# The lattice of subgroups of a free groups: algorithmic aspects

**Enric Ventura**

Departament de Matemàtica Aplicada III

Universitat Politècnica de Catalunya

VIII Encuentro de Teoria de Grupos

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# Outline

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1. The friendly and unfriendly free group
2. The bijection between subgroups and automata
3. Several algorithmic applications
4. Recent applications

# Definitions and notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet ($n$ letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- Usually, $A = \{a, b, c\}$.
- $(A^{\pm 1})^*$ the free monoid on $A^{\pm 1}$ (words on $A^{\pm 1}$).
- 1 denotes the empty word, and we have the notion of length.
- $\sim$ is the eq. rel. generated by $a_i a_i^{-1} \sim a_i^{-1} a_i \sim 1$.
- $F_A = (A^{\pm 1})^* / \sim$ is the free group on $A$ (words on $A^{\pm 1}$ modulo $\sim$).
- Every $w \in A^*$ has a unique reduced form, denoted $\overline{w}$, (clearly $w = \overline{w}$ in $F_A$, and $\overline{w}$ is the shortest word with this property). We also say $\overline{w}$ is a reduced word.

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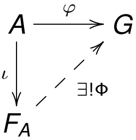
# The universal property

- The universal property: given a group $G$ and a mapping $\varphi \colon A \to G$, there exists a unique group homomorphism $\Phi \colon F_A \to G$ such that the diagram




  commutes (where $\iota$ is the inclusion map).
- Every group is a quotient of a free group

$$G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle = F_A / \ll r_1, \ldots, r_m \gg .$$

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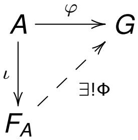


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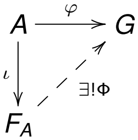


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**vector spaces**

- $K^n$ f.d. $K$-vector space
- Every f.d. $K$-vector space is $\simeq K^n$, some $n$,
- $K^n \simeq K^m \Leftrightarrow n = m$,
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- Steinitz Lemma,
- $F \leqslant E \Rightarrow \dim F \leqslant \dim E$,
- A basis

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*A* *Stallings automata* *is a finite A-labeled oriented graph with a distinguished vertex,* $(X, v)$*, such that:*

1- $X$ *is connected,*

2- *no vertex of degree 1 except possibly* $v$ *(*$X$ *is a* *core-graph**),*

3- *no two edges with the same label go out of (or in to) the same vertex.*

NO :




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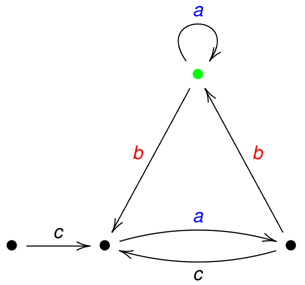


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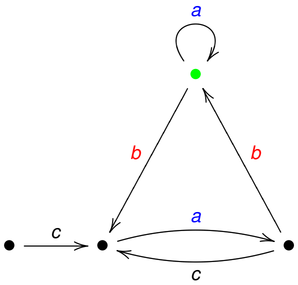


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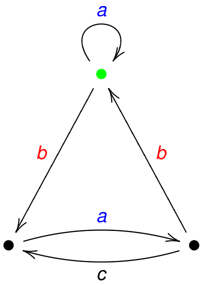

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Stallings (building on previous works) gave a bijection between finitely generated subgroups of $F_A$ and Stallings automata:

$$\{\text{f.g. subgroups of } F_A\} \quad \longleftrightarrow \quad \{\text{Stallings automata}\},$$

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*To any given (Stallings) automaton $(X, v)$, we associate its fundamental group:*

$$\pi(X, v) = \{ \text{labels of closed paths at } v\} \leqslant F_A,$$

*clearly, a subgroup of $F_A$.*




$\pi(X, \bullet) = \{1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \ldots\}$

$\pi(X, \bullet) \not\ni bc^{-1}bcaa$

Membership problem in $\pi(X, \bullet)$ is solvable.

