Twisted conjugacy for free groups and the conjugacy problem for some extensions of groups

Enric Ventura

Departament de Matemàtica Aplicada III

Universitat Politècnica de Catalunya

Algebra Topology Colloquium Series

Bates college, March 24th, 2009.

Outline

Outline

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\varphi\colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta\colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u\colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\varphi\colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta\colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u\colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\varphi \colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta \colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u \colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\varphi \colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta \colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u \colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\varphi\colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta\colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u\colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}(F_n) \subseteq \mathrm{End}(F_n)$.
- I let endomorphisms $\varphi \colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta \colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u \colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\varphi \colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta \colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u \colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- F_n is the free group on A.
- $\mathrm{Aut}\,(F_n) \subseteq \mathrm{End}\,(F_n)$.
- I let endomorphisms $\varphi\colon F_n \to F_n$ act on the right, $x \mapsto x\varphi$.
- So, compositions are $\alpha\beta\colon F_n \xrightarrow{\alpha} F_n \xrightarrow{\beta} F_n$, $x \mapsto x\alpha \mapsto x\alpha\beta$.
- conjugations: $\gamma_u\colon F_n \to F_n$, $x \mapsto u^{-1}xu$.
- $\mathrm{Fix}\,(\phi) = \{x \in F_n \mid x\phi = x\} \leqslant F_n$.

Definition

Two elements $u, v \in G$ are said to be conjugated, denoted $u \sim v$, if $v = g^{-1}ug$ for some $g \in G$.

Definition

The conjugacy problem for G, denoted $CP(G)$: "Given $u, v \in G$ decide whether $u \sim v$".

Definition

For $\varphi \in Aut(G)$, two elements $u, v \in G$ are said to be φ-twisted conjugated, denoted $u \sim_{\varphi} v$, if $v = (g\varphi)^{-1}ug$ for some $g \in G$.

Definition

The twisted conjugacy problem for G, denoted $TCP(G)$: "Given $\varphi \in Aut(G)$ and $u, v \in G$ decide whether $u \sim_{\varphi} v$".

Definition

Two elements $u, v \in G$ are said to be conjugated, denoted $u \sim v$, if $v = g^{-1}ug$ for some $g \in G$.

Definition

The conjugacy problem for G, denoted $CP(G)$: "Given $u, v \in G$ decide whether $u \sim v$".

Definition

For $\varphi \in Aut(G)$, two elements $u, v \in G$ are said to be φ-twisted conjugated, denoted $u \sim_{\varphi} v$, if $v = (g\varphi)^{-1}ug$ for some $g \in G$.

Definition

The twisted conjugacy problem for G, denoted $TCP(G)$: "Given $\varphi \in Aut(G)$ and $u, v \in G$ decide whether $u \sim_{\varphi} v$".

Definition

Two elements $u, v \in G$ are said to be conjugated, denoted $u \sim v$, if $v = g^{-1}ug$ for some $g \in G$.

Definition

The conjugacy problem for G, denoted $CP(G)$: "Given $u, v \in G$ decide whether $u \sim v$".

Definition

For $\varphi \in Aut(G)$, two elements $u, v \in G$ are said to be φ-twisted conjugated, denoted $u \sim_\varphi v$, if $v = (g\varphi)^{-1}ug$ for some $g \in G$.

Definition

The twisted conjugacy problem for G, denoted $TCP(G)$: "Given $\varphi \in Aut(G)$ and $u, v \in G$ decide whether $u \sim_\varphi v$".

Definition

Two elements $u, v \in G$ are said to be conjugated, denoted $u \sim v$, if $v = g^{-1}ug$ for some $g \in G$.

Definition

The conjugacy problem for G, denoted $CP(G)$: "Given $u, v \in G$ decide whether $u \sim v$".

Definition

For $\varphi \in Aut(G)$, two elements $u, v \in G$ are said to be φ-twisted conjugated, denoted $u \sim_\varphi v$, if $v = (g\varphi)^{-1}ug$ for some $g \in G$.

Definition

The twisted conjugacy problem for G, denoted $TCP(G)$: "Given $\varphi \in Aut(G)$ and $u, v \in G$ decide whether $u \sim_\varphi v$".

Not much is known about twisted conjugacy problem:

Theorem

Every finitely generated, virtually

(i) *abelian, or*

(ii) *free, or*

(iii) *surface, or*

(iv) *polycyclic*

group has solvable twisted conjugacy problem.

Theorem (work in progress)

- (w. J.Burillo & F.Matucci) *Thomson's group has solvable TCP,*
- (w. J.González-Meneses) *Braid group has solvable TCP,*
- (with V. Romankov) *Nilpotent and metabelian groups have solvable TCP.*

Not much is known about twisted conjugacy problem:

Theorem

Every finitely generated, virtually

(i) *abelian, or*

(ii) *free, or*

(iii) *surface, or*

(iv) *polycyclic*

group has solvable twisted conjugacy problem.

Theorem (work in progress)

- *(w. J.Burillo & F.Matucci) Thomson's group has solvable TCP,*
- *(w. J.González-Meneses) Braid group has solvable TCP,*
- *(with V. Romankov) Nilpotent and metabelian groups have solvable TCP.*

Not much is known about twisted conjugacy problem:

Theorem

Every finitely generated, virtually

(i) *abelian, or*

(ii) *free, or*

(iii) *surface, or*

(iv) *polycyclic*

group has solvable twisted conjugacy problem.

Theorem (work in progress)

- *(w. J.Burillo & F.Matucci) Thomson's group has solvable TCP,*
- *(w. J.González-Meneses) Braid group has solvable TCP,*
- *(with V. Romankov) Nilpotent and metabelian groups have solvable TCP.*

Not much is known about twisted conjugacy problem:

Theorem

Every finitely generated, virtually

(i) *abelian, or*

(ii) *free, or*

(iii) *surface, or*

(iv) *polycyclic*

group has solvable twisted conjugacy problem.

Theorem (work in progress)

- (w. J.Burillo & F.Matucci) *Thomson's group has solvable TCP,*
- (w. J.González-Meneses) *Braid group has solvable TCP,*
- (with V. Romankov) *Nilpotent and metabelian groups have solvable TCP.*

Not much is known about twisted conjugacy problem:

Theorem

Every finitely generated, virtually

(i) *abelian, or*
(ii) *free, or*
(iii) *surface, or*
(iv) *polycyclic*

group has solvable twisted conjugacy problem.

Theorem (work in progress)

- (w. J.Burillo & F.Matucci) *Thomson's group has solvable TCP,*
- (w. J.González-Meneses) *Braid group has solvable TCP,*
- (with V. Romankov) *Nilpotent and metabelian groups have solvable TCP.*

Not much is known about twisted conjugacy problem:

Theorem

Every finitely generated, virtually

(i) *abelian, or*
(ii) *free, or*
(iii) *surface, or*
(iv) *polycyclic*

group has solvable twisted conjugacy problem.

