Algebraic extensions in free groups with two applications

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Outline

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Definitions and notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- Usually, $A = \{a, b, c\}$.
- $(A^{\pm 1})^*$ the free monoid on $A^{\pm 1}$ (words on $A^{\pm 1}$).
- $F_A = (A^{\pm 1})^* / \sim$ is the free group on A (words on $A^{\pm 1}$ modulo reduction).
- Every $w \in A^*$ has a unique reduced form,
- 1 denotes the empty word, and $|\cdot|$ the (shortest) length in F_A:
  $|1| = 0, \quad |aba^{-1}| = |abbb^{-1}a^{-1}| = 3, \quad |uv| \leqslant |u| + |v|$.

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Motivation

- In basic linear algebra:

$$U \leqslant V \leqslant K^n \quad \Rightarrow \quad V = U \oplus L.$$

- In $\mathbb{Z}^n$, the analog is almost true:

$$U \leqslant V \leqslant \mathbb{Z}^n \quad \Rightarrow \quad \exists\ U \leq_{fi} U' \leqslant V \text{ s.t. } V = U' \oplus L.$$

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almost true again, ... in the sense of Takahasi.

Algebraic and transcendental elements

Mimicking field theory...

Definition

Let $H \leqslant F_A$ and $w \in F_A$. We say that w is

- *algebraic over H if $\exists\ 1 \neq e_H(x) \in H * \langle x \rangle$ such that $e_H(w) = 1$;*
- *transcendental over H otherwise.*

Observation

w is transcendental over $H \iff \langle H, w \rangle \simeq H * \langle w \rangle$
$\iff H$ is contained in a proper f.f. of $\langle H, w \rangle$.

Problem

w_1, w_2 algebraic over $H \not\Rightarrow w_1 w_2$ algebraic over H.

$H = \langle a, \overline{b}ab, \overline{c}ac \rangle \leqslant \langle a, b, c \rangle$, and $w_1 = b$, $w_2 = \overline{c}$

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A relative notion works better...

Definition

Let $H \leqslant K \leqslant F_A$ and $w \in K$. We say that w is

- *K-algebraic over H if $\forall$ free factorization $K = K_1 * K_2$ with $H \leqslant K_1$, we have $w \in K_1$;*
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Observation

w is algebraic over H if and only if it is $\langle H, w \rangle$-algebraic over H.

Observation

If w_1 and w_2 are K-algebraic over H, then so is $w_1 w_2$.

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We say that $H \leqslant K$ is an algebraic extension, denoted $H \leq_{alg} K$,
$\iff$ *every $w \in K$ is K-algebraic over H,*
$\iff$ *H is not contained in any proper free factor of K,*
$\iff$ *$H \leqslant K_1 \leqslant K_1 * K_2 = K$ implies $K_2 = 1$.*

We say that $H \leqslant K$ is a free extension, denoted $H \leq_{ff} K$,
$\iff$ *every $w \in K \setminus H$ is K-transcendental over H,*
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- if $r(H) \geqslant 2$ and $r(K) \leqslant 2$ then $H \leqslant_{alg} K$.
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Can we compute them all ?

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Takahasi's Theorem

Theorem (Takahasi, 1951)

For every $H \leqslant_{fg} F_A$, the set of algebraic extensions, denoted $\mathcal{AE}(H)$, is finite.

- Original proof by Takahasi was combinatorial and technical,
- Modern proof, using Stallings automata, is much simpler, and due independently to Ventura (1997), Margolis-Sapir-Weil (2001) and Kapovich-Miasnikov (2002).
- Additionally, $\mathcal{AE}(H)$ is computable.

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Outline

Definition

A Stallings automaton is a finite A-labeled oriented graph with a distinguished vertex, (X, v)*, such that:*

1- X *is connected,*

2- *no vertex of degree 1 except possibly* v *(*X *is a core-graph),*

3- *no two edges with the same label go out of (or in to) the same vertex.*

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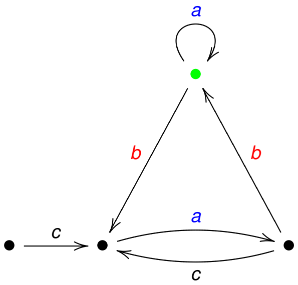

YES :