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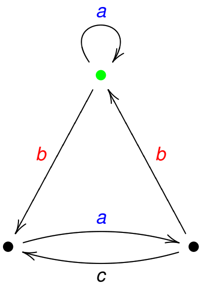


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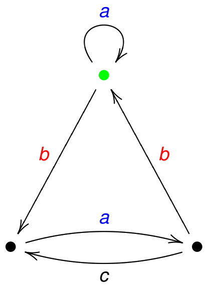


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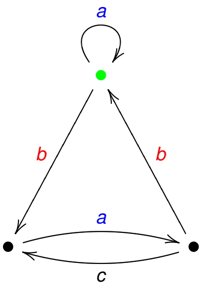
$X=$ 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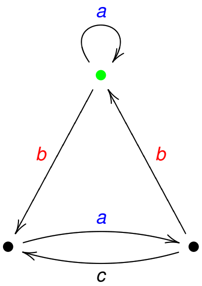
$X=$ 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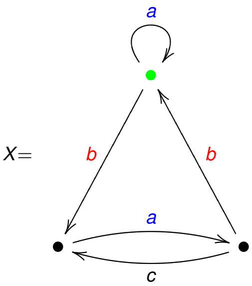


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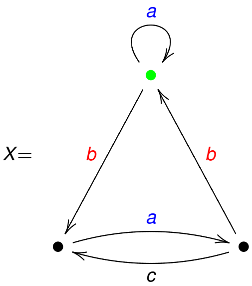


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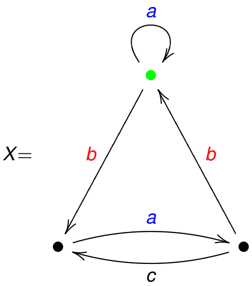


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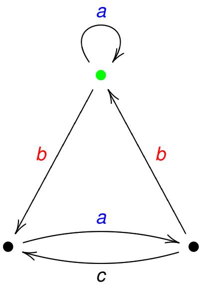


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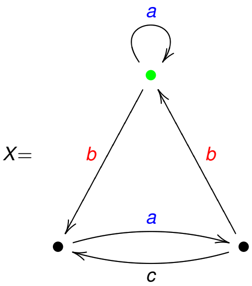


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## Proposition

*For every Stallings automaton $(X, v)$, the group $\pi(X, v)$ is free of rank $rk(\pi(X, v)) = 1 - |VX| + |EX|$.*

**Proof:**

- Take a maximal tree $T$ in $X$.
- Write $T[p, q]$ for the geodesic (i.e. the unique reduced path) in $T$ from $p$ to $q$.
- For every $e \in EX - ET$, $x_e = label(T[v, \iota e] \cdot e \cdot T[\tau e, v])$ belongs to $\pi(X, v)$.
- Not difficult to see that $\{x_e \mid e \in EX - ET\}$ is a basis for $\pi(X, v)$.
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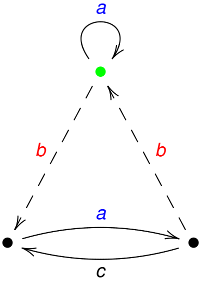


$H = \langle \quad \rangle$

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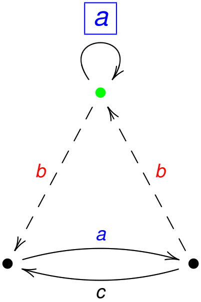


$H = \langle a, \quad \rangle$

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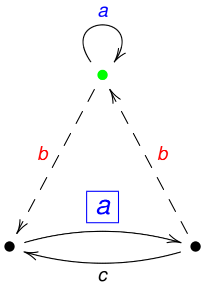


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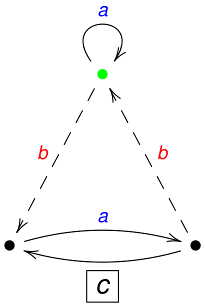


$H = \langle a,\ bab,\ b^{-1}cb^{-1} \rangle$

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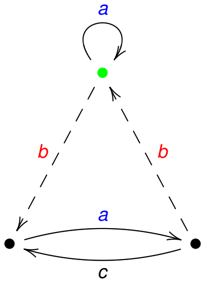
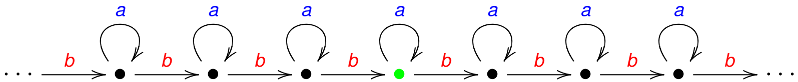