Theorem (work in progress)

- *(w. J.Burillo & F.Matucci) Thomson's group has solvable TCP,*
- *(w. J.González-Meneses) Braid group has solvable TCP,*
- *(with V. Romankov) Nilpotent and metabelian groups have solvable TCP.*

Not much is known about twisted conjugacy problem:

Theorem

Every finitely generated, virtually

(i) *abelian, or*

(ii) *free, or*

(iii) *surface, or*

(iv) *polycyclic*

group has solvable twisted conjugacy problem.

Theorem (work in progress)

- *(w. J.Burillo & F.Matucci) Thomson's group has solvable TCP,*
- *(w. J.González-Meneses) Braid group has solvable TCP,*
- *(with V. Romankov) Nilpotent and metabelian groups have solvable TCP.*

Not much is known about twisted conjugacy problem:

Theorem

Every finitely generated, virtually

(i) *abelian, or*

(ii) *free, or*

(iii) *surface, or*

(iv) *polycyclic*

group has solvable twisted conjugacy problem.

Theorem (work in progress)

- *(w. J.Burillo & F.Matucci) Thomson's group has solvable TCP,*
- *(w. J.González-Meneses) Braid group has solvable TCP,*
- *(with V. Romankov) Nilpotent and metabelian groups have solvable TCP.*

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

Let G be a group (given as a finite presentation) and $K \leqslant G$ a finite index subgroup (given by generators). Then,

- *if K is characteristic and $TCP(K)$ is solvable, then $TCP(G)$ is solvable,*
- *if K is normal and $TCP(K)$ is solvable, then $CP(G)$ is solvable.*

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

There exists a group G with $CP(G)$ solvable but $TCP(G)$ unsolvable.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

Let G be a group (given as a finite presentation) and $K \leqslant G$ a finite index subgroup (given by generators). Then,

- *if K is characteristic and $TCP(K)$ is solvable, then $TCP(G)$ is solvable,*
- *if K is normal and $TCP(K)$ is solvable, then $CP(G)$ is solvable.*

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

There exists a group G with $CP(G)$ solvable but $TCP(G)$ unsolvable.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

Let G be a group (given as a finite presentation) and $K \leqslant G$ a finite index subgroup (given by generators). Then,

- *if K is characteristic and $TCP(K)$ is solvable, then $TCP(G)$ is solvable,*
- *if K is normal and $TCP(K)$ is solvable, then $CP(G)$ is solvable.*

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

There exists a group G with $CP(G)$ solvable but $TCP(G)$ unsolvable.

Definition

Two automorphisms $\alpha, \beta \in Aut(G)$ are isogradient, denoted $\alpha \approx \beta$, if they are conjugated by a conjugation, i.e. $\exists\ g \in G$ such that

$$\begin{array}{ccc} G & \stackrel{\gamma_g}{\longrightarrow} & G \\ \alpha \downarrow & \equiv & \downarrow \beta \\ G & \stackrel{\gamma_g}{\longrightarrow} & G \end{array}$$

Observation

If G has trivial center, then

$$u \sim_{\varphi} v \quad \Leftrightarrow \quad \varphi\gamma_u \approx \varphi\gamma_v \quad \Leftrightarrow \quad \exists g \in G \text{ s.t. } \begin{array}{ccc} G & \stackrel{\gamma_g}{\longrightarrow} & G \\ \varphi\gamma_u \downarrow & \equiv & \downarrow \varphi\gamma_v . \\ G & \stackrel{\gamma_g}{\longrightarrow} & G \end{array}$$

Definition

Two automorphisms $\alpha, \beta \in Aut(G)$ are isogradient, denoted $\alpha \approx \beta$, if they are conjugated by a conjugation, i.e. $\exists\, g \in G$ such that

$$\begin{array}{ccc} G & \stackrel{\gamma_g}{\longrightarrow} & G \\ \alpha \downarrow & \equiv & \downarrow \beta \\ G & \stackrel{\gamma_g}{\longrightarrow} & G \end{array}$$

Observation

If G has trivial center, then

$$u \sim_\varphi v \;\Leftrightarrow\; \varphi\gamma_u \approx \varphi\gamma_v \;\Leftrightarrow\; \exists g \in G \text{ s.t. } \begin{array}{ccc} G & \stackrel{\gamma_g}{\longrightarrow} & G \\ \varphi\gamma_u \downarrow & \equiv & \downarrow \varphi\gamma_v \\ G & \stackrel{\gamma_g}{\longrightarrow} & G \end{array}.$$

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ is solvable.

Proof.

- $\varphi\gamma_u \approx \varphi\gamma_v \quad \Longrightarrow \quad (\operatorname{Fix}\varphi\gamma_u)\gamma_g = \operatorname{Fix}\varphi\gamma_v$.
- The convers is not true in general, but...
- In the diagram $\begin{array}{ccc} F_n & \stackrel{\gamma_g}{\to} & F_n \\ \varphi\gamma_u \downarrow & & \downarrow \varphi\gamma_v, \\ F_n & \stackrel{\gamma_g}{\to} & F_n \end{array}$

  a) $x(\varphi\gamma_u)\gamma_g = x\gamma_g(\varphi\gamma_v) \quad \Longleftrightarrow \quad x\varphi \in C(v^{-1}(g\varphi)^{-1}ug)$;
  b) $\{x \in F_n \mid x(\varphi\gamma_u)\gamma_g = x\gamma_g(\varphi\gamma_v)\}$ is either cyclic, or the whole F_n.
- So, $\varphi\gamma_u \approx \varphi\gamma_v \quad \Longleftarrow \quad (\operatorname{Fix}\varphi\gamma_u)\gamma_g = \operatorname{Fix}\varphi\gamma_v$ plus rank $\geqslant 2$.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ is solvable.

Proof.

- $\varphi\gamma_u \approx \varphi\gamma_v \quad \Longrightarrow \quad (\operatorname{Fix}\varphi\gamma_u)\gamma_g = \operatorname{Fix}\varphi\gamma_v.$
- The convers is not true in general, but...
- In the diagram $\begin{array}{ccc} F_n & \stackrel{\gamma_g}{\rightarrow} & F_n \\ \varphi\gamma_u \downarrow & & \downarrow \varphi\gamma_v, \\ F_n & \stackrel{\gamma_g}{\rightarrow} & F_n \end{array}$
  - a) $x(\varphi\gamma_u)\gamma_g = x\gamma_g(\varphi\gamma_v) \quad \Longleftrightarrow \quad x\varphi \in C(v^{-1}(g\varphi)^{-1}ug);$
  - b) $\{x \in F_n \mid x(\varphi\gamma_u)\gamma_g = x\gamma_g(\varphi\gamma_v)\}$ is either cyclic, or the whole F_n.
- So, $\varphi\gamma_u \approx \varphi\gamma_v \quad \Longleftarrow \quad (\operatorname{Fix}\varphi\gamma_u)\gamma_g = \operatorname{Fix}\varphi\gamma_v$ plus rank $\geqslant 2$.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ is solvable.

