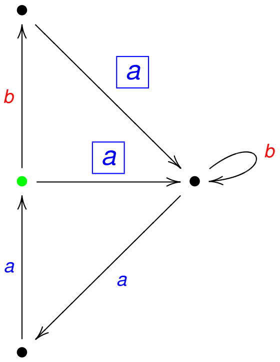

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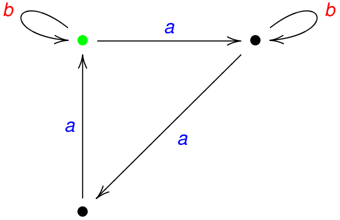

Folding #3. $\Gamma(H)$

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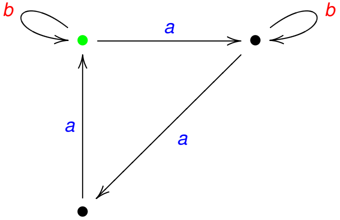

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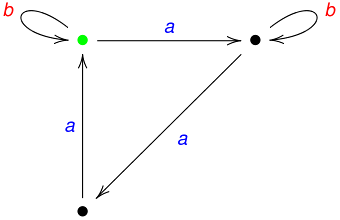

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Local confluence

Proposition

The automaton $\Gamma(H)$ does not depend on the sequence of foldings

Proof:

- Suppose $(X, v) \rightsquigarrow (X', v')$ is a single folding of 2 edges
- If $\begin{array}{l} p \xrightarrow{x} q \\ p \xrightarrow{x} r \end{array}$ in (X, v), then $\begin{array}{l} p' \xrightarrow{x} q' \\ p' \xrightarrow{x} r' \end{array}$ in (X', v') (possibly with $q' = r'$).
- So, we get local confluence:

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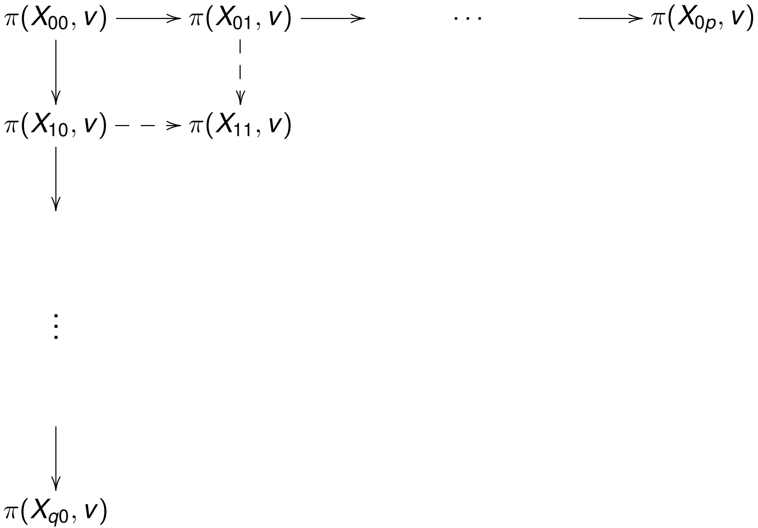
The confluence grid



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Independence from the generators

Proposition

The automaton $\Gamma(H)$ does not depend on the generators of H.

Proof:

- Suppose $H = \langle w_1, \ldots, w_p \rangle = \langle w'_1, \ldots, w'_q \rangle$ and let $\Gamma(H)$ and $\Gamma'(H)$ be the Stallings automata obtained from each set of generators.
- Consider the double flower

whose fundamental group is $\langle w_1, \ldots, w_p, w'_1, \ldots, w'_q \rangle = H$.

- Now, fold in the two natural ways:

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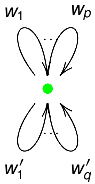

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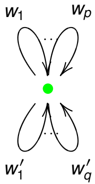

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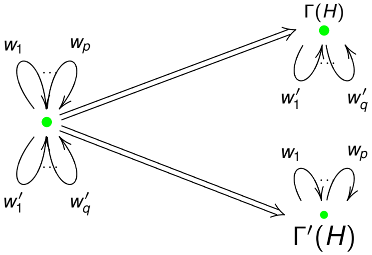

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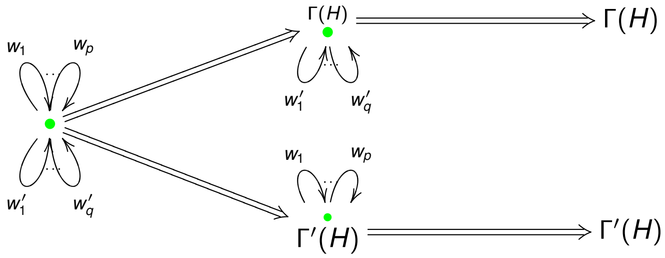

Lemma (Useless-w)

If $H \leqslant_{fg} F_A$ and $w \in H$ then, attaching a petal labeled w to the basepoint of $\Gamma(H)$ and folding, we obtain again $\Gamma(H)$.