$H = \langle a,\ bab,\ b^{-1}cb^{-1} \rangle$
$rk(H) = 1 - 3 + 5 = 3.$

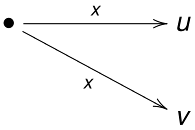
# Example-2




$$F_{\aleph_0} \simeq H = \langle \ldots, b^{-2}ab^2, b^{-1}ab, a, bab^{-1}, b^2ab^{-2}, \ldots \rangle \leqslant F_2.$$

# Constructing the automata from the subgroup

In any automaton containing the following situation, for $x \in A^{\pm 1}$,

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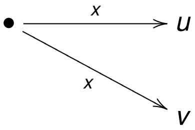


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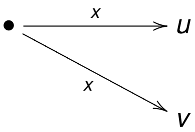


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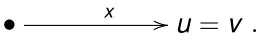


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*If $(X, v) \rightsquigarrow (X', v')$ is a Stallings folding then $\pi(X, v) = \pi(X', v')$.*

*Given a f.g. subgroup $H = \langle w_1, \ldots, w_m \rangle \leqslant F_A$ (we assume $w_i$ are reduced words), do the following:*

1- *Draw the flower automaton,*

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Well defined?

Need to see that the output does not depend on the process...

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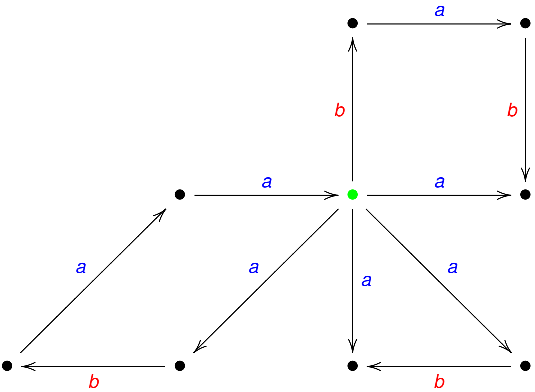


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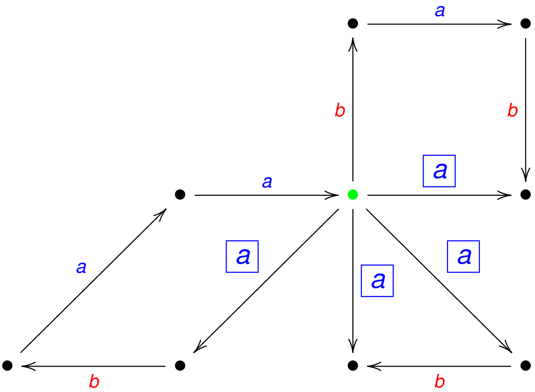


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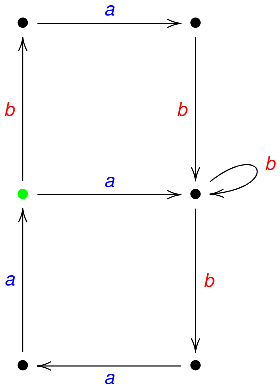


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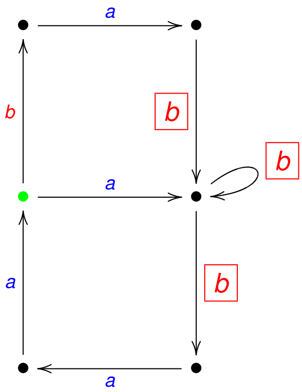


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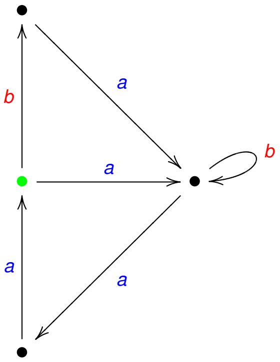


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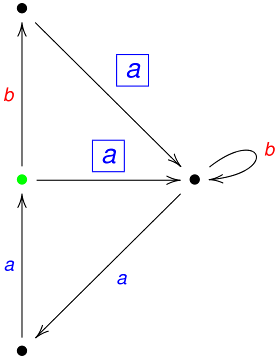


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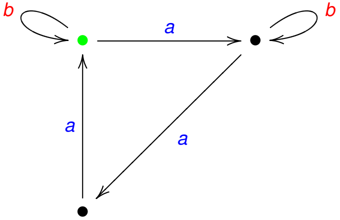