Proof.

- $\varphi\gamma_u \approx \varphi\gamma_v \quad \Longrightarrow \quad (\operatorname{Fix}\varphi\gamma_u)\gamma_g = \operatorname{Fix}\varphi\gamma_v$.
- The convers is not true in general, but...
- In the diagram $\begin{array}{ccc} F_n & \stackrel{\gamma_g}{\to} & F_n \\ \varphi\gamma_u \downarrow & & \downarrow \varphi\gamma_v, \\ F_n & \stackrel{\gamma_g}{\to} & F_n \end{array}$
  a) $x(\varphi\gamma_u)\gamma_g = x\gamma_g(\varphi\gamma_v) \quad \Longleftrightarrow \quad x\varphi \in C(v^{-1}(g\varphi)^{-1}ug)$;
  b) $\{x \in F_n \mid x(\varphi\gamma_u)\gamma_g = x\gamma_g(\varphi\gamma_v)\}$ is either cyclic, or the whole F_n.
- So, $\varphi\gamma_u \approx \varphi\gamma_v \quad \Longleftarrow \quad (\operatorname{Fix}\varphi\gamma_u)\gamma_g = \operatorname{Fix}\varphi\gamma_v$ plus rank $\geqslant 2$.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ is solvable.

Proof.

- $\varphi\gamma_u \approx \varphi\gamma_v \quad \Longrightarrow \quad (\mathrm{Fix}\,\varphi\gamma_u)\gamma_g = \mathrm{Fix}\,\varphi\gamma_v.$
- The convers is not true in general, but...
- In the diagram $\begin{array}{ccc} F_n & \stackrel{\gamma_g}{\to} & F_n \\ \varphi\gamma_u\downarrow & & \downarrow\varphi\gamma_v, \\ F_n & \stackrel{\gamma_g}{\to} & F_n \end{array}$
  - a) $x(\varphi\gamma_u)\gamma_g = x\gamma_g(\varphi\gamma_v) \quad \Longleftrightarrow \quad x\varphi \in C(v^{-1}(g\varphi)^{-1}ug);$
  - b) $\{x \in F_n \mid x(\varphi\gamma_u)\gamma_g = x\gamma_g(\varphi\gamma_v)\}$ is either cyclic, or the whole F_n.
- So, $\varphi\gamma_u \approx \varphi\gamma_v \quad \Longleftarrow \quad (\mathrm{Fix}\,\varphi\gamma_u)\gamma_g = \mathrm{Fix}\,\varphi\gamma_v$ plus rank $\geqslant 2$.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ is solvable.

Proof.

- $\varphi\gamma_u \approx \varphi\gamma_v \quad \Longrightarrow \quad (\mathrm{Fix}\,\varphi\gamma_u)\gamma_g = \mathrm{Fix}\,\varphi\gamma_v.$
- The convers is not true in general, but...
- In the diagram $\begin{array}{ccc} F_n & \xrightarrow{\gamma_g} & F_n \\ \varphi\gamma_u \downarrow & & \downarrow \varphi\gamma_v, \\ F_n & \xrightarrow{\gamma_g} & F_n \end{array}$
  - a) $x(\varphi\gamma_u)\gamma_g = x\gamma_g(\varphi\gamma_v) \quad \Longleftrightarrow \quad x\varphi \in C(v^{-1}(g\varphi)^{-1}ug);$
  - b) $\{x \in F_n \mid x(\varphi\gamma_u)\gamma_g = x\gamma_g(\varphi\gamma_v)\}$ is either cyclic, or the whole F_n.
- So, $\varphi\gamma_u \approx \varphi\gamma_v \quad \Longleftarrow \quad (\mathrm{Fix}\,\varphi\gamma_u)\gamma_g = \mathrm{Fix}\,\varphi\gamma_v$ plus rank $\geqslant 2$.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

$TCP(F_n)$ is solvable.

Proof.

- $\varphi\gamma_u \approx \varphi\gamma_v \quad \Longrightarrow \quad (\mathrm{Fix}\,\varphi\gamma_u)\gamma_g = \mathrm{Fix}\,\varphi\gamma_v.$
- The convers is not true in general, but...
- In the diagram $\begin{array}{ccc} F_n & \xrightarrow{\gamma_g} & F_n \\ \varphi\gamma_u \downarrow & & \downarrow \varphi\gamma_v, \\ F_n & \xrightarrow{\gamma_g} & F_n \end{array}$
  - a) $x(\varphi\gamma_u)\gamma_g = x\gamma_g(\varphi\gamma_v) \quad \Longleftrightarrow \quad x\varphi \in C(v^{-1}(g\varphi)^{-1}ug);$
  - b) $\{x \in F_n \mid x(\varphi\gamma_u)\gamma_g = x\gamma_g(\varphi\gamma_v)\}$ is either cyclic, or the whole F_n.
- So, $\varphi\gamma_u \approx \varphi\gamma_v \quad \Longleftarrow \quad (\mathrm{Fix}\,\varphi\gamma_u)\gamma_g = \mathrm{Fix}\,\varphi\gamma_v$ plus rank $\geqslant 2$.

Idea is to force the diagram to commute.

- Extend to $F_n * \langle z \rangle$ and $\hat{\varphi}\colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.
- Now, if $z\gamma_g \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v)$ $\Rightarrow$ z commutes in the diagram .
  $\Rightarrow$ $z \in C(v^{-1}(g\varphi)^{-1}ug)$
  $\Rightarrow$ $v^{-1}(g\varphi)^{-1}ug = 1$
  $\Rightarrow$ $u \sim_\varphi v$
- Hence, $u \sim_\varphi v \iff z\gamma_g \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v)$ for some $g \in F_n$.

Idea is to force the diagram to commute.

- Extend to $F_n * \langle z \rangle$ and $\hat{\varphi} \colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.
- Now, if $z\gamma_g \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v)$ $\Rightarrow$ z commutes in the diagram .
$\Rightarrow z \in C(v^{-1}(g\varphi)^{-1}ug)$
$\Rightarrow v^{-1}(g\varphi)^{-1}ug = 1$
$\Rightarrow u \sim_\varphi v$
- Hence, $u \sim_\varphi v \iff z\gamma_g \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v)$ for some $g \in F_n$.

Idea is to force the diagram to commute.