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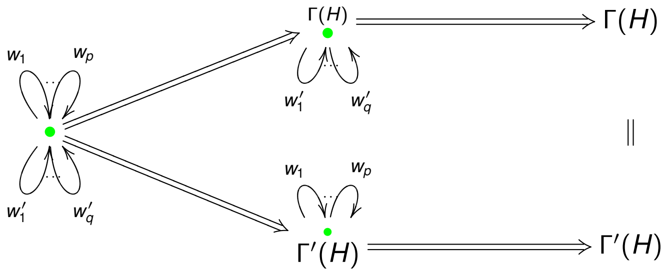

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Theorem

The following is a bijection:

$$\begin{array}{rcl} \{\textit{f.g. subgroups of } F_A\} & \longleftrightarrow & \{\textit{Stallings automata}\} \\ H & \to & \Gamma(H) \\ \pi(X, v) & \leftarrow & (X, v) \end{array}$$

Proof:

- By Stallings Lemma, it is clear that $\pi(\Gamma(H)) = H$.
- Let (X, v) be a Stallings automata, and $\pi(X, v) = = \langle w_1, \ldots, w_p \rangle$.
- Let (Y, v) be the automata obtained by attaching petals labeled $w_1, \ldots, w_p$ to the vertex v of (X, v).
- By the useless-w Lemma, (Y, v) can be folded to both (X, v) and $\Gamma(\pi(X, v))$. And both are completely folded. Hence, $\Gamma(\pi(X, v)) = (X, v)$.

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Corollary (Nielsen-Schreier)

Every subgroup of F_A is free.

- We have proved the finitely generated case, but everything extends easily to the general case.
- The original proof (1920's) is combinatorial and much more technical.

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Outline

(Membership)

Does w belong to $H = \langle w_1, \ldots, w_m \rangle$?

- Construct $\Gamma(H)$,
- Check whether w is readable as a closed path in $\Gamma(H)$ (at the basepoint).

(Containment)

Given $H = \langle w_1, \ldots, w_m \rangle$ and $K = \langle v_1, \ldots, v_n \rangle$, is $H \leqslant K$?

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(Computing a basis)

Given $H = \langle w_1, \ldots, w_m \rangle$, find a basis for H.

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Given $H = \langle w_1, \ldots, w_m \rangle$ and $K = \langle v_1, \ldots, v_n \rangle$, are they conjugate (i.e. $H^x = K$ for some $x \in F_A$) ?

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Given $H = \langle w_1, \ldots, w_m \rangle$, is $H \leqslant_{f.i.} F_A$? If yes, find a set of coset representatives.

→ For $u \in V\Gamma(H)$, choose p (the label of) a path from • to u; then,

$$\{\text{labels of paths from } \bullet \text{ to } u\} = \pi(\Gamma(H), \bullet) \cdot p = H \cdot p$$

is a coset of F_A/H.

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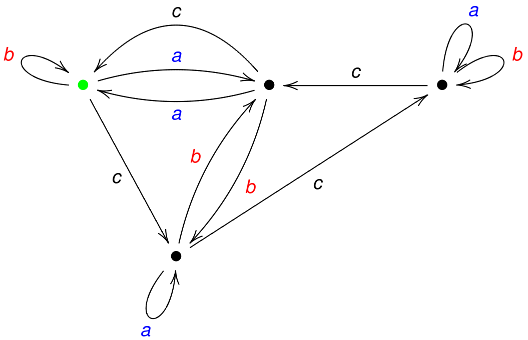
Example

$$H = \langle b, ac, c^{-1}a, cac^{-1}, c^{-1}bc^{-1}, cbc, c^4, c^2ac^{-2}, c^2bc^{-2}\rangle$$