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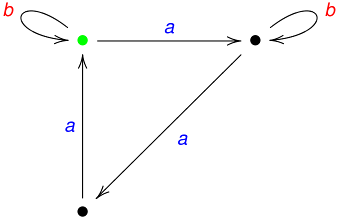


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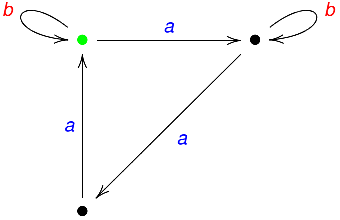


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By Stallings Lemma, $\pi(\Gamma(H), \bullet) = \langle baba^{-1}, aba^{-1}, aba^2 \rangle$
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# The bijection

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*The automaton $\Gamma(H)$ does not depend on the sequence of foldings*

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*The following is a bijection between f.g subgroups and Stallings automata:*

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## Corollary (Nielsen-Schreier)

*Every subgroup of $F_A$ is free.*

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# Outline

# Membership & containment

**(Membership)**

*Does $w$ belong to $H = \langle w_1, \ldots, w_m \rangle$ ?*

- Construct $\Gamma(H)$,
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# Finite index subgroups

## (Finite index)

*Given $H = \langle w_1, \ldots, w_m \rangle$, is $H \leqslant_{f.i.} F_A$ ? If yes, find a set of coset representatives.*

→ For $u \in V\Gamma(H)$, choose $p$ (the label of) a path from • to $u$; then,

$$\{\text{labels of paths from } \bullet \text{ to } u\} = \pi(\Gamma(H), \bullet) \cdot p = H \cdot p$$

is a coset of $F_A/H$.

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$$H = \langle b, ac, c^{-1}a, cac^{-1}, c^{-1}bc^{-1}, cbc, c^4, c^2ac^{-2}, c^2bc^{-2} \rangle \leqslant \langle a,\ b,\ c \rangle$$

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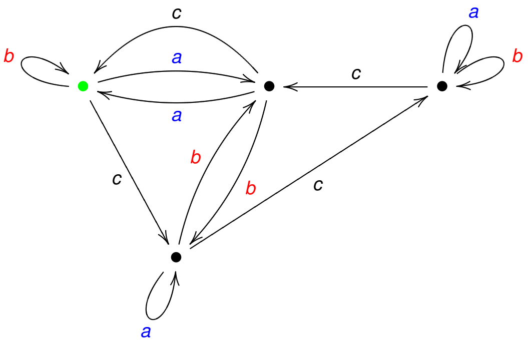
$\Gamma(H) =$ 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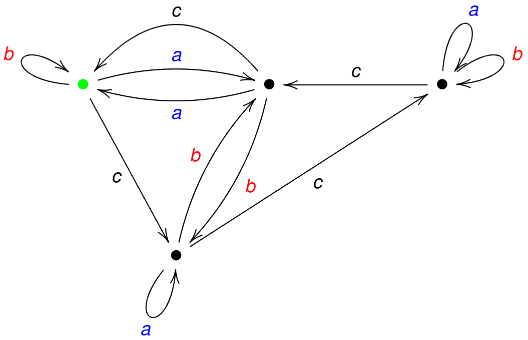
$\Gamma(H) =$ 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# More on finite index

## (Schreier index formula)

*If $H \leqslant_{f.i.} F_A$ is of index $[F : H]$, then $r(H) = 1 + [F : H] \cdot (r(F_A) - 1)$.*

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$H = \langle b, cbc, c^2bc^{-2} \rangle \leqslant \langle a, b, c \rangle = F_3$

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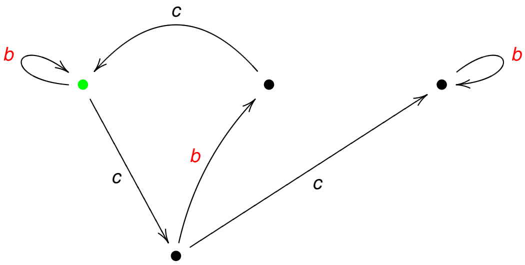


