Isomorphisms between Graph Products of Groups

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Abstract

We say a group is rigid if its decomposition as a graph product of groups is essentially unique. This property is known to fail in general. In this paper, rigidity is proved when vertex groups satisfy: 1. Serre's (FA) property, 2. are well behaved under direct products (in a sense which must be precised).

1 Definitions and notation.

We start fixing some terminology about graphs and graph products (sections 1.1 and 1.2). Afterwards we formally introduce and state the problem of rigidity (sections 1.3 and 1.4). There, we also provide a quick overview of the most important known results on this problem. Finally, section 1.5 is devoted to make a quick introduction to Bass-Serre theory, focused on groups which satisfy the (FA) property.

1.1 Graphs

We start this section providing the most elemental definitions of graph theory. Next the modular expansion of a graph is defined. This last notion will prove to be natural when studying rigidity of graph products of groups.

Definition 1. A *graph* X is a tuple $(V(X), E(X))$ with $V(X) \neq \emptyset$, and $E(X) \subseteq V(X) \times V(X)$ such that $(v,u) \in E(X)$ if and only if $(u,v) \in E(X)$.

Usually $V(X)$ and $E(X)$ are called the vertex and the edge set of the graph, respectively. It is common practice to denote the cardinality of $V(X)$ by $|X|$. The graph is called *finite* if $|X|$ is finite. Two vertices v and $u \in V(X)$ are said to be adjacent, or connected, if $(v,u) \in E(X)$. Given an edge $(v,u) \in E(X)$, then (u,v) is said to be its inverse edge. Edges will sometimes be denoted by single roman letters, for example e. If, for every pair of vertices v and $u \in V(X)$, we have $(v,u) \in E(X)$, then X is said to be *complete*.

Definition 2. Let X be a graph, and $v \in V(X)$ a vertex such that $(v,v) \in E(X)$. Then the edge (v,v) is called a *loop*.

DEFINITION 3. Let X and Y be two graphs, and f an application $f : V(X) \to V(Y)$. Then f is called a *graph morphism* if, for every pair of different vertices v, $u \in V(X)$ such that $f(v) \neq f(u)$, we have $(f(v), f(u)) \in E(Y)$ if and only if $(v,u) \in E(X)$.

Remark. Sometimes a graph morphism is *only* required to preserve the adjacency of adjacent vertices. That is, $(f(v), f(u)) \in E(Y)$ whenever $(v,u) \in E(X)$ (even if $f(u) = f(v)$, in which case a loop is formed). Since we shall not need this kind of application, we will stick to the more restrictive definition we just introduced.

As usual, a graph morphism f is called a graph monomorphism, epimorphism, or isomorphism if f is injective, exhaustive, or bijective, respectively.

DEFINITION 4. Let X and Z be two graphs. Then Z is said to be a *full subgraph of* X if there exists a graph monomorphism $f : V(Z) \hookrightarrow V(X)$.

A full subgraph isomorphic to a complete graph is called a *clique*.

DEFINITION 5. Let n be an integer greater or equal to 1. An n-cycle C_n is a graph consisting of n vertices, $V(C_n) = \{v_1, ..., v_n\}$, and edges $E(C_n) = \{(v_1, v_2), ..., (v_{n-1}, v_n), (v_n, v_1)\}$ and its inverses. Similarly, an m-path P_m, $m \geq 1$, is a graph with n vertices, $\{v_1, ..., v_n\}$, and edges $\{(v_1, v_2), ..., (v_{n-1}, v_n))\}$ together with its inverses.

Remark. By the way we have defined our concepts, there do not exist 2-cycles. A loop (v, v), together with v, is a 1-cycle.

DEFINITION 6. A graph X is said to be *connected* if, for every two vertices v, $u \in V(X)$, there exists an integer $n \geq 1$ and an n-path P_n with vertices $\{w_1, ..., w_n\}$ such that $v = w_1$ and $u = w_n$. In this case, v and u are said to be *connected by* P_n.

DEFINITION 7. A graph X is said to be a *tree* if it is connected and there is no $n \geq 1$ such that C_n is a full subgraph of X.

Note that, if X is a tree, then for every pair of vertices from $V(X)$ there is a unique path connecting them.

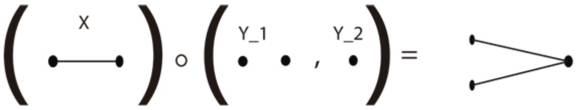
DEFINITION 8. Given a graph X, a full subgraph induced by a subset M of its vertices is called a *module* if, for every vertex $v \in X - M$, v is adjacent to some vertex $u \in M$ if and only if it is adjacent to every vertex in M. M is said to be *trivial* if it consists of a single vertex or if it is equal to the whole X. X is said to be *prime* if all its modules are trivial.

Given a graph X and a module M in it, there is a natural graph epimorphism:

$$[]_M : X \twoheadrightarrow X/M$$

This epimorphism consists in replacing the module M by a single vertex $v(M)$, and connecting $v(M)$ to another vertex $w \in V(X) - V(M)$ if some vertex in M (and hence all) is adjacent to w. This is clearly well defined and a graph epimorphism.

DEFINITION 9. Let X be a graph with n vertices $\{v_1, ..., v_n\}$, and $\mathscr{Y} = (Y_1, ..., Y_n)$ an n-tuple of graphs. Suppose the vertices of Y_i are $\{u_{i1}, ..., u_{it_i}\}$. Then the *modular expansion of* X *with respect to* $\mathscr{Y}$, $X \circ \mathscr{Y}$, is the graph obtained from X by replacing each vertex v_i by the graph Y_i, and connecting u_{ij} with u_{sr} if and only if v_i and v_s were adjacent in X (see the next figure).



Clearly, the modular expansion $X \circ \mathscr{Y}$ contains Y_1, ..., Y_n as disjoint modules, and the following sequence of epimorphisms is well defined:

$$X \circ \mathscr{Y} \xrightarrow{[]_{Y_1}} (X \circ \mathscr{Y})/Y_1 \xrightarrow{[]_{Y_2}} \ldots \xrightarrow{[]_{Y_n}} X$$

The reader is refered to [7], [15] for more information on modules.

1.2 Graph products of groups.

In this section we introduce the notion of a graph product of groups, together with some fundamentl results which will be required later at some point.

When talking about graph products of groups, we will always assume graphs are finite and without loops, and vertex groups are nontrivial and finitely generated.

DEFINITION 10. Let $\mathscr{G} = (G_1, ..., G_n)$ be an n- tuple of (nontrivial and finitely generated) groups, and X a (finite) graph (without loops) with n vertices. Then $X\mathscr{G} = X(G_1, ..., G_n)$ will denote the *graph product of the groups* G_1, ..., G_n *with respect to the graph* X. This is the group obtained when taking the quotient of the free product $G_1 * ... * G_n$ by the normal clousure of the set

$$\{[G_i, G_j] | v_i \sim v_j\}$$

where $[G_i, G_j] = \{[a,b] | a \in G_i, b \in G_j\}$, and $[a,b] = aba^{-1}b^{-1}$.

In other words, the graph product of the groups G_1, ..., G_n, associated to the vertices v_1, ..., v_n, in X is the group obtained from their free product once we allow to commute all pairs of elements which lie in groups corresponding to adjacent vertices.