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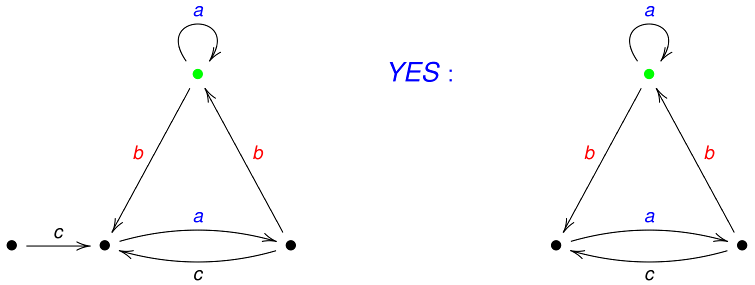

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Stallings (building on previous works) gave a bijection between finitely generated subgroups of F_A and Stallings automata:

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Reading the subgroup from the automata

Definition

To any given (Stallings) automaton (X, v), we associate its fundamental group:

$$\pi(X, v) = \{ \textit{ labels of closed paths at } v\} \leqslant F_A,$$

clearly, a subgroup of F_A.

$$\pi(X, \bullet) = \{1,\ a,\ a^{-1},\ bab,\ bc^{-1}b,\ babab^{-1}cb^{-1}, \ldots\}$$

$$\pi(X, \bullet) \not\ni bc^{-1}bcaa$$

Membership problem in $\pi(X, \bullet)$ is solvable.

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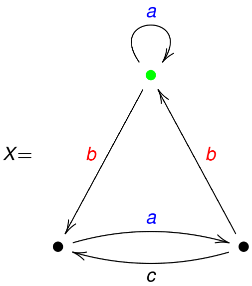

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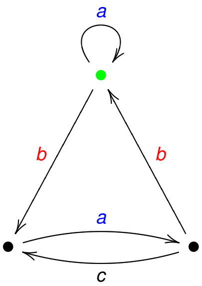

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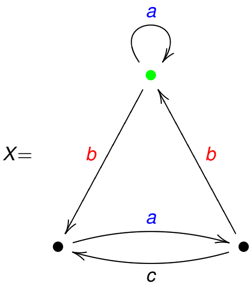

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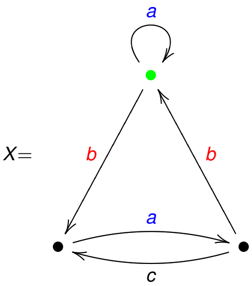

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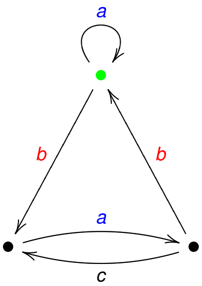
$X=$ 

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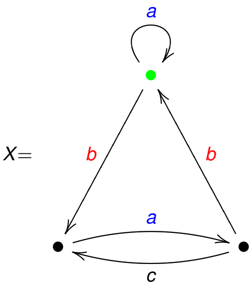

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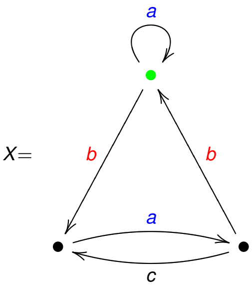

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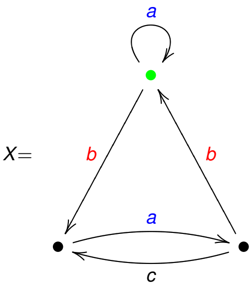

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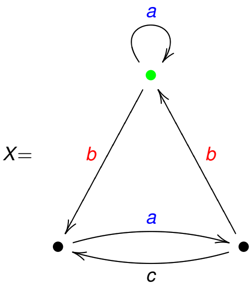

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A basis for $\pi(X, v)$

Proposition

For every Stallings automaton (X, v), the group $\pi(X, v)$ is free of rank $rk(\pi(X, v)) = 1 - |VX| + |EX|$.

Proof:

- Take a maximal tree T in X.
- Write $T[p, q]$ for the geodesic (i.e. the unique reduced path) in T from p to q.
- For every $e \in EX - ET$, $x_e = label(T[v, \iota e] \cdot e \cdot T[\tau e, v])$ belongs to $\pi(X, v)$.
- Not difficult to see that $\{x_e \mid e \in EX - ET\}$ is a basis for $\pi(X, v)$.
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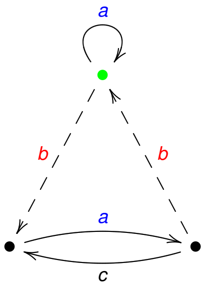
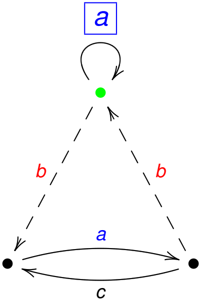