- Extend to $F_n * \langle z \rangle$ and $\hat{\varphi}\colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.
- Now, if $z\gamma_g \in \mathrm{Fix}(\hat{\varphi}\gamma_v) \Rightarrow z$ commutes in the diagram
  $\Rightarrow z \in C(v^{-1}(g\varphi)^{-1}ug)$
  $\Rightarrow v^{-1}(g\varphi)^{-1}ug = 1$
  $\Rightarrow u \sim_\varphi v$
- Hence, $u \sim_\varphi v \iff z\gamma_g \in \mathrm{Fix}(\hat{\varphi}\gamma_v)$ for some $g \in F_n$.

Idea is to force the diagram to commute.

- Extend to $F_n * \langle z \rangle$ and $\hat{\varphi}\colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.
- Now, if $z\gamma_g \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v) \Rightarrow z$ commutes in the diagram.
  $\Rightarrow z \in C(v^{-1}(g\varphi)^{-1}ug)$
  $\Rightarrow v^{-1}(g\varphi)^{-1}ug = 1$
  $\Rightarrow u \sim_\varphi v$
- Hence, $u \sim_\varphi v \iff z\gamma_g \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v)$ for some $g \in F_n$.

Idea is to force the diagram to commute.

- Extend to $F_n * \langle z \rangle$ and $\hat{\varphi} \colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.
- Now, if $z\gamma_g \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v)$ $\Rightarrow$ z commutes in the diagram
  $\Rightarrow$ $z \in C(v^{-1}(g\varphi)^{-1}ug)$
  $\Rightarrow$ $v^{-1}(g\varphi)^{-1}ug = 1$
  $\Rightarrow$ $u \sim_{\varphi} v$
- Hence, $u \sim_{\varphi} v \iff z\gamma_g \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v)$ for some $g \in F_n$.

Idea is to force the diagram to commute.

- Extend to $F_n * \langle z \rangle$ and $\hat{\varphi}\colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.
- Now, if $z\gamma_g \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v)$ $\Rightarrow$ z commutes in the diagram .

$$\begin{aligned} &\Rightarrow z \in C(v^{-1}(g\varphi)^{-1}ug) \\ &\Rightarrow v^{-1}(g\varphi)^{-1}ug = 1 \\ &\Rightarrow u \sim_{\varphi} v \end{aligned}$$

- Hence, $u \sim_{\varphi} v \iff z\gamma_g \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v)$ for some $g \in F_n$.

Idea is to force the diagram to commute.

- Extend to $F_n * \langle z \rangle$ and $\hat{\varphi} \colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.
- Now, if $z\gamma_g \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v)$ $\Rightarrow$ z commutes in the diagram .
$$\begin{aligned} &\Rightarrow z \in C(v^{-1}(g\varphi)^{-1}ug) \\ &\Rightarrow v^{-1}(g\varphi)^{-1}ug = 1 \\ &\Rightarrow u \sim_\varphi v \end{aligned}$$
- Hence, $u \sim_\varphi v \iff z\gamma_g \in \mathrm{Fix}\,(\hat{\varphi}\gamma_v)$ for some $g \in F_n$.

Algorithm is the following: given $\varphi \in \mathrm{Aut}(F_n)$ and $u, v \in F_n$:

1- Extend φ to $\hat{\varphi}\colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.

2- Compute a basis for $\mathrm{Fix}(\hat{\varphi}\gamma_v)$ using

Theorem (Maslakova)

Fixed subgroups of automorphisms of free groups are computable.

3- Check whether $\mathrm{Fix}(\hat{\varphi}\gamma_v)$ contains $z\gamma_g$ for some $g \in F_n$, using Stallings' automata. □

Algorithm is the following: given $\varphi \in \mathrm{Aut}(F_n)$ and $u, v \in F_n$:

1- Extend φ to $\hat{\varphi}\colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.

2- Compute a basis for $\mathrm{Fix}(\hat{\varphi}\gamma_v)$ using

Theorem (Maslakova)

Fixed subgroups of automorphisms of free groups are computable.

3- Check whether $\mathrm{Fix}(\hat{\varphi}\gamma_v)$ contains $z\gamma_g$ for some $g \in F_n$, using Stallings' automata. □

Algorithm is the following: given $\varphi \in \mathrm{Aut}(F_n)$ and $u, v \in F_n$:

1- Extend φ to $\hat{\varphi}\colon F_n * \langle z \rangle \to F_n * \langle z \rangle$, sending z to uzu^{-1}.

2- Compute a basis for $\mathrm{Fix}(\hat{\varphi}\gamma_v)$ using

Theorem (Maslakova)

Fixed subgroups of automorphisms of free groups are computable.

3- Check whether $\mathrm{Fix}(\hat{\varphi}\gamma_v)$ contains $z\gamma_g$ for some $g \in F_n$, using Stallings' automata. $\square$

Algorithm is the following: given $\varphi \in \mathrm{Aut}(F_n)$ and $u, v \in F_n$:

1- Extend φ to $\hat{\varphi}\colon F_n * \langle z\rangle \to F_n * \langle z\rangle$, sending z to uzu^{-1}.

2- Compute a basis for $\mathrm{Fix}(\hat{\varphi}\gamma_v)$ using

Theorem (Maslakova)

Fixed subgroups of automorphisms of free groups are computable.

3- Check whether $\mathrm{Fix}(\hat{\varphi}\gamma_v)$ contains $z\gamma_g$ for some $g \in F_n$, using Stallings' automata. $\square$

Outline

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \, \rangle$ be a free group on $\{x_1, \ldots, x_n\}$ $(n \geq 2)$, and let $\varphi \in Aut(F_n)$. The free-by-cyclic group $F_n \rtimes_\varphi \mathbb{Z}$ is defined as

$$\begin{aligned} F_n \rtimes_\varphi \mathbb{Z} &= \langle x_1, \ldots, x_n, \, t \mid t^{-1} x_i t = x_i \varphi \rangle \\ &= \langle x_1, \ldots, x_n, \, t \mid x_i t = t(x_i \varphi) \rangle . \end{aligned}$$

Move t's to left & get usual normal forms, $t^r w$, with $r \in \mathbb{Z}$, $w \in F_n$.

Example

Consider $F_3 = \langle a, b, c \mid \, \rangle$ and $\varphi \colon F_3 \to F_3$ given by $a \mapsto a$, $b \mapsto ba$, $c \mapsto b^{-2}cba$. In $F_3 \rtimes_\varphi \mathbb{Z}$ we have

$$\begin{aligned} tbt^2a^{-2}t^{-1}cta^{-1}b^{-1} &= tbt^2a^{-2}t^{-1}tb^{-2}cbaa^{-1}b^{-1} \\ &= tbt^2a^{-2}b^{-2}c \\ &= tt^2ba^2a^{-2}b^{-2}c \\ &= t^3(b^{-1}c). \end{aligned}$$

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \; \rangle$ be a free group on $\{x_1, \ldots, x_n\}$ $(n \geq 2)$, and let $\varphi \in Aut(F_n)$. The free-by-cyclic group $F_n \rtimes_\varphi \mathbb{Z}$ is defined as

$$\begin{aligned} F_n \rtimes_\varphi \mathbb{Z} &= \langle x_1, \ldots, x_n, \; t \mid t^{-1} x_i t = x_i \varphi \rangle \\ &= \langle x_1, \ldots, x_n, \; t \mid x_i t = t(x_i \varphi) \rangle. \end{aligned}$$

Move t's to left & get usual normal forms, $t^r w$, with $r \in \mathbb{Z}$, $w \in F_n$.