It is assumed that the group G_i corresponds to the i-th vertex. In addition, given a vertex $v \in V(X)$, we will write G_v to denote the vertex group corresponding to v. Sometimes we will use the notation $X\mathscr{G}$ even when the number of vertices n in X is not equal to the number m of groups in the tuple $\mathscr{G}$. In this case, if $n < m$, $X\mathscr{G}$ will mean $X(G_1, ..., G_n)$, and, if $n > m$, $X\mathscr{G}$ will mean $X(G_1, ..., G_m, 1, ..., 1)$. Note that the latter is isomorphic to $X'\mathscr{G}$, where X' is obtained from X by removing its last $n - m$ vertices (and the edges incident to them).

Given a graph product $X\mathscr{G}$ with $V(X) = \{v_1, ..., v_n\}$, and Z a full subgraph of X with $V(Z) = \{v_{i_1}, ..., v_{i_m}\}$, we make the convention that $Z\mathscr{G}$ denotes $Z(G_{i_1}, ..., G_{i_m})$. The following proposition, which is of common usage, assures that $Z\mathscr{G}$ is canonicaly embedded in $X\mathscr{G}$.

PROPOSITION 1 (cf. [16], Theorem 3.2). *Let $X\mathscr{G}$ be a graph product of groups, and Z a full subgraph of X. Then the subgroup of $X\mathscr{G}$ generated by $\{G_i | v_i \in Z\}$ is canonically isomorphic to $Z\mathscr{G}$.*

EXAMPLE 1. *Let C be a clique in a graph X with vertices $V(C) = \{v_1, ..., v_t\}$, and $\mathscr{G}$ an $|X|$-tuple of groups. Then the group $G_1 \times ... \times G_t$ is canonically isomorphic to $C(G_1, ..., G_t)$, and canonically embedded in $X\mathscr{G}$.*

Given a graph X and a group G, XG will denote the graph product $X\mathscr{G}$, where $\mathscr{G} = (G, ..., G)$. For example, any right angled Artin group can be written as $X\mathbb{Z}$ for some graph X, and any right angled Coxeter group as $X\mathbb{Z}_2$.

DEFINITION 11. We will say that a graph product of groups is *trivial* if its corresponding graph has only one vertex (recall that we do not allow trivial groups on the vertices, otherwise a trivial graph product of groups would be a graph product with associated graph having either one single vertex or trivial groups on the vertices, except by one vertex). A group will be called *graphologically indecomposable* if it cannot be expressed as a nontrivial graph of groups.

DEFINITION 12. Let x be an element in $X\mathscr{G}$, then we can write $x = g_1...g_t$ with every g_i belonging to a vertex group, say G_{j_i}. The elements g_i will be called syllables and the expression $g_1...g_t$ a *syllable form*. The positive integer t is said to be the *length of the syllable form*. We will say that the vertex v_{j_i} *appears* in the expression $g_1...g_t$.

DEFINITION 13. *Let x be an element in $X\mathscr{G}$, and $x = g_1...g_t$ a syllable form of x. If its lenght t is minimal among all syllable forms for x, then we say that $g_1...g_t$ is a normal form for x.*

THEOREM 1 (NORMAL FORM THEOREM.). *Given any syllable expression $g_1...g_t$ of an element $x \in X\mathscr{G}$, we can reach a normal form of x using only combinations of the following movements:*

1. Permute g_i with g_{i+1} if they belong to adjacent groups.
2. Melt g_i with g_{i+1}, if they belong to the same group G_j, into a single element of the group G_j.
3. Erase any g_i if it is the identity element.

Morevoer, if we define $supp(x)$ as the set of vertices appearing in a particular normal form for x, and $link(x)$ as the set of vertices connected to every vertex in $supp(x)$, then $supp(x)$ and $link(x)$ are well defined: they do not depend on which normal form for x is chosen. Moreover, a normal form for x is unique up to movements of type 1, and the vertices in $supp(x)$ must appear in any expression $g_1...g_k$ of x, normal or not.

COROLLARY 1. *Let $x \in X\mathscr{G}$ be an element in a graph product of groups, and suppose x admits two different syllable forms:*

$$x = a_1...a_r = b_1...b_s$$

Let $\mathscr{A}$ and $\mathscr{B}$ denote the set of groups appearing in the first and second expression of x, respectively. Then $supp(x) \subseteq \mathscr{A} \cap \mathscr{B}$.

DEFINITION 14. Let $X\mathscr{G}$ be a graph product of groups, and $S \subseteq X\mathscr{G}$ a subset. Then $supp(S)$ is defined to be:

$$supp(S) = \bigcup_{x \in S} supp(x).$$

It is clear that, with the above notation, if $S \subseteq T \subseteq X\mathscr{G}$, then $supp(S) \subseteq supp(T)$.

PROPOSITION 2. *Let $\mathscr{G} = (G_1, ..., G_n)$, and $\mathscr{H} = \{H_1, ..., H_n\}$ be two n-tuples of groups such that $H_i \trianglelefteq G_i$ for all i. Let X be a graph with n vertices and denote the normal clousure of $X\mathscr{H} \leq X\mathscr{G}$ by $<< X\mathscr{H} >>$. Then;*

$$X\mathscr{G} / << X\mathscr{H} >> \cong X(\mathscr{G}/\mathscr{H}),$$

where $\mathscr{G}/\mathscr{H} = (G_1/H_1, ..., G_n/H_n)$.

Proof. Let $\iota : X\mathscr{G} \twoheadrightarrow X(\mathscr{G}/\mathscr{H})$ be the canonical epimorphism consisting in adding the relations $\bigcup_i H_i$ (eg, ι is the natural epimorphism arising when one takes the quotient of $X\mathscr{G}$ by the normal clousure of $\bigcup_i H_i$). Then the kernel of ι is $<< \bigcup_i H_i >>$, which is also the normal closure of $< \bigcup_i H_i >$, eg, $ker\iota =<< (< \bigcup_i H_i >) >>$, but, by proposition 1, $X\mathscr{H} =< \bigcup_i H_i >$.

□

1.3 Modules and different graph product decompositions.

Given a graph product, $X\mathscr{G}$, one can naturally obtain another isomorphic graph product of different groups, with respect to different graphs, in the following two ways:

1. Let M be a nontrivial module in X, and denote X' the graph obtained from X by contracting M into a vertex. We can suppose the vertices of M are the first t vertices in X. Then the following is clear;

   $$X(G_1, ..., G_n) \cong X'(M(G_1, ..., G_t), G_{t+1}, ..., G_n)$$

2. Suppose G_1 can be expressed as a nontrivial graph product of groups:

   $$G_1 = Y(H_1, ..., H_r)$$

   Then the following is also clear;

   $$X\mathscr{G} \cong (X \circ (Y, v_2, ..., v_n))(H_1, ..., H_r, G_2, ..., G_n),$$

   where v_i means a graph with a single vertex v_i.

Remark. One could try to define some kind of *normal form* for a graph product, consisting in applying transformations of type 2, until no vertex group is nontrivially graphologically decomposable. In that case, however, it should be proved that the process ends at some point.

1.4 Rigidity.

The question this paper deals with is that of whether a graph product decomposition is unique up to the transformations described above. One of the first results on rigidity, and probably the most known, is Drom's, [6], in which right angled artin groups are shown to be rigid. This result was sharpened by Laurence, [14]. Another first result was given by Green on graph products of finite indecomposable cyclic groups, [8]. Later, in [16], Radcliffe probably uses the term rigid for the first time, and proves that, when all vertex groups are finite and indecomposable as direct products, then the graph product is rigid. Finally, in [9], Gutiérrez and Piggot extend Drom's and Green's result by proving rigidity for graph products of abelian groups.