$H = \langle \quad \rangle$

Example



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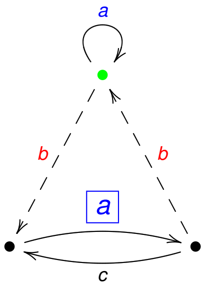
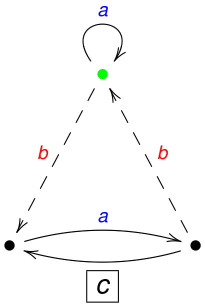


$H = \langle a,\ bab,\ \quad \rangle$

Example



$H = \langle a,\ bab,\ b^{-1}cb^{-1} \rangle$

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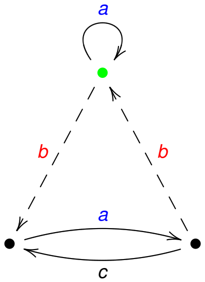
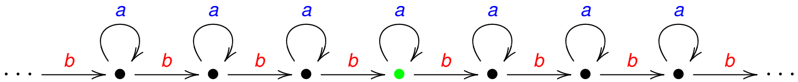


$H = \langle a,\ bab,\ b^{-1}cb^{-1} \rangle$
$rk(H) = 1 - 3 + 5 = 3.$

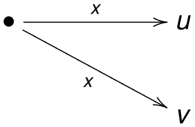
Example-2



$$F_{\aleph_0} \simeq H = \langle \ldots, b^{-2}ab^2, b^{-1}ab, a, bab^{-1}, b^2ab^{-2}, \ldots \rangle \leqslant F_2.$$

Constructing the automata from the subgroup

In any automaton containing the following situation, for $x \in A^{\pm 1}$,



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This operation, $(X, v) \rightsquigarrow (X', v)$, is called a Stallings folding.

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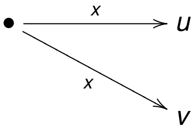

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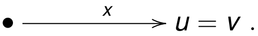

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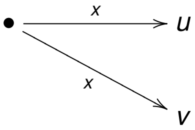

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$$\bullet \xrightarrow{\quad x \quad} u = v \ .$$

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Constructing the automata from the subgroup

Lemma (Stallings)

If $(X, v) \rightsquigarrow (X', v')$ is a Stallings folding then $\pi(X, v) = \pi(X', v')$.

Given a f.g. subgroup $H = \langle w_1, \ldots, w_m \rangle \leqslant F_A$ (we assume w_i are reduced words), do the following:

1- *Draw the flower automaton,*

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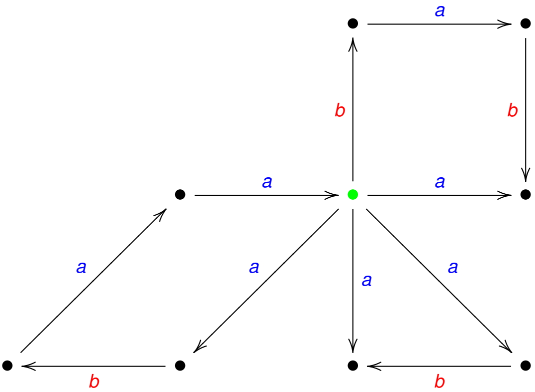
Example: $H = \langle baba^{-1}, aba^{-1}, aba^2 \rangle$



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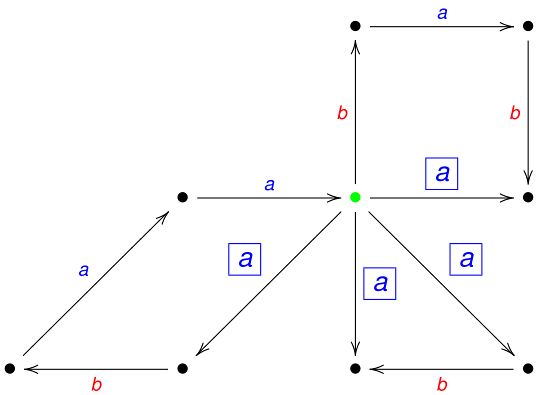

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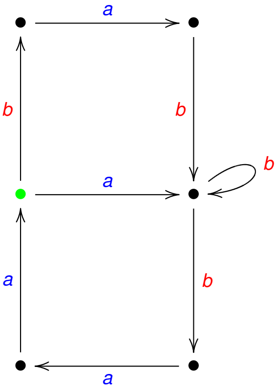