Example

Consider $F_3 = \langle a, b, c \mid \; \rangle$ and $\varphi \colon F_3 \to F_3$ given by $a \mapsto a$, $b \mapsto ba$, $c \mapsto b^{-2}cba$. In $F_3 \rtimes_\varphi \mathbb{Z}$ we have

$$\begin{aligned} tbt^2a^{-2}t^{-1}cta^{-1}b^{-1} &= tbt^2a^{-2}t^{-1}tb^{-2}cbaa^{-1}b^{-1} \\ &= tbt^2a^{-2}b^{-2}c \\ &= tt^2ba^2a^{-2}b^{-2}c \\ &= t^3(b^{-1}c). \end{aligned}$$

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \ \rangle$ be a free group on $\{x_1, \ldots, x_n\}$ $(n \geq 2)$, and let $\varphi \in Aut(F_n)$. The free-by-cyclic group $F_n \rtimes_\varphi \mathbb{Z}$ is defined as

$$\begin{aligned} F_n \rtimes_\varphi \mathbb{Z} &= \langle x_1, \ldots, x_n, t \mid t^{-1} x_i t = x_i \varphi \rangle \\ &= \langle x_1, \ldots, x_n, t \mid x_i t = t(x_i \varphi) \rangle. \end{aligned}$$

Move t's to left & get usual normal forms, $t^r w$, with $r \in \mathbb{Z}$, $w \in F_n$.

Example

Consider $F_3 = \langle a, b, c \mid \ \rangle$ and $\varphi \colon F_3 \to F_3$ given by $a \mapsto a$, $b \mapsto ba$, $c \mapsto b^{-2}cba$. In $F_3 \rtimes_\varphi \mathbb{Z}$ we have

$$\begin{aligned} tbt^2a^{-2}t^{-1}cta^{-1}b^{-1} &= tbt^2a^{-2}t^{-1}tb^{-2}cbaa^{-1}b^{-1} \\ &= tbt^2a^{-2}b^{-2}c \\ &= tt^2ba^2a^{-2}b^{-2}c \\ &= t^3(b^{-1}c). \end{aligned}$$

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \; \rangle$ be a free group on $\{x_1, \ldots, x_n\}$ ($n \geq 2$), and let $\varphi \in Aut(F_n)$. The free-by-cyclic group $F_n \rtimes_\varphi \mathbb{Z}$ is defined as

$$\begin{aligned} F_n \rtimes_\varphi \mathbb{Z} &= \langle x_1, \ldots, x_n, \, t \mid t^{-1} x_i t = x_i \varphi \rangle \\ &= \langle x_1, \ldots, x_n, \, t \mid x_i t = t(x_i \varphi) \rangle. \end{aligned}$$

Move t's to left & get usual normal forms, $t^r w$, with $r \in \mathbb{Z}$, $w \in F_n$.

Example

Consider $F_3 = \langle a, b, c \mid \; \rangle$ and $\varphi \colon F_3 \to F_3$ given by $a \mapsto a$, $b \mapsto ba$, $c \mapsto b^{-2}cba$. In $F_3 \rtimes_\varphi \mathbb{Z}$ we have

$$\begin{aligned} tbt^2a^{-2}t^{-1}cta^{-1}b^{-1} &= tbt^2a^{-2}t^{-1}tb^{-2}cbaa^{-1}b^{-1} \\ &= tbt^2a^{-2}b^{-2}c \\ &= tt^2ba^2a^{-2}b^{-2}c \\ &= t^3(b^{-1}c). \end{aligned}$$

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \ \rangle$ be a free group on $\{x_1, \ldots, x_n\}$ ($n \geq 2$), and let $\varphi \in Aut(F_n)$. The free-by-cyclic group $F_n \rtimes_\varphi \mathbb{Z}$ is defined as

$$\begin{aligned} F_n \rtimes_\varphi \mathbb{Z} &= \langle x_1, \ldots, x_n, t \mid t^{-1}x_i t = x_i\varphi \rangle \\ &= \langle x_1, \ldots, x_n, t \mid x_i t = t(x_i\varphi) \rangle. \end{aligned}$$

Move t's to left & get usual normal forms, $t^r w$, with $r \in \mathbb{Z}$, $w \in F_n$.

Example

Consider $F_3 = \langle a, b, c \mid \ \rangle$ and $\varphi \colon F_3 \to F_3$ given by $a \mapsto a$, $b \mapsto ba$, $c \mapsto b^{-2}cba$. In $F_3 \rtimes_\varphi \mathbb{Z}$ we have

$$\begin{aligned} tbt^2a^{-2}t^{-1}cta^{-1}b^{-1} &= tbt^2a^{-2}t^{-1}tb^{-2}cbaa^{-1}b^{-1} \\ &= tbt^2a^{-2}b^{-2}c \\ &= tt^2ba^2a^{-2}b^{-2}c \\ &= t^3(b^{-1}c). \end{aligned}$$

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \ \rangle$ be a free group on $\{x_1, \ldots, x_n\}$ ($n \geq 2$), and let $\varphi \in Aut(F_n)$. The free-by-cyclic group $F_n \rtimes_\varphi \mathbb{Z}$ is defined as

$$\begin{aligned} F_n \rtimes_\varphi \mathbb{Z} &= \langle x_1, \ldots, x_n, \ t \mid t^{-1}x_i t = x_i\varphi \rangle \\ &= \langle x_1, \ldots, x_n, \ t \mid x_i t = t(x_i\varphi) \rangle. \end{aligned}$$

Move t's to left & get usual normal forms, $t^r w$, with $r \in \mathbb{Z}$, $w \in F_n$.