Rigidity for general graph products of groups is known to be false. Indeed, Ruth Green, in her thesis [8], where the notion of graph product is first introduced, wonders about general rigidity, and provides the following construction from [13] as a negative answer:

PROPOSITION 3. *Consider the following groups:*

$$A =< a_1, a_2 | {a_1}^2 = {a_2}^2 >$$

$$B =< b_1, b_2 | {b_1}^3 = {b_2}^3 >$$

$$C =< c_1, c_2, c_3, c_4 | {c_1}^2 = {c_2}^2 = {c_3}^3 = {c_4}^3 >$$

$$D =< d >$$

Then, $A \times B \cong C \times D$, but the four groups are graphologically indecomposable and A, B are not isomorphic to C or D.

1.5 Groups acting on trees.

In this section we introduce the theory of groups acting on trees without inverting any edge. This is commonly known as Bass-Serre theory. It was developed by Jean-Pierre Serre in 1970's and first introduced in his monograph *Trees*, [18], which was written in collaboration with Hyman Bass. Nowadays it is a common and important tool in group theory. The exelent introductory book by Bogopolski, [3], deals neatly with the vast majority of concepts and results we will be using. The classical but more advanced books by Serre, [18], and Dicks and Dunwoody, [5], are also good references.

We start the section with basic definitions, followed by the statement of the two fundamental results in Bass-Serre theory, (theorems 2 and 3). Afterwards we study groups which cannot be expressed as a nontrivial fundamental group of a graph of groups (in a sense which must be precised), and provide a result which will prove to be key in the following sections, (propositions 5).

Before starting, recall that the edges of a graph X can be denoted either by ordered pairs of vertices, (v,u), or by a single roman letter e. The nature of the concepts this section deals with makes sometimes convenient to adopt the second kind of notation. In this case, given an edge $e=(v,u)\in E(X)$, $\bar{e}$, $\alpha(e)$ and $\omega(e)$ will denote, respectively, the inverse edge (u,v), the initial vertex v, and the final vertex u.

Remark. When talking about graph products of groups, loops on the vertices were forbidden. It is, however, not the case in Bass-Serre theory, where they play an important role.

DEFINITION 15. Let X be a graph (possibly with loops) and G a group. We say that G *acts on the graph* if there exists an action of G on the vertex set $V(X)$, $G\times V(X)\to V(X)$, $(g,v)\mapsto gv$, such that, for all $g\in G$, the application $\mu_g : V(X)\to V(X)$ defined by $\mu_g(v)=gv$ is a graph automorphism. This action is said to be *without inversion of edges* if, for every edge $(v,u)\in E(X)$, one has $(gv,gu)\neq(u,v)$ for all $g\in G$.

DEFINITION 16. Given a graph X , a group G acting on it, and a vertex or edge $x\in V(X)\cup E(X)$, the *stabilizer subgroup of* x is defined to be the following set;

$$St_G(x)=\{g\in G|gx=x\}$$

DEFINITION 17. A graph of groups $(\mathscr{G},X)$ consists of a connected graph X (possibly with loops), a *vertex group* G_v for every vertex $v\in V(X)$, an *edge group* $G_{(v,u)}$ for each edge $(v,u)\in E(X)$, and group monomorphisms $\alpha_{v,u}:G_{(v,u)}\hookrightarrow G_v$ whenever $(v,u)\in E(X)$. We require in addition that $G_{(v,u)}=G_{(u,v)}$.

Let $(\mathscr{G},X)$ be a graph of groups, and let T be the set $\{t_e|e\in E(X)\}$, where each t_e is a formal element corresponding to an edge in X. Let G be the free product of the groups $\{G_v|v\in V(X)\}$ and the free group with basis T, and let N be the following set:

$$N:=\{{t_e}^{-1}\alpha_e(g)t_e\left(\alpha_{\bar{e}}(g)\right)^{-1}, t_e{t_{\bar{e}}}^{-1}|e\in E(X)\text{ and }g\in G_e\}$$

We will denote the factor group of G by the normal clousure of N by $F(\mathscr{G},X)$.

DEFINITION 18. Let $(\mathscr{G},X)$ be a graph of groups, and T a maximal subtree of the graph X. The *fundamental group* $\pi_1(\mathscr{G},X,T)$ *of the graph of groups* $(\mathscr{G},X)$ *with respect to the subtree* T is the factor group of $F(\mathscr{G},X)$ by the normal clousure of the set $\{t_e|e\in E(T)\}$.

Given a graph of groups $(\mathscr{G},X)$ and two maximal subtrees T_1 and T_2 of X, it can be shown that $\pi_1(\mathscr{G},X,T_1)\cong\pi_1(\mathscr{G},X,T_2)$ (cf. [3], Theorem 16.5). Bearing this in mind, we will often speak about the fundamental group $\pi_1(\mathscr{G},X,T)$ without specifiying anything else about T.

EXAMPLE 2 (Amalgamated free products)**.** Let G and H be two groups with distinguished isomorphic subgroups $A\leq G$ and $B\leq G$. Fix an isomorphism $\phi : A\to B$. *The free product of G and H with amalgamtion of A and B by the isomorphism* ϕ is the factor group of $G*H$ by the normal clousure of the set $\{\phi(a)a^{-1}|a\in A\}$. Whenever there is no risk of ambiguity, this product will be denoted by $G*_A H$ without any mention of B or ϕ. Informally speaking, $G*_A H$ is the group obtained from $G*H$ when identifying A and B.

In case $A = G$ or $B = H$, $G *_A H$ will be called a *trivial amalgamated product*. It turns out to be that $G *_A H$ is canonically isomorphic to the fundamental group of the graph of groups $(\mathscr{G}, Y)$, where Y consists of two diferent vertices v, u, and two edges (v,u), (u,v), with $G_v = G$, $G_u = H$, $G_{(v,u)} = G_{(u,v)} = C$, where C is a group isomorphic to A and B, and monomorphisms $\alpha_{(v,u)}(C) = A$, $\alpha_{(u,v)}(C) = B$.

We will omit the proof of the following three results, which can be found in [3], Theorem 16.10, Theorem 18.2, and Theorem 18.5, respectively.

LEMMA 1. *Given a graph of groups $(\mathscr{G}, X)$, and a vertex or edge group G_x, the homomorphism $i : G_x \to \pi_1(\mathscr{G}, X, T)$ defined by $i(g) = g$ is injective. In other words, vertex and edge groups are canonically embedded in the fundamental group of a graph of groups.*

THEOREM 2. *Let $G = \pi_1(\mathscr{G}, X, T)$ be the fundamental group of a graph of groups $(\mathscr{G}, X)$ with respect to a maximal subtree T. Then the group G acts without inversion of edges on a tree Y, called the* Bass-Serre tree *of G, such that the following holds:*

1. The factor graph Y/G is isomorphic to the graph X.
2. For every vertex $v \in V(Y)$ and edge $e \in E(Y)$, the stabilizers $St_G(v)$, $St_G(e)$ are conjugate to the canonical images of the vertex and edge groups in $\mathscr{G}$, respectively.

The following is the converse of the previous result. Usually it is stated in much more precise terms. We provide a simpler version which is strictly fited to our purposes.