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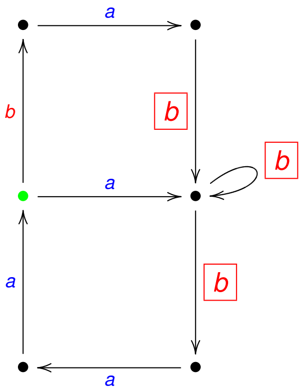

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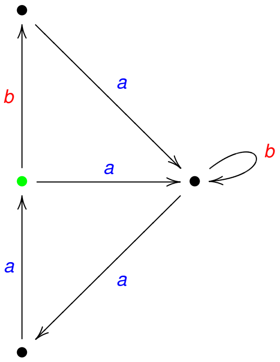

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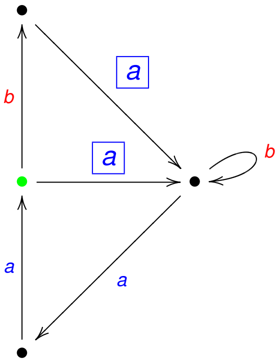

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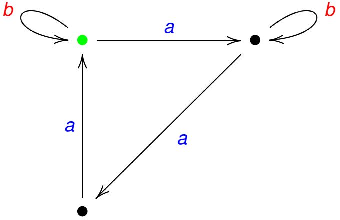

Folding #3. $\Gamma(H)$

By Stallings Lemma, $\pi(\Gamma(H), \bullet) = \langle baba^{-1}, aba^{-1}, aba^2 \rangle$

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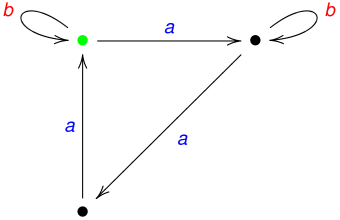

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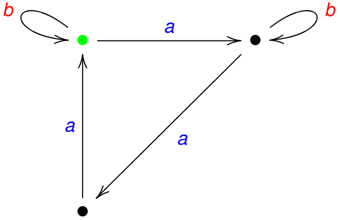
Example: $H = \langle baba^{-1}, aba^{-1}, aba^2 \rangle$



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Local confluence

It can be shown that

Proposition

The automaton $\Gamma(H)$ does not depend on the sequence of foldings

Proposition

The automaton $\Gamma(H)$ does not depend on the generators of H.

Theorem

The following is a bijection:

$$\begin{array}{rcl} \{\text{f.g. subgroups of } F_A\} & \longleftrightarrow & \{\text{Stallings automata}\} \\ H & \to & \Gamma(H) \\ \pi(X, v) & \leftarrow & (X, v) \end{array}$$

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Nielsen-Schreier Theorem

Corollary (Nielsen-Schreier)

Every subgroup of F_A is free.

- Finite automata work for the finitely generated case, but everything extends easily to the general case (using infinite graphs).
- The original proof (1920's) is combinatorial and much more technical.

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Outline

Takahasi's theorem

Definition

Let $H \leqslant K \leqslant F_A$. Then, $H \leqslant K$ is algebraic if and only if H is not contained in any proper free factor of K.

Theorem (Takahasi, 1951)

For every $H \leqslant_{fg} F_A$, the set of algebraic extensions, $\mathcal{AE}(H)$, is finite.

Proof (Ventura; Margolis-Sapir-Weil; Kapovich-Miasnikov):

- Consider $\tilde{\Gamma}(H)$, the result of attaching all possible (infinite) "hairs" to $\Gamma(H)$ (i.e. the covering of the bouquet corresponding to H).
- Given $H \leqslant K$ (both f.g.), we can obtain $\tilde{\Gamma}(K)$ from $\tilde{\Gamma}(H)$ by performing the appropriate identifications of vertices (plus subsequent foldings).

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Takahasi's theorem

- Hence, if $H \leqslant K$ (both f.g.) then $\Gamma(K)$ contains as a subgraph either $\Gamma(H)$ or some quotient of it (i.e. $\Gamma(H)$ after some identifications of vertices, $\Gamma(H)/\sim$).
- The overgroups of H: $\mathcal{O}(H) = \{\pi(\Gamma(H)/\sim, \bullet) \mid \sim \text{ is a partition of } V\Gamma(H)\}$.
- Hence, for every $H \leqslant K$, there exists $L \in \mathcal{O}(H)$ such that $H \leqslant L \leqslant_{ff} K$.
- Thus, $\mathcal{AE}(H) \subseteq \mathcal{O}(H)$ and so, it is finite. $\square$