$\Gamma(H) =$



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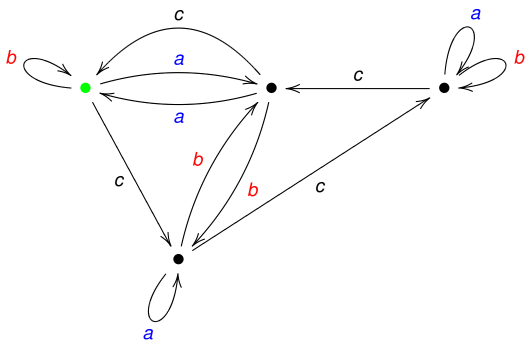
$\Gamma(H) =$ 

$F_3 = H \sqcup Hc \sqcup Ha \sqcup Hac^{-1}.$

More on finite index

(Schreier index formula)

If $H \leqslant_{f.i.} F_A$ is of index $[F:H]$, then $r(H) = 1 + [F:H] \cdot (r(F_A) - 1)$.

Proof:

$$\begin{aligned} r(H) &= 1 - |V\Gamma(H)| + |E\Gamma(H)| = 1 - |V\Gamma(H)| + |A| \cdot |V\Gamma(H)| \\ &= 1 + |V\Gamma(H)| \cdot (|A| - 1) = 1 + [F:H] \cdot (r(F_A) - 1). \quad \square \end{aligned}$$

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Every f.g. subgroup $H \leqslant_{fg} F_A$ is a free factor of a finite index one, $H \leqslant_{ff} H * L \leqslant_{f.i.} F_A$.

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Every f.g. subgroup $H \leqslant_{fg} F_A$ is a free factor of a finite index one, $H \leqslant_{ff} H * L \leqslant_{f.i.} F_A$.

Proof:

- Compute $\Gamma(H)$ from a generating set,
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More on finite index

(Schreier index formula)

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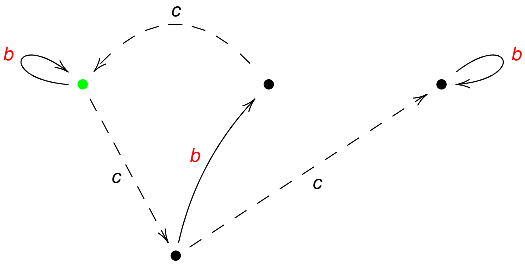
$H = \langle b, cbc, c^2bc^{-2} \rangle$

$\Gamma(H) =$ 

$H \leqslant_{ff} H * \langle \quad \rangle$

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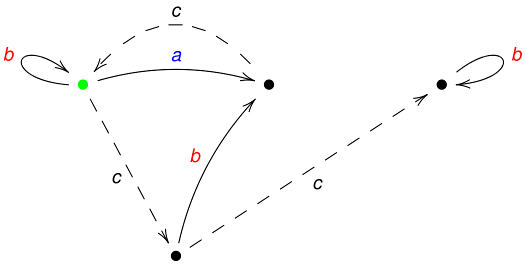
$\Gamma(H) =$ 

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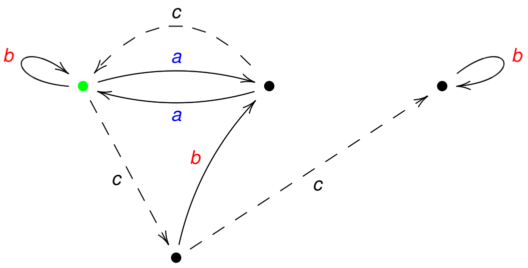

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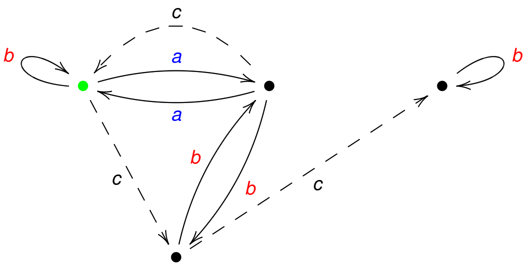

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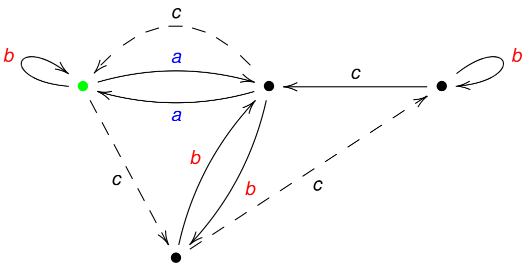
$\Gamma(H) =$ 