THEOREM 3. *Let a group G act without inversion of edges on a tree Y. Then there exists a graph of groups $(\mathscr{G}, X)$ (X may have loops) such that:*

1. For every $x \in V(X) \cup E(X)$, $G_x = St_G(y(x)) \subseteq G$ for some $y(x) \in V(Y) \cup E(Y)$. Morevoer, $y(x) \in V(Y)$ if and only if $x \in V(X)$.
2. There exists a canonical isomorphism from G onto the group $\pi_1(\mathscr{G}, X, T)$. This isomorphism extends the identity isomorphisms $St_G(y(x)) \to G_x \subseteq \pi_1(\mathscr{G}, XT)$, $x \in V(X)$.

COROLLARY 2. *Let $G = \pi_1(\mathscr{G}, X, T)$ be a fundamental group of a graph of groups $(\mathscr{G}, X)$, and H a subgroup of $\pi_1(\mathscr{G}, X, T)$. Then H is isomorphic to the fundamental group of a graph of groups $(\mathscr{H}, Z)$, with vertex and edge groups conjugate to subgroups of the groups in $\mathscr{G}$. That is, for every $z \in V(Z) \cup E(Z)$, there exists $g(z) \in G$ and $x(z) \in V(X) \cup E(X)$ such that $H_z = g(z)\bar{G}_{x(z)}g(z)^{-1}$, where $\bar{G}_{x(z)}$ is a subgroup of $G_{x(z)}$. Moreover, $x(z) \in V(X)$ if and only if $z \in V(z)$.*

Proof. By theorem 2, G acts on a tree Y without inversion of edges, and the stabilizers of this action are conjugates of the groups in $\mathscr{G}$. Since H is a subgroup of G, it acts also on Y without inversion of edges. It is easy to see that, for every $y \in V(Y) \cup E(Y)$ there exists an element $g \in G$ and $x \in V(X) \cup E(X)$ such that;

$$St_H(y) = H \cap St_G(y) = H \cap gG_xg^{-1} \subseteq gG_xg^{-1}$$

Now the assertion follows from theorem 3.

□

DEFINITION 19. Let G be a group, and X a graph on which G acts. The action is said to *fix a point* if there is a vertex $v \in V(X)$ (the *fixed point*) such that $gv = v$ for all $g \in G$. Equivalently, if $Gv := \{gv | g \in G\} = v$.

DEFINITION 20. A group G is said to satisfy the *(FA) property* if every action of G on a tree without inversion of edges has a fixed point.

Next we provide some examples and non-examples of groups satisfying the (FA) property.

1. Finite groups satisfy the (FA) property (cf. [3], corollary 2.6).
2. The free group F_n of rank $n \geq 1$ does not satisfy the (FA) property, for it acts without inversion of edges and without fixed points on its Cayley graph. In particular, $\mathbb{Z}$ does not satisfy the (FA) property.
3. Any finitely generated torsion group satisfies the (FA) property (a torsion group is one in which every element has finite order). [18].
4. Satisfying the (FA) property is closed under extensions, under taking quotients, and under taking subgroups. Also, if a group G acts without inversion of edges on a tree, and there exists a finite index subgroup $H \subseteq G$ satisfying the (FA) property, then G satisfies it too, [18].
5. $SL(n, \mathbb{Z})$ and $Aut(F_n)$ satisfy the (FA) property if $n \geq 2$ (cf. [18], [4], respectively).
6. Any group satisfying Kazhdan's property (T) satisfies the (FA) property (cf. [19]).

THEOREM 4 (cf. [18])**.** *Let G be a group, and $(\mathcal{G}, X)$ a graph of groups such that there exists an isomorphism ϕ from G into $\pi_1(\mathcal{G}, X, T)$. Then G satisfies the (FA) property if and only if $X = T$ and there exists a vertex $v \in V(G)$ such that $\phi(G) = G_v$.*

Proof. We only outline the proof. It is clear, from theorem 2, that $\phi(G)$ is isomorphic to some vertex group G_v, namely, one corresponding to a fixed vertex. Now, X must be a tree, because if it had some embedded n-cycle, C_n, with $E(C_n) = \{e_1, ..., e_n\}$, then the element $t_{e_1} ... t_{e_n}$ would be an element in $\phi(G)$ not contained in G_v (see [18] for more details).

□

COROLLARY 3. *Let $G = \pi_1(\mathcal{G}, X, T)$ be a fundamental group of a graph of groups $(\mathcal{G}, X)$, and H a subgroup of G satisfying the (FA) property. Then H is a subgroup of the conjugate of some vertex group in $(\mathcal{G}, Y)$.*

Proof. It follows immediately from corollary 2 and theorem 4.

□

It is important to remark that, in the previous corollary, H is not only isomorphic to a subgroup of the conjugate of some vertex or edge group, but a subset of it (it is embedded in it by the identity morphism). This is crucial for our purposes.

Lemma 2. *Let A and B be two groups such that $A \times B$ acts on a tree X without inversion of edges. Suppose there exists vertices v and $u \in V(X)$ fixed by A and B, respectively. Then there exists a vertex $w \in V(X)$ fixed by $A \times B$. (v, u, and w are not necessarily different.)*

Before starting the proof we make the following definition: for any two vertices v and u in a connected graph, $l(v,u)$ will denote the minimum integer n such that there exists a path P_n connecting v and u, *minus one*.

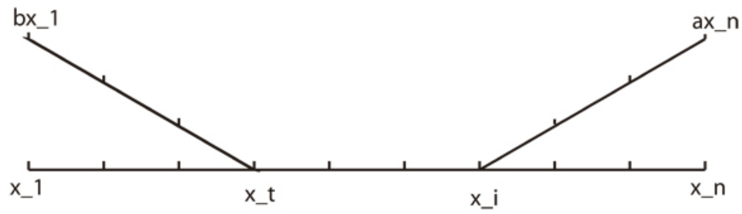
Proof. We can suppose $l(v,u)$ is minimal among the set $\{l(x,y)|Ax = x, By = y, x, y \in V(X)\}$. We can also assume v and u are two different vertices, for if $v = u$ then clearly $A \times B$ fixes v. Hence suppose $l(v,u) \geq 1$ and let P be the unique path connecting v and u (recall that trees are connected graphs).

$$P = (x_1, x_2, ..., x_{n-1}, x_n)$$

Where $x_0 = v$ and $x_n = u$.

Now take any $b \in B$, $b \neq 1$, and consider the path $bP = (bx_1, bx_2, ..., bx_n)$. Since $bx_n = x_n$, there exists a unique i such that $bx_j = x_j$ for all $j \geq i$, and $bx_j \neq x_j$ for all $j < i$. We claim that $i = 1$. If the claim was true, then b would fix v. Since b can be any element in B, B, together with A, would fix v, and hence $A \times B$ would fix v.

Next the claim is proved. Take any $a \in A$ and consider the path aP. As before, there exists a unique t such that $aP = (x_1, x_2, ..., x_{t-1}, ax_t, ..., ax_n)$ with $ax_j = x_j$ for all $j \leq t$ and $ax_j \neq x_j$ for all $j > t$. Observe that $abv = bv$ and $abu = au$. Hence $n - 1 = l(v,u) = l(ab \cdot v, ab \cdot u) = l(b \cdot v, a \cdot u) = i + (n - t) + |t - i|$, from which it is deduced that $t \geq i$. Hence, any $a \in A$ fixes x_i, and, by the minimality of $l(v,u)$, it follows that $i = 1$.



□

Remark. Several alternative proofs of this previous lemma can be derived from stronger results in, for example, [18], and [3].