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- In basic linear algebra:

$$U \leqslant V \leqslant K^n \quad \Rightarrow \quad V = U \oplus L.$$

- In $\mathbb{Z}^n$, the analog is almost true:

$$U \leqslant V \leqslant \mathbb{Z}^n \quad \Rightarrow \quad \exists\ U \leq_{fi} U' \leqslant V \text{ s.t. } V = U' \oplus L.$$

- In F_A, the following analog is true:

$$H \leqslant K \leqslant F_A \quad \Rightarrow \quad \exists\ H \leq_{alg} H_i \leqslant K \text{ s.t. } K = H_i * L.$$

Computing $\mathcal{AE}(H)$

Corollary

$\mathcal{AE}(H)$ is computable.

Proof:

- Compute $\Gamma(H)$,
- Compute $\Gamma(H)/\sim$ for all partitions $\sim$ of $V\Gamma(H)$,
- Compute $\mathcal{O}(H)$,
- Clean $\mathcal{O}(H)$ by detecting all pairs $K_1, K_2 \in \mathcal{O}(H)$ such that $K_1 \leqslant_{ff} K_2$ and deleting K_2.
- The resulting set is $\mathcal{AE}(H)$. $\square$

For the cleaning step we need:

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For the cleaning step we need:

Deciding free-factorness

Proposition

Given $H, K \leqslant F_A$, it is algorithmically decidable whether $H \leqslant_{ff} K$ or not.

Proved by:

- Whitehead 1930's (classical and exponential),
- Silva-Weil 2006 (graphical algorithm, faster but still exponential),
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The algebraic closure

Observation

If $H \leqslant_{alg} K_1$ and $H \leqslant_{alg} K_2$ then $H \leqslant_{alg} \langle K_1 \cup K_2 \rangle$.

Corollary

For every $H \leqslant K \leqslant F_A$ (all f.g.), $\mathcal{AE}_K(H)$ has a unique maximal element, called the K-algebraic closure of H, and denoted $Cl_K(H)$.

Corollary

Every extension $H \leqslant K$ of f.g. subgroups of F_A splits, in a unique way, in an algebraic part and a free part, $H \leqslant_{alg} Cl_K(H) \leqslant_{ff} K$.

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Outline

Pseudo-varieties of finite groups

Definition

A pseudo-variety of groups $\mathcal{V}$ is a class of finite groups closed under taking subgroups, quotients and finite direct products.

- $\mathcal{G}$ = all finite groups,
- $\mathcal{G}_p$ = all finite p-groups,
- $\mathcal{G}_{nil}$ = all finite nilpotent groups,
- $\mathcal{G}_{sol}$ = all finite soluble groups,
- $\mathcal{G}_{ab}$ = all finite abelian groups,
- for a finite group V, $[V]$ = all quotients of subgroups of V^k, $k \geqslant 1$.
- $\dots$

Definition

$\mathcal{V}$ is *extension-closed* if $V \lhd W$ with $V, W/V \in \mathcal{V}$ imply $W \in \mathcal{V}$.

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The pro-$\mathcal{V}$ topology in G

Definition

Let G be a group, and $\mathcal{V}$ be a pseudo-variety of finite groups. The pro-$\mathcal{V}$ topology on G can be defined in several equivalent ways:

- *it is the smallest topology making all the morphisms from G into all $V \in \mathcal{V}$ (with the discrete topology) continuous,*
- *a basis of open sets is given by $\varphi^{-1}(x)$, for all morphism $\varphi\colon G \to V \in \mathcal{V}$,*
- *the normal (finite index) subgroups $K \trianglelefteq G$ such that $G/K \in \mathcal{V}$ form a basis of neighborhoods of 1,*
- *it is the topology given by the pseudo-ultra-metric $d(x, y) = 2^{-r(x,y)}$, where $r(x, y) = \min\{|V| \mid V \in \mathcal{V}$ and separates x and y $\}$.*

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This topology is Hausdorf $\Longleftrightarrow$ d is an ultra-metric $\Longleftrightarrow$ G is residually-$\mathcal{V}$.

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$\mathcal{V}$-closures in F_A

Proposition (Ribes, Zaleskiĭ)

Let $\mathcal{V}$ be an extension-closed pseudo-variety, and consider F_A the free group on A with the pro-$\mathcal{V}$ topology. Then, for $H \leq_{ff} K \leqslant F_A$, both f.g.,

$$K \quad \mathcal{V}-closed \implies H \quad \mathcal{V}-closed.$$

Corollary

For an extension-closed $\mathcal{V}$ and a $H \leq_{fg} F_A$, we have $H \leq_{alg} cl_{\mathcal{V}}(H)$.