Example

Consider $F_3 = \langle a, b, c \mid \ \rangle$ and $\varphi \colon F_3 \to F_3$ given by $a \mapsto a$, $b \mapsto ba$, $c \mapsto b^{-2}cba$. In $F_3 \rtimes_\varphi \mathbb{Z}$ we have

$$\begin{aligned} tbt^2a^{-2}t^{-1}cta^{-1}b^{-1} &= tbt^2a^{-2}t^{-1}tb^{-2}cbaa^{-1}b^{-1} \\ &= tbt^2a^{-2}b^{-2}c \\ &= tt^2ba^2a^{-2}b^{-2}c \\ &= t^3(b^{-1}c). \end{aligned}$$

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \ \rangle$ be a free group on $\{x_1, \ldots, x_n\}$ ($n \geq 2$), and let $\varphi \in Aut(F_n)$. The free-by-cyclic group $F_n \rtimes_\varphi \mathbb{Z}$ is defined as

$$\begin{aligned} F_n \rtimes_\varphi \mathbb{Z} &= \langle x_1, \ldots, x_n, \, t \mid t^{-1} x_i t = x_i\varphi \rangle \\ &= \langle x_1, \ldots, x_n, \, t \mid x_i t = t(x_i\varphi) \rangle. \end{aligned}$$

Move t's to left & get usual normal forms, $t^r w$, with $r \in \mathbb{Z}$, $w \in F_n$.

Example

Consider $F_3 = \langle a, b, c \mid \ \rangle$ and $\varphi \colon F_3 \to F_3$ given by $a \mapsto a$, $b \mapsto ba$, $c \mapsto b^{-2}cba$. In $F_3 \rtimes_\varphi \mathbb{Z}$ we have

$$\begin{aligned} tbt^2a^{-2}t^{-1}cta^{-1}b^{-1} &= tbt^2a^{-2}t^{-1}tb^{-2}cbaa^{-1}b^{-1} \\ &= tbt^2a^{-2}b^{-2}c \\ &= tt^2ba^2a^{-2}b^{-2}c \\ &= t^3(b^{-1}c). \end{aligned}$$

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $(g^{-1}t^{-k})(t^r u)(t^k g) = g^{-1}t^r(u\varphi^k)g = t^r(g\varphi^r)^{-1}(u\varphi^k)g.$
- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.

▶ **Case 1:** $r \neq 0$

- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

  $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $(g^{-1}t^{-k})(t^r u)(t^k g) = g^{-1}t^r(u\varphi^k)g = t^r(g\varphi^r)^{-1}(u\varphi^k)g.$
- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.

▶ **Case 1:** $r \neq 0$

- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

$t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $(g^{-1}t^{-k})(t^r u)(t^k g) = g^{-1}t^r(u\varphi^k)g = t^r(g\varphi^r)^{-1}(u\varphi^k)g$.
- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.

Case 1: $r \neq 0$

- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

  $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $(g^{-1}t^{-k})(t^r u)(t^k g) = g^{-1}t^r(u\varphi^k)g = t^r(g\varphi^r)^{-1}(u\varphi^k)g.$
- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.

Case 1: $r \neq 0$

- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

  $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $(g^{-1}t^{-k})(t^r u)(t^k g) = g^{-1}t^r(u\varphi^k)g = t^r(g\varphi^r)^{-1}(u\varphi^k)g$.
- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.

▶ **Case 1:** $r \neq 0$

- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

  $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $(g^{-1}t^{-k})(t^r u)(t^k g) = g^{-1}t^r(u\varphi^k)g = t^r(g\varphi^r)^{-1}(u\varphi^k)g.$
- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.

▶ **Case 1:** $r \neq 0$

- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

$t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $(g^{-1}t^{-k})(t^r u)(t^k g) = g^{-1}t^r(u\varphi^k)g = t^r(g\varphi^r)^{-1}(u\varphi^k)g$.
- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.

▶ **Case 1:** $r \neq 0$

- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

  $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

Theorem (Bogopolski-Martino-Maslakova-V., 2005)

For every $\varphi \in Aut(F_n)$, $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable.

Proof. Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $(g^{-1}t^{-k})(t^r u)(t^k g) = g^{-1}t^r(u\varphi^k)g = t^r(g\varphi^r)^{-1}(u\varphi^k)g.$
- $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.

▶ **Case 1:** $r \neq 0$

- To reduce to finitely many k's, note that $u \sim_\varphi u\varphi$ (because $u = (u\varphi)^{-1}(u\varphi)u$), so $u\varphi^k \sim_{\varphi^r} u\varphi^{k\pm\lambda r}$; hence,

  $t^r u$ and $t^s v$ conj. in M_φ $\iff$ $r = s$, $v \sim_{\varphi^r} (u\varphi^k)$ for some $k = 0, \ldots, r-1$.

- Thus, $CP(M_\varphi)$ reduces to finitely many checks of $TCP(F_n)$.

- **Case 2: $r = 0$**
- Still infinitely many k's:

$$u \text{ and } v \text{ conj. in } M_\varphi \iff v \sim u\varphi^k \text{ for some } k \in \mathbb{Z}.$$

- This is precisely Brinkmann's result:

Theorem (Brinkmann, 2006)

Given $\phi\colon F_n \to F_n$ and $u, v \in F_n$, it is decidable whether $v \sim u\phi^k$ for some $k \in \mathbb{Z}$.

- Hence, $CP(M_\varphi)$ is solvable. □

- **Case 2: $r = 0$**
- Still infinitely many k's:

$$u \text{ and } v \text{ conj. in } M_\varphi \iff v \sim u\varphi^k \text{ for some } k \in \mathbb{Z}.$$

- This is precisely Brinkmann's result:

Theorem (Brinkmann, 2006)

Given $\phi\colon F_n \to F_n$ and $u, v \in F_n$, it is decidable whether $v \sim u\phi^k$ for some $k \in \mathbb{Z}$.

- Hence, $CP(M_\varphi)$ is solvable. □

- **Case 2: $r = 0$**
- Still infinitely many k's:

$$u \text{ and } v \text{ conj. in } M_\varphi \iff v \sim u\varphi^k \text{ for some } k \in \mathbb{Z}.$$

- This is precisely Brinkmann's result:

Theorem (Brinkmann, 2006)

Given $\phi\colon F_n \to F_n$ and $u, v \in F_n$, it is decidable whether $v \sim u\phi^k$ for some $k \in \mathbb{Z}$.

- Hence, $CP(M_\varphi)$ is solvable. □

- **Case 2:** $r = 0$

- Still infinitely many k's:

$$u \text{ and } v \text{ conj. in } M_\varphi \iff v \sim u\varphi^k \text{ for some } k \in \mathbb{Z}.$$

- This is precisely Brinkmann's result:

Theorem (Brinkmann, 2006)

Given $\phi \colon F_n \to F_n$ and $u, v \in F_n$, it is decidable whether $v \sim u\phi^k$ for some $k \in \mathbb{Z}$.

- Hence, $CP(M_\varphi)$ is solvable. □

Outline

Theorem (Bogopolski-Martino-V., 2008)

Let

$$1 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$$

be an algorithmic short exact sequence of groups such that

(i) *$TCP(F)$ is solvable,*

(ii) *$CP(H)$ is solvable, and*

(iii) *there is an algorithm which, given an input $1 \neq h \in H$, computes a finite set of elements $z_{h,1}, \ldots, z_{h,t_h} \in H$ such that*

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

Then,

$$CP(G) \text{ is solvable} \iff A_G = \left\{ \begin{array}{rcl} \gamma_g : F & \to & F \\ x & \mapsto & g^{-1}xg \end{array} \;\middle|\; g \in G \right\}$$

$\leqslant Aut(F)$ *is orbit decidable.*

Definition

A subgroup $A \leqslant Aut(F)$ is said to be orbit decidable (O.D.) if $\exists$ an algorithm s.t., given $u, v \in F$ decides whether $v \sim u\alpha$ for some $\alpha \in A$.