Proposition 4. *Let A and B be two groups. Then $A \times B$ satisfies the (FA) property if and only if both A and B satisfy it too.*

Proof. First assume that both A and B satisfy the (FA) property, and let X be a tree on which $A \times B$ acts without inversion of edges. In particular, both A and B act on X without inversion of edges, and therefore there exists two vertices v and $u \in X$ such that $Av = v$ and $Bu = u$. Now the previous lemma tells us that $A \times B$ fixes at least one vertex of X.

Conversely, assume $A \times B$ satisfies the (FA) property, and let A act on a tree without inversion of edges, $(a,v) \mapsto a \cdot v$. The action of $A \times B$ on X defined by $(ab,v) \mapsto ab * v = a \cdot v$ is a well defined action without inversion of edges, which extends the action of A on X. Since $A \times B$ fixes a vertex, A fixes it too.

□

2 Rigidity for graph products of groups satisfying the (FA) property.

In this section we prove rigidity for graph products of groups satisfying the (FA) property. The key step is lemma 5, in which an isomorphism between such graph products is shown to carry bijectively maximal cliques into maximal cliques (in a sense which must be precised). The remainding of the proof studies the behavior of this bijection when making set operations with maximal cliques, and, with the help of a key observation from [16], we end up seeing that the isomorphism carries nicely some maximal modules into maximal modules. At this point the result follows easily.

The following lemma will only be used during the proof of lemma 5.

LEMMA 3. *Let $X\mathcal{G}$ be a graph product of groups and C, D two cliques in X such that, for some $g \in X\mathcal{G}$, $C\mathcal{G} \subseteq gD\mathcal{G}g^{-1}$. Then $C\mathcal{G} \subseteq D\mathcal{G}$.*

Proof. Taking supports (see section 1, definition 11), we have;

$$V(C) = supp(C\mathcal{G}) \subseteq supp(g) \cup supp(D\mathcal{G}) = supp(g) \cup V(D).$$

First of all observe that, if $V(C) \subseteq V(D)$, then clearly $C\mathcal{G} \subseteq D\mathcal{G}$.

Now suppose there is a vertex $v \in V(C) - V(D)$. For any $1 \neq x \in G_v$ we have a syllable form, $a_1...a_r$, such that;

$$x = ga_1...a_rg^{-1}$$

with $supp(a_i) \in V(D)$. As a consequence of the normal form theorem, and since $v \notin supp(a_1...a_r)$, v must lie in $supp(g)$, and all the syllables g_i in $g = g_1...g_t$ (normal form) such that $supp(g_i) = v$ can only cancel with syllables inside g or g^{-1} with support also equal to v. However, there is an even number of such syllables, counting among g and g^{-1}, and cancellation procedures between them must yield only one such syllable, which is not possible.

□

A proof of the following lemma can be found in [8].

LEMMA 4 (cf. [8], Lemma 3.20)**.** *Let $X\mathcal{G}$ be a graph product of groups. Then, if X is not a complete graph, $X\mathcal{G}$ admits a nontrivial amalgamated product decomposition:*

$$X\mathcal{G} = X_1\mathcal{G} *_{X_2\mathcal{G}} X_3\mathcal{G},$$

where X_1 is the full subgraph induced by a vertex v and $link(v)$, $X_2 = link(v)$, and $X_3 = X - v$. More precisely, v is a vertex such that $V(X) \neq v \cup link(v)$ (the existence of such vertex is guaranteed by the fact that X is not complete).

Given a graph X, we will denote the set of its cliques by $C(X)$, and the set of its maximal cliques by $MC(X)$.

The next lemma is fundamental.

LEMMA 5. *Let $\mathscr{A}$ and $\mathscr{B}$ be n and m-tuples of groups such that, for every i, j, A_i and B_j satisfy the (FA) property, and let X and Y be two graphs with n and m vertices. Let $MC(X) = \{C_1, ..., C_r\}$, and $MC(Y) = \{D_1, ..., D_s\}$, respectively. Suppose $\phi : X\mathscr{A} \to Y\mathscr{B}$ is an isomorphism. Then there exists a bijection $f : MC(X) \to MC(Y)$ such that C_iA is carried isomorphically by ϕ onto $f(C_i)B$.*

Proof. First of all suppose Y is not a complete graph. By lemma 4, $Y\mathscr{B}$ admits a nontrivial amalgamated free product decomposition:

$$Y\mathscr{B} = Y_1\mathscr{B} *_{Y_2\mathscr{B}} Y_3\mathscr{B}$$

Now take a clique from $C(X)$, C_i, and consider the subgroup $(C_i\mathscr{A})^\phi \subseteq Y\mathscr{B}$. By proposition 5, $(C_i\mathscr{A})^\phi$ satisfies the (FA) property, and, by corollary 3, since $Y_1\mathscr{B} *_{Y_2\mathscr{B}} Y_3\mathscr{B}$ is the fundamental group of a graph of groups with vertex groups $Y_1\mathscr{B}$, $Y_3\mathscr{B}$, and edge group $Y_2\mathscr{B}$ (see example 1), $(C_i\mathscr{A})^\phi$ is a subgroup of some conjugate of $Y_1\mathscr{B}$, or $Y_3\mathscr{B}$.

In both cases, if Y_i $(i = 1, 3)$ is not complete, one can repeat the process until a complete full subgraph of Y, say K, is reached, obtaining, for some $y \in Y\mathscr{B}$,

$$(C_i\mathscr{A})^\phi \leq y(K\mathscr{B})y^{-1}$$

So, what we have proved so far is that, for every clique C_i in $C(X)$, there exists a clique $f(C_i)$ in $C(Y)$, and an element $y_i \in Y\mathscr{B}$, such that;

$$(C_i\mathscr{A})^\phi \subseteq y_i(f(C_i)\mathscr{B})y_i^{-1}.$$

Of course, the same is true for the isomorphism ϕ^{-1}, so for every clique D_i in $C(Y)$, there exists a clique $g(D_i)$ in $C(X)$ and an element $x_i \in X\mathscr{A}$ such that;

$$(D_i\mathscr{B})^{\phi^{-1}} \subseteq x_i(g(D_i)\mathscr{A})x_i^{-1}$$

Now, from these two previous considerations, we get;

$$C_i\mathscr{A} \subseteq y_i^{\phi^{-1}}(f(C_i)\mathscr{B})^{\phi^{-1}}y_i^{-\phi^{-1}} \subseteq \left(y_i^{\phi^{-1}}x_i\right)(g(f(C_i))\mathscr{A})\left(x_i^{-1}y_i^{-\phi^{-1}}\right)$$

(Note this inclusion is a subset inclusion, eg, the first group is set-theoretically contained in the last one. This is much stronger than simply being isomorphic to a subgroup, and crucial for our purposes.)

So, for every clique C_i in $C(X)$, there exists another clique $h(C_i) = g(f(C_i))$ also in $C(X)$, and an element $g_i \in X\mathscr{A}$, such that;

$$C_i\mathscr{A} \subseteq g_i\,(h(C_i)\mathscr{A})\,g_i{}^{-1}$$

Now, by lemma 3, $C_i\mathscr{A} \subseteq h(C_i)\mathscr{A}$. This implies C_i is a complete subgraph of $h(C_i)$. In particular, if C_i is a maximal clique from $MC(X)$, then $C_i = h(C_i)$. Moreover, let D be a clique in $MC(Y)$ such that $f(C_i) \subseteq D$. Then $f(C)\mathscr{B} \subseteq D\mathscr{B}$, and, from;

$$C_i\mathscr{A} \subseteq y_i{}^{\phi^{-1}}\,(f(C_i)\mathscr{B})^{\phi^{-1}}\,y_i{}^{-\phi^{-1}} \subseteq y_i{}^{\phi^{-1}}(D\mathscr{B})^{\phi^{-1}}y_i{}^{-\phi^{-1}} \subseteq s(g(D)\mathscr{A})s^{-1} = C_i\mathscr{A}$$

(where s is some element in $Y\mathscr{B}$), we get $f(C)\mathscr{B} = D\mathscr{B}$, which implies $f(C) = D$ (The last equality follows form lemma 3 and the maximality of C). Therefore f is bijective and maps maximal cliques in $MC(X)$ to maximal cliques in $MC(Y)$.