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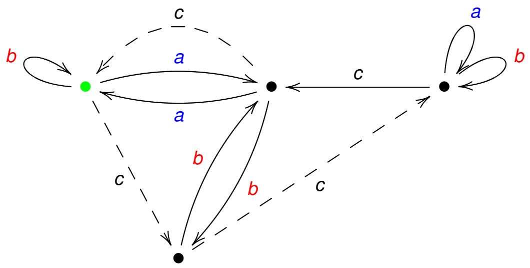

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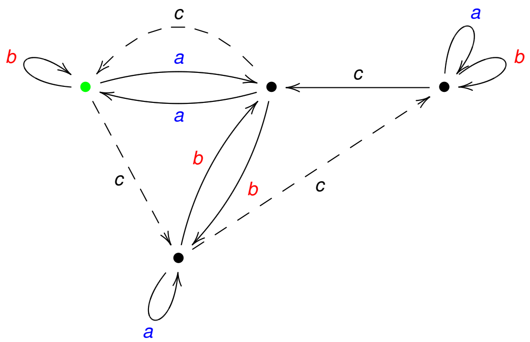
$\Gamma(H) =$ 

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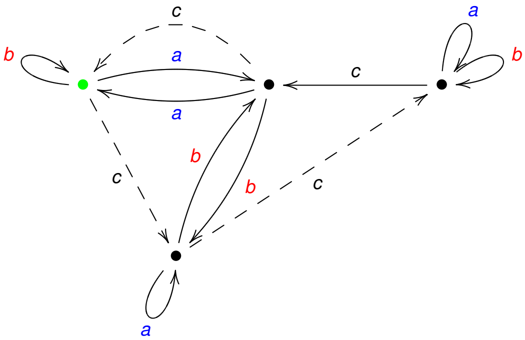

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Pull-back of automata

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The pull-back of two Stallings automata, (X, v) and (Y, w), is the cartesian product $(X \times Y, (v, w))$ (respecting labels). This is not in general connected, neither without degree 1 vertices, but it is folded.

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For every f.g. subgroups $H, K \leqslant_{fg} F_A$, $\Gamma(H \cap K)$ coincides with the connected component of $\Gamma(H) \times \Gamma(K)$ containing the basepoint, after trimming.

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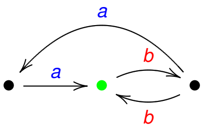

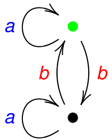

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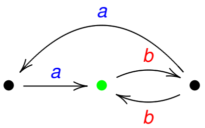

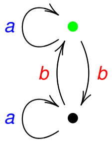

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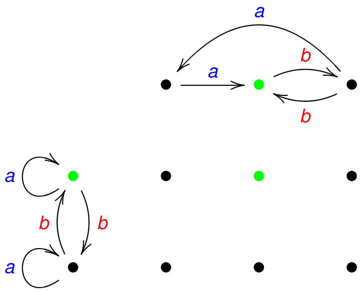

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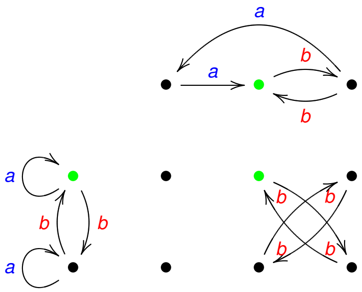

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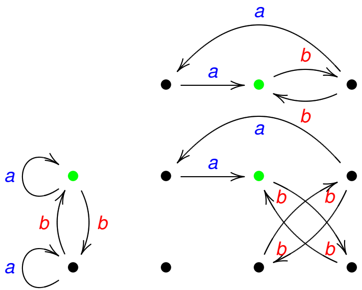

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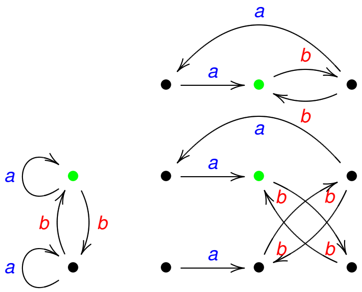