Previous result in this language:

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Definition

A subgroup $A \leqslant Aut(F)$ is said to be orbit decidable (O.D.) if $\exists$ an algorithm s.t., given $u, v \in F$ decides whether $v \sim u\alpha$ for some $\alpha \in A$.

Previous result in this language:

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Definition

A subgroup $A \leqslant Aut(F)$ is said to be orbit decidable (O.D.) if $\exists$ an algorithm s.t., given $u, v \in F$ decides whether $v \sim u\alpha$ for some $\alpha \in A$.

Previous result in this language:

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Outline

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \rangle$ be the free group on $\{x_1, \ldots, x_n\}$ $(n \geq 2)$, and let $\varphi_1, \ldots, \varphi_m \in Aut(F_n)$. The free-by-free group $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ is

$$F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m = \langle x_1, \ldots, x_n,\ t_1, \ldots, t_m \mid t_j^{-1} x_i t_j = x_i \varphi_j \rangle.$$

And this sequence satisfies (i), (ii) and (iii):

$$1 \longrightarrow F_n \longrightarrow F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m \longrightarrow F_m \longrightarrow 1$$

So,

$$CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m) \text{ is solvable } \Leftrightarrow \langle \varphi_1, \ldots, \varphi_m \rangle \leqslant \text{Aut}(F_n) \text{ is O.D.}$$

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \; \rangle$ be the free group on $\{x_1, \ldots, x_n\}$ ($n \geq 2$), and let $\varphi_1, \ldots, \varphi_m \in Aut(F_n)$. The free-by-free group $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ is

$$F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m = \langle x_1, \ldots, x_n, \; t_1, \ldots, t_m \mid t_j^{-1} x_i t_j = x_i\varphi_j \rangle.$$

And this sequence satisfies (i), (ii) and (iii):

$$1 \longrightarrow F_n \longrightarrow F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m \longrightarrow F_m \longrightarrow 1$$

So,

$CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$ is solvable $\Leftrightarrow \langle \varphi_1, \ldots, \varphi_m \rangle \leqslant \mathrm{Aut}(F_n)$ is O.D.

Definition

Let $F_n = \langle x_1, \ldots, x_n \mid \ \rangle$ be the free group on $\{x_1, \ldots, x_n\}$ $(n \geq 2)$, and let $\varphi_1, \ldots, \varphi_m \in Aut(F_n)$. The ***free-by-free*** *group $F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m$ is*

$$F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m = \langle x_1, \ldots, x_n,\ t_1, \ldots, t_m \mid t_j^{-1} x_i t_j = x_i \varphi_j \rangle.$$

And this sequence satisfies (i), (ii) and (iii):

$$1 \longrightarrow F_n \longrightarrow F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m \longrightarrow F_m \longrightarrow 1$$

So,

$CP(F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m)$ is solvable $\Leftrightarrow \langle \varphi_1, \ldots, \varphi_m \rangle \leqslant \mathrm{Aut}(F_n)$ is O.D.

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead)

The full $Aut(F_n)$ is O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle = Aut(F_n)$ then $F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m$ has solvable conjugacy problem.

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead)

The full $Aut(F_n)$ is O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle = Aut(F_n)$ then $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead)

The full $Aut(F_n)$ is O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle = Aut(F_n)$ then $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

Theorem (Brinkmann, 2006)

Cyclic subgroups of $Aut(F_n)$ are O.D.

Corollary (Bogopolski-Martino-Maslakova-V., 2005)

Free-by-cyclic groups have solvable conjugacy problem.

Theorem (Whitehead)

The full $Aut(F_n)$ is O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle = Aut(F_n)$ then $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $Aut(F_n)$ are O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle$ is of finite index in $Aut(F_n)$ then $F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $Aut(F_2)$ is O.D.

Corollary

Every F_2-by-free group has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $Aut(F_n)$ are O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle$ is of finite index in $Aut(F_n)$ then $F_n \rtimes_{\varphi_1, \ldots, \varphi_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $Aut(F_2)$ is O.D.

Corollary

Every F_2-by-free group has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $Aut(F_n)$ are O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle$ is of finite index in $Aut(F_n)$ then $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $Aut(F_2)$ is O.D.

Corollary

Every F_2-by-free group has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $Aut(F_n)$ are O.D.

Corollary

If $\langle \varphi_1, \ldots, \varphi_m \rangle$ is of finite index in $Aut(F_n)$ then $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $Aut(F_2)$ is O.D.

Corollary

Every F_2-by-free group has solvable conjugacy problem.

But...

Theorem (Miller, 70's)

There are free-by-free groups with unsolvable conjugacy problem.

Corollary

There exist 14 automorphisms $\varphi_1, \ldots, \varphi_{14} \in Aut(F_3)$ such that $\langle \varphi_1, \ldots, \varphi_{14} \rangle \leqslant Aut(F_3)$ is orbit undecidable.

But...

Theorem (Miller, 70's)

There are free-by-free groups with unsolvable conjugacy problem.

Corollary

There exist 14 automorphisms $\varphi_1, \ldots, \varphi_{14} \in Aut(F_3)$ such that $\langle \varphi_1, \ldots, \varphi_{14} \rangle \leqslant Aut(F_3)$ is orbit undecidable.

But...

Theorem (Miller, 70's)

There are free-by-free groups with unsolvable conjugacy problem.