□

DEFINITION 21. *Let $\mathscr{A} = (A_1, ..., A_n)$ and $\mathscr{B} = (B_1, ..., B_m)$ be n and m-tuples of groups. Then $\{A_1, ..., A_n\} \cong_\sigma \{B_1, ..., B_m\}$ denotes $n = m$ and that there exists a permutation σ such that $A_i \cong B_{\sigma(i)}$.*

The permutation σ will be omitted whenever it is not relevant.

DEFINITION 22. A family of groups $\mathscr{T}$ is said to be *rigid with respect to the direct product* if every group $G \in \mathscr{T}$ is not isomorphic to a nontrivial direct product of groups, and, for any four groups A, B, C, and $D \in \mathscr{T}$, the following holds:

$$A \times B \cong C \times D \Leftrightarrow \{A, B\} \cong \{C, D\}$$

A good criterion for identifying rigid families of groups is provided by the following classical theorem.

THEOREM 5 (Krull-Schmidt, cf. [17], 6.36)**.** *Let G be a group satisfying the ascending chain coindition (every ascending chain of subgroups $1 = H_1 \subseteq H_2 \subseteq ...$ is eventually stationary), and the descending chain condition (every descending chain of subgroups $G = H_1 \supseteq H_2 \supseteq ...$ is eventually stationary). Then there is essentially a unique way of decomposing G as a direct product of directly idnecomposable groups. This meaning that in case there is a finite number of directly indecomposable groups A_i and B_j such that the following holds:*

$$G \cong A_1 \times ... \times A_n \cong B_1 \times ... \times B_m$$

then $\{A_1, ..., A_n\} \cong \{B_1, ..., B_m\}$.

As a direct consequence of Krull-Schmidt theorem we have that indecomposable finite groups form a rigid family of groups with respect to the direct product.

Remark. Indecomposable cyclic groups, finite or not, also form a rigid family with respect to the direct product. It is interesting to note that $\mathbb{Z}$ does not satisfy the descending chain condition.

The reader is reffered to the papers by R.Hirshon, [10], [11], [12], for further information and results regarding such rigid families.

THEOREM 6. *Let A and B be n and m-tuples of groups, all of them satisfying the (FA) property and belonging to a rigid family of groups with respect to the direct product. Let X and Y be graphs with n and m vertices, respectively. Then there exists an isomorphism $\phi : X\mathscr{A} \to Y\mathscr{B}$ if and only if $n = m$ and there exists a graph isomorphism $\eta : V(X) \to V(Y)$ such that*

$$\{A_1, ..., A_n\} \cong_\eta \{B_1, ..., B_m\}.$$

The following lemmas will be used during the proof of theorem 6.

LEMMA 6. *Let C_1 and C_2 be two cliques in a graph X, and consider a graph product of groups $X\mathscr{G}$. Then:*

$$(C_1 \cap C_2)\mathscr{G} = (C_1\mathscr{G}) \cap (C_2\mathscr{G})$$

Proof. We claim that $supp((C_1\mathscr{G}) \cap (C_2\mathscr{G})) = supp(C_1\mathscr{G}) \cap supp(C_2\mathscr{G}) = C_1 \cap C_2$. If the claim is true, then $(C_1\mathscr{G}) \cap (C_2\mathscr{G}) \subseteq (C_1 \cap C_2)\mathscr{G}$, and it is easy to see the reverse inclusion.

Next the claim is proved. Any element x in $(C_1\mathscr{G}) \cap (C_2\mathscr{G})$ admits two syllable forms: $x = a_1...a_t$, and $x = b_1...b_t$, where a_i are syllables in C_1, and b_i are syllables in C_2. Now, $supp(a_1...a_t) = supp(b_1...b_t)$, and therefore a normal form for x must be formed with syllables belonging both to C_1 and C_2, by corollary 1. Thus $supp((C_1\mathscr{G}) \cap (C_2\mathscr{G})) \subseteq V(C_1) \cap V(C_2) = supp(C_1\mathscr{G}) \cap supp(C_2\mathscr{G})$. The other inclusion follows easily: for every vertex v both in C_1 and C_2, there is an element x in $(C_1\mathscr{G}) \cap (C_2\mathscr{G})$ such that $supp(x) = v$, namely, any element in G_v (which is not trivial by hypotesis).

□

LEMMA 7. *Let C_1 and C_2 be two maximal cliques in a graph X, and $\mathscr{A}$, $\mathscr{B}$ two n and m-tuples of groups satisfying the (FA) property. Then any isomorphism $\phi : X\mathscr{A} \to Y\mathscr{B}$ carries $(C_1 \cap C_2)\mathscr{A}$ isomorphically onto $(f(C_1) \cap f(C_2))\mathscr{B}$, where f is the induced bijection betweem maximal cliques given by lemma 5.*

Proof. It is immediate from lemma 5 and 6:

$$((C_1 \cap C_2)\mathscr{A})^\phi = (C_1\mathscr{A} \cap C_2\mathscr{A})^\phi = (C_1\mathscr{A})^\phi \cap (C_2\mathscr{A})^\phi =$$
$$= f(C_1)\mathscr{B} \cap f(C_2)\mathscr{B} = (f(C_1) \cap f(C_2))\mathscr{B}.$$

□

LEMMA 8. *Let $X\mathscr{A}$ be a graph product of groups, and Z_1, Z_2 be two full subgraphs in X. Then;*

$$(Z_1 - Z_2)\mathscr{A} = Z_1\mathscr{A} / << (Z_1 \cap Z_2)\mathscr{A} >>$$

As a consequence, let $\mathscr{A}$ and $\mathscr{B}$ be n and m-tuples of groups satisfying the (FA) property. Let ϕ be an isomorphism between them, and f the bijection between maximal cliques given by lemma 5. Let also C_1, ... C_t and $D_1, ..., D_r$ be some maximal cliques in X. Then;

$$\left(\left(\left(\bigcap C_i\right)-\left(\bigcup D_i\right)\right)\mathscr{A}\right)^{\phi}=\left(\left(\bigcap f(C_i)\right)-\left(\bigcup f(D_i)\right)\right)\mathscr{B}$$

Proof. The first part follows from proposition 2 when taking $\mathscr{H}=(G_{i_1},...,G_{i_t})$, where $\{v_{i_1},...,v_{i_t}\}=V(Z_1\cap Z_2)$.