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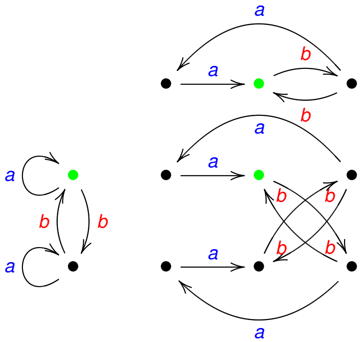

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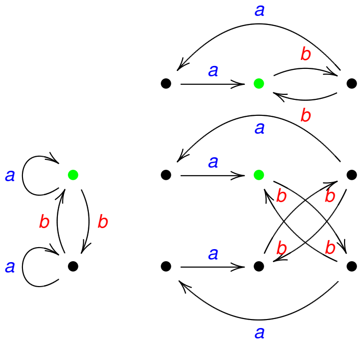

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But the intersection can have bigger rank: "$3 = 3 \cap 2 \leqslant 2$"

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- HNC holds if H (or K) has rank 1 (immediate),
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Outline

Free and algebraic extensions

Definition

And extension of subgroups $H \leqslant K$, in F_A is called

- *a free extension if H is a free factor of K (i.e. $K = H * L$ for some $L \leqslant F_A$), denoted $H \leqslant_{ff} K$;*
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- $\langle a \rangle \leqslant_{ff} \langle a, b \rangle \leqslant_{ff} \langle a, b, c \rangle$, and $\langle x^r \rangle \leqslant_{alg} \langle x \rangle$, $\forall x \in F_A \, \forall r \in \mathbb{Z}$.
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Free and algebraic extensions

Definition

And extension of subgroups $H \leqslant K$, in F_A is called

- *a free extension if H is a free factor of K (i.e. $K = H * L$ for some $L \leqslant F_A$), denoted $H \leqslant_{ff} K$;*
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Theorem (Takahasi, 1951)

For every $H \leqslant_{fg} F_A$, the set of algebraic extensions, denoted $\mathcal{AE}(H)$, is finite.

- Original proof by Takahasi was combinatorial and technical,
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The modern proof

Proof:

- Let us (temporarily) attach some "hairs" to $\Gamma(H)$ an denote the resulting (folded) automata by $\tilde{\Gamma}(H)$.
- Given $H \leqslant K$ (both f.g.), we can obtain $\Gamma(K)$ from $\Gamma(H)$ by 1) adding the appropriate hairs, 2) identifying several vertices to $\bullet$, 3) folding; (note that adding extra hairs, the result will be the same if we 4) trim at the end).
- Hence, if $H \leqslant K$ (both f.g.) then $\Gamma(K)$ contains as a subgraph either $\Gamma(H)$ or some quotient of it (i.e. $\Gamma(H)$ after identifying several sets of vertices $(\sim)$ and then folding, $\Gamma(H)/\sim$).
- The overgroups of H: $\mathcal{O}(H) = \{\pi(\Gamma(H)/\sim, \bullet) \mid \sim \text{ is a partition of } V\Gamma(H)\}$.
- Hence, for every $H \leqslant K$, there exists $L \in \mathcal{O}(H)$ such that $H \leqslant L \leqslant_{ff} K$.
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Corollary

$\mathcal{AE}(H)$ is computable.

Proof:

- Compute $\Gamma(H)$,
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- Compute $\mathcal{O}(H)$,
- Clean $\mathcal{O}(H)$ by detecting all pairs $K_1, K_2 \in \mathcal{O}(H)$ such that $K_1 \leqslant_{ff} K_2$ and deleting K_2.
- The resulting set is $\mathcal{AE}(H)$. $\square$

For the cleaning step we need:

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Deciding free-factorness

Proposition

Given $H, K \leqslant F_A$, it is algorithmically decidable whether $H \leqslant_{ff} K$ or not.

Proved by:

- Whitehead 1930's (classical and exponential),
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The algebraic closure

Observation

If $H \leqslant_{alg} K_1$ *and* $H \leqslant_{alg} K_2$ *then* $H \leqslant_{alg} \langle K_1 \cup K_2 \rangle$.

Corollary

For every $H \leqslant K \leqslant F_A$ *(all f.g.),* $\mathcal{AE}_K(H)$ *has a unique maximal element, called the* K*-algebraic closure of* H*, and denoted* $Cl_K(H)$.

Corollary

Every extension $H \leqslant K$ *of f.g. subgroups of* F_A *splits, in a unique way, in an algebraic part and a free factor part,* $H \leqslant_{alg} Cl(H) \leqslant_{ff} K$.

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