Furthermore,

Proposition (Ribes, Zaleskiĭ)

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Let $\mathcal{V}$ be an extension-closed pseudo-variety, and consider F_A the free group on A with the pro-$\mathcal{V}$ topology. Then, for $H \leq_{ff} K \leqslant F_A$, both f.g.,

$$K \quad \mathcal{V}-closed \implies H \quad \mathcal{V}-closed.$$

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For an extension-closed $\mathcal{V}$ and a $H \leq_{fg} F_A$, we have $H \leq_{alg} cl_{\mathcal{V}}(H)$.

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Proposition (Margolis-Sapir-Weil)

The p-closure of $H \leq_{fg} F_A$ is effectively computable, for all primes p.

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The nil-closure of $H \leq_{fg} F_A$ is the intersection, over all primes, of the p-closure of H. Hence, it is effectively computable.

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Is the sol-closure of $H \leq_{fg} F_A$ effectively computable ?

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Outline

The four families

Definition

A subgroup $H \leqslant F_A$ is said to be

- *1-auto-fixed if $H = Fix(\phi)$ for some $\phi \in Aut(F_A)$,*
- *1-endo-fixed if $H = Fix(\phi)$ for some $\phi \in End(F_A)$,*
- *auto-fixed if $H = Fix(S) = \cap_{\phi \in S} Fix(\phi)$ for some $S \subseteq Aut(F_A)$,*
- *endo-fixed if $H = Fix(S) = \cap_{\phi \in S} Fix(\phi)$ for some $S \subseteq End(F_A)$,*

Easy to see that 1-mono-fixed = 1-auto-fixed.

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Relations between them

$$\begin{array}{ccc}
\boxed{1-auto-fixed} & \subseteq & \boxed{1-endo-fixed} \\
\cap\!| & & \cap\!| \\
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$$\boxed{1-auto-fixed} \quad \underset{\neq}{\subseteq} \quad \boxed{1-endo-fixed}$$

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Example (Martino-V., 03; Ciobanu-Dicks, 06)

Let $F_3 = \langle a, b, c\rangle$ *and* $H = \langle b, cacbab^{-1}c^{-1}\rangle \leqslant F_3$. *Then,* $H = Fix(a \mapsto 1,\ b \mapsto b,\ c \mapsto cacbab^{-1}c^{-1})$, *but H is NOT the fixed subgroup of any set of automorphism of* F_3.

Relations between them

$$\boxed{1-auto-fixed} \quad \overset{\subseteq}{\neq} \quad \boxed{1-endo-fixed}$$

$$\cap| \quad \| ? \qquad\qquad\qquad \cap| \quad \| ?$$

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Problem

Vertical inclusions are equalities ?

In other words,

Are the families of 1-auto-fixed and 1-endo-fixed subgroups of F_A closed under intersection ?

Yes, up to free complements

Theorem (Martino-V., 00)

Let $S \subseteq End(F_A)$. Then, $\exists \phi \in \langle S \rangle$ such that $Fix(S) \leq_{ff} Fix(\phi)$.

Sketch. One can reduce the problem to

- $S \subseteq \mathrm{Aut}(F_A)$,
- $|S| = 2$, say $S = \{\alpha, \beta\}$,
- $\mathrm{Per}(\beta) = \mathrm{Fix}(\beta)$.

Now, take $H = \mathrm{Fix}(\alpha) \cap \mathrm{Fix}(\beta)$ and we'll see $H \leq_{ff} \mathrm{Fix}(\alpha\beta^n)$ for some n:

- Clearly, $H \leqslant \mathrm{Fix}(\alpha\beta^n)$, for every n.
- $\forall n, \exists H_n \in \mathcal{AE}(H)$ such that $H \leqslant H_n \leq_{ff} \mathrm{Fix}(\alpha\beta^n)$.
- Take $n < m$ with $H_n = H_m$ (recall that $\mathcal{AE}(H)$ is finite).

$$\mathrm{Fix}(\alpha)\cap\mathrm{Fix}(\beta) = H \leqslant H_n \cap H_m \leqslant \mathrm{Fix}(\alpha\beta^n)\cap\mathrm{Fix}((\alpha\beta^m)) \leqslant \mathrm{Fix}(\alpha)\cap\mathrm{Fix}(\beta).$$

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