For the second part, write $Z_1:=\bigcap C_i$, $Z_2:=\bigcup D_i$, $Z_1{}^{\phi}=\bigcap f(C_i)$, and $Z_2{}^{\phi}=\bigcup f(D_i)$. Now observe that an isomorphism ϕ induces a natural isomorphism

$$\bar{\phi}:Z_1\mathscr{A}/<<(Z_1\cap Z_2)\mathscr{A}>>\to(Z_1\mathscr{A})^{\phi}/<<((Z_1\cap Z_2)\mathscr{A})^{\phi}>>=Z_1{}^{\phi}\mathscr{B}/<<(Z_1{}^{\phi}\cap Z_2{}^{\phi})\mathscr{B}>>.$$

We have, therefore, the following sequence of isomorphisms:

$$(Z_1-Z_2)\mathscr{A}\to Z_1\mathscr{A}/<<(Z_1\cap Z_2)\mathscr{A}>>\xrightarrow{\bar{\phi}}Z_1{}^{\phi}\mathscr{B}/<<(Z_1{}^{\phi}\cap Z_2{}^{\phi})\mathscr{B}>>\to(Z_1{}^{\phi}-Z_2{}^{\phi})\mathscr{B}$$

$$x\mapsto x<<(Z_1\cap Z_2)\mathscr{A}>>\mapsto x^{\phi}<<(Z_1{}^{\phi}\cap Z_2{}^{\phi})\mathscr{B}>>\mapsto x^{\phi}$$

Hence, ϕ carries $(Z_1-Z_2)\mathscr{A}$ isomorphically into $(Z_1{}^{\phi}-Z_2{}^{\phi})\mathscr{B}$, as claimed.

□

LEMMA 9. *Let $X\mathscr{A}$ and $Y\mathscr{B}$ be two graph products of groups, ϕ an isomorphism between $X\mathscr{A}$ and $Y\mathscr{B}$, and f a bijection $f:X\to Y$. Suppose that, for every vertex $v\in V(X)$, we have $(A_v)^{\phi}=B_{f(v)}$. Then f is a graph isomorphism.*

Proof. It suffices to check that f preserves adjacencies. Let v, u be two vertices in $V(X)$, and g_v, g_u two elements from A_v, A_u, respectively. Then, since $g_v{}^{\phi}\in B_{f(v)}$ and $g_u{}^{\phi}\in B_{f(u)}$;

$$(v,u)\in E(X)\Leftrightarrow[g_v,g_u]=1\Leftrightarrow[g_v,g_u]^{\phi}=[g_v{}^{\phi},g_u{}^{\phi}]=1\Leftrightarrow(f(v),f(u))\in E(Y)$$

□

The following idea, consisting in considering maximal modules which are complete graphs (or maximal complete modules), is a key step which can be found in [16].

LEMMA 10. *Let X be a graph, and let $\mathscr{C}$ be a set of maximal cliques in X such that*

$$K:=\left(\bigcap_{C\in\mathscr{C}}C\right)-\left(\bigcup_{C\in MC(X)-\mathscr{C}}C\right)\neq\emptyset.$$

Then K is a maximal complete module in X.

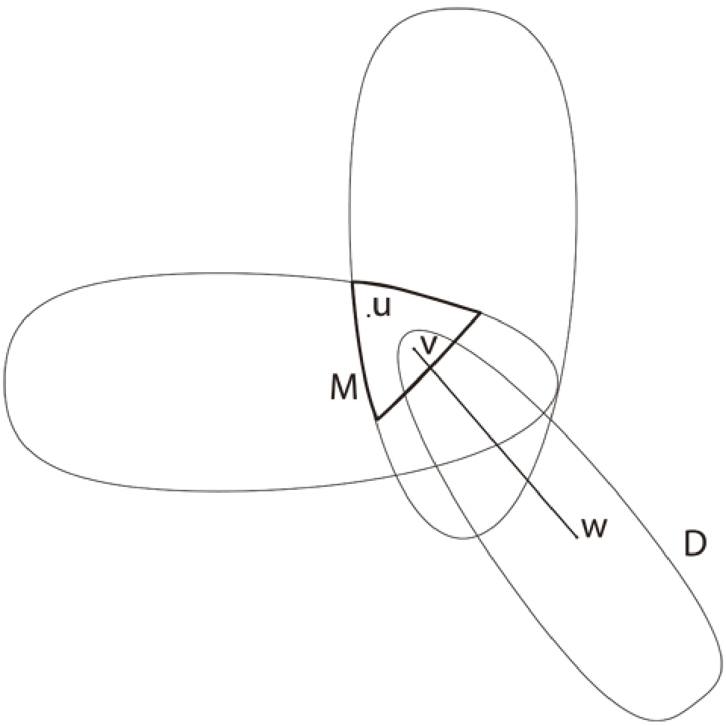
Conversely, suppose M is a maximal complete module in X. Denote by M^{MC} the set of all maximal cliques containing M. Then,

$$M=\left(\bigcap_{C\in M^{MC}}C\right)-\left(\bigcup_{C\in MC(X)-M^{MC}}C\right).$$

Proof. First we prove that the full subgraph $K = \bigcap_{C\in\mathscr{C}} C - \bigcup_{C\in MC(X)-\mathscr{C}} C \neq \emptyset$ is a complete module. The key observation for the whole proof is that every maximal clique either contains K or intersects K trivially. Indeed, if C is a maximal clique in X, then it either belongs to $\mathscr{C}$, in which case it contains K, or it does not belong to $\mathscr{C}$, in which case it cannot share any vertex with K by the way we defined it.

Now, K is cearly complete. To see it is a module note the following: for every vertex u adjacent to some vertex $v \in V(K)$, there is a maximal clique D containing the edge (v,u). This clique D intersects K nontrivially and therefore must contain M; in particular, it contains K and thus u is adjacent to every vertex in K.

Now take a maximal complete module M and let $\mathscr{C} = M^{MC}$. We are going to show that, in this situation, K contains M. Therefore K is nonempty and, by maximality, $K = M$. Suppose K does not contain M, so there exists a vertex $v \in M$, and a maximal clique D not containing M and containing v. Then there exists a vertex $u \in M - D$, and, since D is maximal, $D - M \neq \emptyset$. Now, for any $w \in D - M$, w is adjacent to v because D is a clique. Since M is a module and $v \in M$, w is adjacent to every vertex in M, in particular it is adjacent to u. Therefore the full subgraph induced by D and u is complete, contradicting the maximality of D. (See Figure).



It remains to see that, for any $\mathscr{C}$ such that $K \neq \emptyset$,K is maximal. Suppose therefore that M is a maximal complete module containing K. By the previous observations, M is the intersection of all maximal cliques belonging to M^{MC} minus the union of the remaining maximal cliques. Equivalently, every maximal clique either contains M or intersects M

trivially. We claim that $\mathscr{C} = M^{MC}$. Clearly, if the claim is true, then the proof is complete.

To prove the claim, suppose first that $C \in \mathscr{C} - M^{MC}$. Then C intersects K nontrivially, and therefore it also intersects M nontrivially. This is a contradiction and hence $\mathscr{C} \subseteq M^{MC}$. Conversely, if $C \in M^{MC}$, then it contains K and thus must belong to $\mathscr{C}$.

$\square$

Note that the vertices of a graph X can always be partitioned into maximal complete modules. Indeed, every vertex belongs to a complete module (itself), and therefore to a maximal complete one, and two such modules never intersect, for otherwise at least one of them would not be maximal. We will implicitly be using this considerations without making any mention of them.