Corollary

There exist 14 automorphisms $\varphi_1, \ldots, \varphi_{14} \in Aut(F_3)$ *such that* $\langle \varphi_1, \ldots, \varphi_{14} \rangle \leqslant Aut(F_3)$ *is orbit undecidable.*

Outline

Definition

Let $\mathbb{Z}^n = \langle x_1, \ldots, x_n \mid [x_i, x_j] \rangle$ be the free abelian group of rank $n \geq 2$, and let $M_1, \ldots, M_m \in Aut(\mathbb{Z}^n) = GL_n(\mathbb{Z})$. The (free abelian)-by-free group $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ is defined as

$$F_n \rtimes_{M_1,\ldots,M_m} F_m = \langle x_1, \ldots, x_n,\ t_1, \ldots, t_m \mid t_j^{-1} x_i t_j = x_i M_j,\ [x_i, x_j] = 1 \rangle.$$

And this sequence, again, satisfies (i), (ii) and (iii):

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m \longrightarrow F_m \longrightarrow 1$$

So,

$$CP(\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m) \text{ is solvable } \Leftrightarrow \langle M_1, \ldots, M_m \rangle \leqslant GL_n(\mathbb{Z}) \text{ is O.D.}$$

Definition

Let $\mathbb{Z}^n = \langle x_1, \ldots, x_n \mid [x_i, x_j] \rangle$ be the free abelian group of rank $n \geq 2$, and let $M_1, \ldots, M_m \in Aut(\mathbb{Z}^n) = GL_n(\mathbb{Z})$. The (free abelian)-by-free group $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ is defined as

$$F_n \rtimes_{M_1,\ldots,M_m} F_m = \langle x_1, \ldots, x_n,\ t_1, \ldots, t_m \mid t_j^{-1} x_i t_j = x_i M_j,\ [x_i, x_j] = 1 \rangle.$$

And this sequence, again, satisfies (i), (ii) and (iii):

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m \longrightarrow F_m \longrightarrow 1$$

So,

$$CP(\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m) \text{ is solvable } \Leftrightarrow \langle M_1, \ldots, M_m \rangle \leqslant GL_n(\mathbb{Z}) \text{ is O.D.}$$

Definition

Let $\mathbb{Z}^n = \langle x_1, \ldots, x_n \mid [x_i, x_j] \rangle$ be the free abelian group of rank $n \geq 2$, and let $M_1, \ldots, M_m \in Aut(\mathbb{Z}^n) = GL_n(\mathbb{Z})$. The (free abelian)-by-free group $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ is defined as

$$F_n \rtimes_{M_1,\ldots,M_m} F_m = \langle x_1, \ldots, x_n,\ t_1, \ldots, t_m \mid t_j^{-1} x_i t_j = x_i M_j,\ [x_i, x_j] = 1 \rangle.$$

And this sequence, again, satisfies (i), (ii) and (iii):

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m \longrightarrow F_m \longrightarrow 1$$

So,

$$CP(\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m) \text{ is solvable } \Leftrightarrow \langle M_1, \ldots, M_m \rangle \leqslant GL_n(\mathbb{Z}) \text{ is O.D.}$$

Theorem (linear algebra)

Cyclic subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

$\mathbb{Z}^n$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

Theorem (elementary)

The full $GL_n(\mathbb{Z})$ is O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (linear algebra)

Cyclic subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

$\mathbb{Z}^n$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

Theorem (elementary)

The full $GL_n(\mathbb{Z})$ is O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (linear algebra)

Cyclic subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

$\mathbb{Z}^n$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

Theorem (elementary)

The full $GL_n(\mathbb{Z})$ is O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (linear algebra)

Cyclic subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

$\mathbb{Z}^n$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

Theorem (elementary)

The full $GL_n(\mathbb{Z})$ is O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle = GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $GL_2(\mathbb{Z})$ is O.D.

Corollary

Every $\mathbb{Z}^2$-by-free group has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $GL_2(\mathbb{Z})$ is O.D.

Corollary

Every $\mathbb{Z}^2$-by-free group has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $GL_2(\mathbb{Z})$ is O.D.

Corollary

Every $\mathbb{Z}^2$-by-free group has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Finite index subgroups of $GL_n(\mathbb{Z})$ are O.D.

Corollary

If $\langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_n(\mathbb{Z})$ then $\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

Theorem (Bogopolski-Martino-V., 2008)

Every finitely generated subgroup of $GL_2(\mathbb{Z})$ is O.D.

Corollary

Every $\mathbb{Z}^2$-by-free group has solvable conjugacy problem.

But...

Theorem (Bogopolski-Martino-V., 2008)

There are exist 14 matrices $M_1, \ldots, M_{14} \in GL_n(\mathbb{Z})$, for $n \geqslant 4$, such that $\langle M_1, \ldots, M_{14} \rangle \leqslant GL_n(\mathbb{Z})$ is orbit undecidable.

Corollary

There exists a $\mathbb{Z}^4$-by-F_{14} group with unsolvable conjugacy problem.

Question

Does $GL_3(\mathbb{Z})$ contain orbit undecidable subgroups ?

Question

Does there exist $\mathbb{Z}^3$-by-free groups with unsolvable conjugacy problem ?

But...

Theorem (Bogopolski-Martino-V., 2008)

There are exist 14 matrices $M_1, \ldots, M_{14} \in GL_n(\mathbb{Z})$, for $n \geqslant 4$, such that $\langle M_1, \ldots, M_{14} \rangle \leqslant GL_n(\mathbb{Z})$ is orbit undecidable.

Corollary

There exists a $\mathbb{Z}^4$-by-F_{14} group with unsolvable conjugacy problem.

Question

Does $GL_3(\mathbb{Z})$ contain orbit undecidable subgroups ?

Question

Does there exist $\mathbb{Z}^3$-by-free groups with unsolvable conjugacy problem ?

But...

Theorem (Bogopolski-Martino-V., 2008)

There are exist 14 matrices $M_1, \ldots, M_{14} \in GL_n(\mathbb{Z})$, for $n \geqslant 4$, such that $\langle M_1, \ldots, M_{14} \rangle \leqslant GL_n(\mathbb{Z})$ is orbit undecidable.

Corollary

There exists a $\mathbb{Z}^4$-by-F_{14} group with unsolvable conjugacy problem.

Question

Does $GL_3(\mathbb{Z})$ contain orbit undecidable subgroups ?

Question

Does there exist $\mathbb{Z}^3$-by-free groups with unsolvable conjugacy problem ?

But...

Theorem (Bogopolski-Martino-V., 2008)

There are exist 14 matrices $M_1, \ldots, M_{14} \in GL_n(\mathbb{Z})$, for $n \geqslant 4$, such that $\langle M_1, \ldots, M_{14} \rangle \leqslant GL_n(\mathbb{Z})$ is orbit undecidable.

Corollary

There exists a $\mathbb{Z}^4$-by-F_{14} group with unsolvable conjugacy problem.

Question

Does $GL_3(\mathbb{Z})$ contain orbit undecidable subgroups ?

Question

Does there exist $\mathbb{Z}^3$-by-free groups with unsolvable conjugacy problem ?

But...

Theorem (Bogopolski-Martino-V., 2008)

There are exist 14 matrices $M_1, \ldots, M_{14} \in GL_n(\mathbb{Z})$, for $n \geqslant 4$, such that $\langle M_1, \ldots, M_{14} \rangle \leqslant GL_n(\mathbb{Z})$ is orbit undecidable.

Corollary

There exists a $\mathbb{Z}^4$-by-F_{14} group with unsolvable conjugacy problem.

Question

Does $GL_3(\mathbb{Z})$ contain orbit undecidable subgroups ?

Question

Does there exist $\mathbb{Z}^3$-by-free groups with unsolvable conjugacy problem ?

THANKS