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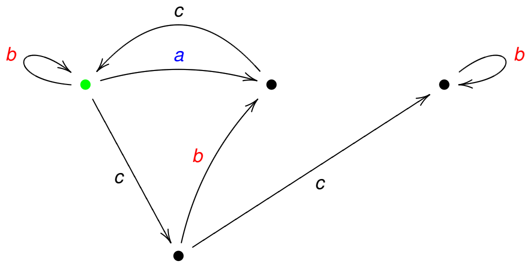
$\Gamma(H) =$ 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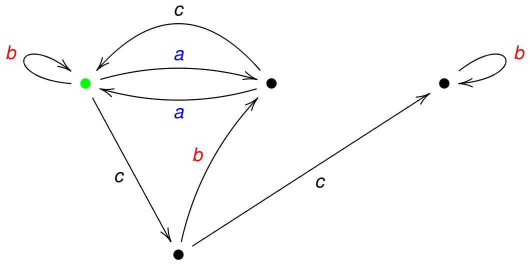
$\Gamma(H) =$ 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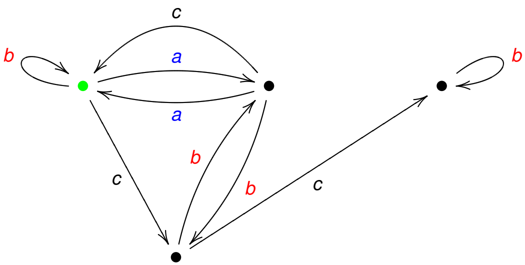


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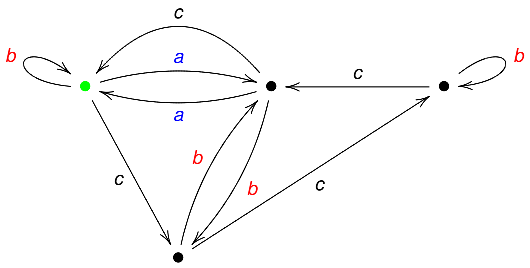
$\Gamma(H) =$ 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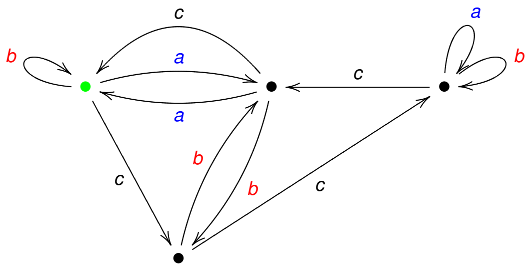
$\Gamma(H) =$ 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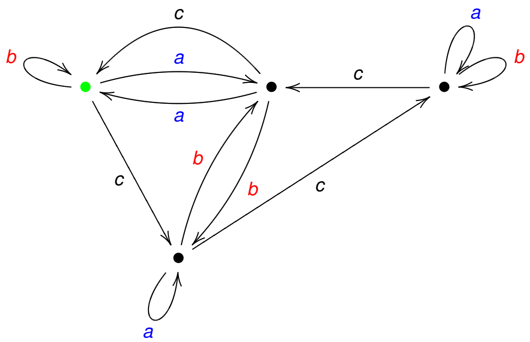


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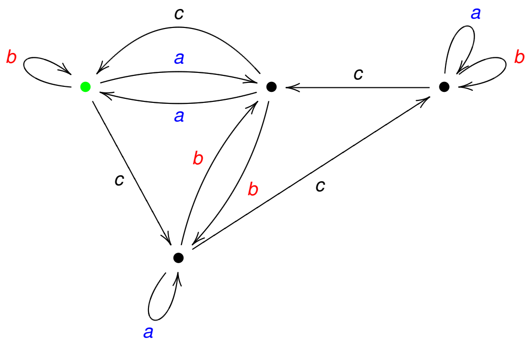
$\Gamma(H) =$ 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## Definition

*The pull-back of two Stallings automata, $(X, v)$ and $(Y, w)$, is the cartesian product $(X \times Y, (v, w))$, respecting labels. This is not in general connected, neither without degree 1 vertices, but it is folded.*

## Theorem (H. Neumann-Stallings)

*For every f.g. subgroups $H, K \leqslant_{fg} F_A$, $\Gamma(H \cap K)$ coincides with the connected component of $\Gamma(H) \times \Gamma(K)$ containing the basepoint, after trimming.*

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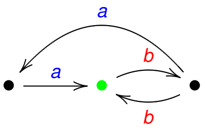


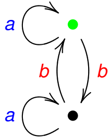


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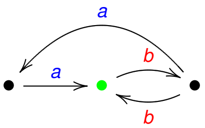


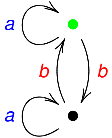


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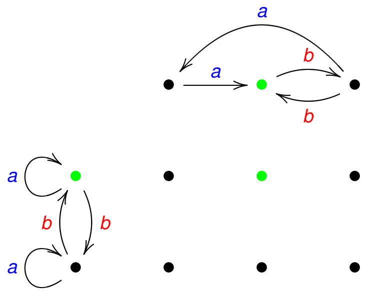


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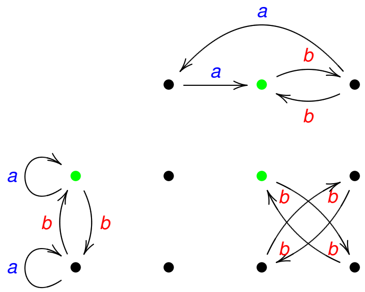


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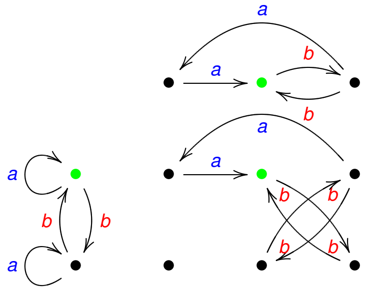


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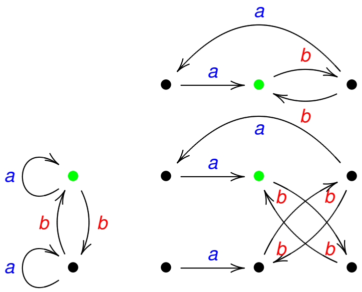


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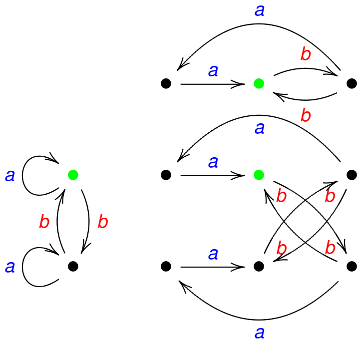


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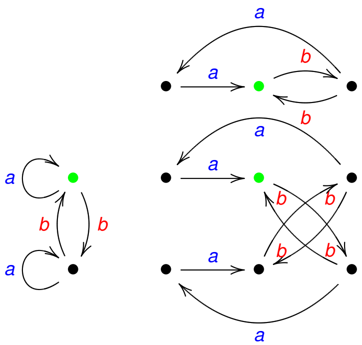


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**Theorem (Howson)**

*The intersection of finitely generated subgroups of $F_A$ is again finitely generated.*

But the intersection can have bigger rank: "$3 = 3 \cap 2 \leqslant 2$"

**Theorem (H. Neumann)**

$\tilde{r}(H \cap K) \leqslant 2\tilde{r}(H)\tilde{r}(K)$, *where* $\tilde{r}(H) = \max\{0, r(H) - 1\}$.

**Conjecture (H. Neumann)**

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# Outline

# Algebraic applications

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*Every extension $H \leqslant K$ of f.g. subgroups of $F_A$ splits, in a unique way, in an algebraic part and a free factor part, $H \leqslant_{alg} Cl(H) \leqslant_{ff} K$.*

**Theorem (Whitehead, '30)**

*Given $H, K \leqslant F_A$, it is algorithmically decidable whether $H \leqslant_{ff} K$ or not.*

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*There exist subgroups of $F_A$ which are fixed subgroups of endomorphisms of $F_A$ but are not the fixed subgroups of any automorphism.*

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*Given $H \leqslant_{f.g.} F_A$ it is decidable whether $H$ is the fixed subgroup of some family of automorphisms (endomorphisms) of $F_A$ and, in the affirmative case, find one.*

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*The p-closure of $H \leqslant_{f.g.} F_A$ is effectively computable, for all primes p.*

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*The nil-closure of $H \leqslant_{f.g.} F_A$ is the intersection, over all primes, of the p-closure of H. Hence, it is effectively computable.*

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