LEMMA 11. *Let $X\mathscr{A}$ and $Y\mathscr{B}$ be two graph products of groups satisfying the (FA) property, and ϕ an isomorphism betwen them. Let M be a maximal complete module in X. Then ϕ carries $M\mathscr{A}$ into $s(M)\mathscr{B}$, where $s(M)$ is a maximal complete module in Y. Moreover, s is a bijection between the set of maximal complete modules in X and the set of maximal complete modules in Y.*

Proof. Let f be the bijection between maximal cliques in X and Y given by lemma 5, and let M be as in the statement. Then, following the notation in the previous lemma,

$$M = \left(\bigcap_{C \in M^{MC}} C \right) - \left(\bigcup_{C \notin M^{MC}} C \right)$$

Now, by lemma 8,

$$(M\mathscr{A})^{\phi} = \left(\left(\left(\bigcap_{C \in M^{MC}} C \right) - \left(\bigcup_{C \notin M^{MC}} C \right) \right) \mathscr{A} \right)^{\phi} =$$

$$= \left(\left(\bigcap_{C \in M^{MC}} f(C) \right) - \left(\bigcup_{C \notin M^{MC}} f(C) \right) \right) \mathscr{B} := s(M)\mathscr{B}$$

Since M is nonempty, $s(M)$ is nonempty too, and, by the previous corollary, it is a maximal complete module in Y.

It remains to see s is a bijection. Since every vertex belongs to a maximal complete module, $(X\mathscr{A})^{\phi}$ is generated by the set $\{s(M_i)\mathscr{B} | M_i \in MC(X)\}$, and therefore by the vertex groups B_v such that $v \in s(M_i)$ for some $M_i \in MC(X)$. Now, if s was to be not exhaustive, there would be some $v \in V(Y)$ not belonging to any $s(M_i)$, and therefore $(X\mathscr{A})^{\phi}$ would be contained in $(Y - v)\mathscr{B}$, contradicting the fact that ϕ is exhaustive. Moreover, s is injective; if not, there would be two maximal complete modules, M_1 and M_2, such that $s(M_1) = s(M_2)$, and two elements g_1 and g_2 belonging to $M_1\mathscr{A}$ and $M_2\mathscr{B}$, respectively, such that ${g_1}^{\phi} = {g_2}^{\phi}$, because ϕ carries isomorphically $M_i\mathscr{A}$ into $s(M_i)\mathscr{B}$.

$\square$

Proof of theorem 6. The implication from right to left is clear. Suppose now that ϕ is an isomorphism between $X\mathscr{A}$ and $Y\mathscr{B}$, and let $M(X)$ be the set of maximal modules of X

which are complete graphs. Note that every vertex belongs to one and only one such module. Let $s : M(X) \to M(Y)$ be the bijection as in the previous lemma. Now consider the graph products of groups $X'\mathscr{A}'$ and $Y'\mathscr{B}'$ obtained by contracting the modules in $M(X)$ and $M(Y)$ as in section 1.3. ϕ induces a natural isomorphism between $X'\mathscr{A}'$ and $Y'\mathscr{B}'$, which will still be denoted ϕ. This isomorphism carries bijectively vertex groups into vertex groups, so lemma 8 assures that ϕ induces a graph isomorphism between X' and Y'.

Thus, what we have seen so far is that there exists a graph isomorphism $f : X' \to Y'$ such that, if $V(X') = \{s_1, ..., s_t\}$, and $M(X) = \{M_{s_1}, ..., M_{s_t}\}$, then $\phi(M_{v_i}\mathscr{A}) = M_{f(v_i)}\mathscr{B}$. By rigidity with respect to the direct product, $|M_{v_i}| = |M_{f(v_i)}|$, and f induces an isomorphism $\bar{f}$ between X and Y. This $\bar{f}$ can be defined in the following way. Take a maximal complete module M_{v_i} in X, and let $v_1, ..., v_k$ be its vertices. Let σ be any permutation of k elements. Suppose the vertices of $M_{s(v_i)}$ are $\{u_1, ..., u_k\}$. Then define $\bar{f}(v_i) = u_{\sigma(i)}$. This is clearly well defined and a graph isomorphism. Moreover, $\{A_1, ..., A_n\} \cong_{\bar{f}} \{B_1, ..., B_m\}$.

□

We now recover Radcliffe's result [16].

COROLLARY 4 (Radcliffe, [16])**.** *Any graph product of groups with directly indecomposable finite vertex groups is rigid.*

Proof. We have already stated that finite groups satisfy the (FA) property. It has also been seen that directly indecomposable finite groups form a rigid family of groups with respect to the direct product (as a consequence of theorem 5). The assertion now follows from theorem 6.

□

2.1 Comments on the hypothesis.

There are two natural hypotesis when requiring rigidity of a graph product:

1. Vertex groups must be graphologically indecomposable. This is clear and by no means removable.
2. Vertex groups must belong to a rigid family of groups with respect to direct product. This is because rigidity for direct products of directly indecomposable groups does not always hold. One could try to avoid this hypotesis by adding conditions to the associated graphs. For example, if we only require the graphs not to be complete, is it strictly necessary that the vertex group belong to a rigid family with respect to the direct product? This is most likely to have a positive answer: groups will always have to belong to a rigid family, except when the graphs have no complete maximal modules with more than one vertex.

Note that groups satisfying the hypotesis in the statement of theorem 6 satisfy 1) and 2) in the previous enumeration. Indeed, a group satisfying the (FA) property is not isomorphic to a nontrivial graph product of groups with an uncomplete associated graph, for then it would decompose as a nontrivial free amalgamated product. Also, by definition, a group in a rigid family of groups with respect to direct products is indecomposable as a nontrivial direct product.

It is suspicious that some of the counterexample groups in proposition 4 are nontrivial amalgamated free products (for axample, $A = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$). (FA) property, however, seems too much. Probably rigidity could still be assured by requiring vertex groups not to be amalgamated products, instead of asking (FA) property, or maybe this hypotesis could be completely dropped (then, of course, one would have to ask for graphologically indecomposable groups).

3 Further work.

We are currently working on a variation of the proof presented in this paper. This variation would allow infinite cylic vertex groups (together with groups satisfying the (FA) property). The strategy is the same: once we have seen the isomorphism is well behaved on subgroups generated by maximal cliques, it is easy to infer the remaining part of the proof. However, this well behavior is much more tricky in this new situation, for the subgroups induced by cliques with the group $\mathbb{Z}$ in all its vertices can act on trees in a way that the information about one of the $\mathbb{Z}$'s is completely lost. However, some more subtle considerations about the decomposition of a graph product as an amalgamated product, together with a couple of results of Y.Antolin and A. Minasyan, [1], have allowed us to succesfully deal with maximal cliques with infinite cyclic groups on its vertices, except when the maximal clique has exactly two vertices. Luckily, some recent considerations also point towards a posibly effective method for this kind of cliques.

If completed, this new proof would be interesting because it would imply the three mentioned results about rigidity, [6], [9], [16]. In particular, it would imply Droms' theorem without making any use of the algebras that appear in his paper.

References

[1] Yago Antolin, Ashot Minasyan, *Tits alternatives for graph products*, (2011).

[2] Donald A. Barkauskas, *Centralizers in Graph Products of Groups*, Journal of Algebra 312 9-32, (2007).

[3] Oleg Bogopolski, *Introduction to Group Theory*, European Mathematical Society, (2008).

[4] O. V. Bogopolski, *Arboreal decomposability of the group of automorphisms of a free group*, Algebra i Logica 26, no. 2, 131-149, (1987).

[5] W. Dicks, M. J. Dunwoody, *Groups acting on graphs*, Cambridge Studies in Advanced Mathematics, 17. Cambridge University Press, Cambridge, (1989).

[6] Carl Droms, *Isomorphisms of Graph Groups*, Proceedings of the American Mathematical Society 100-3, (1987).

[7] Tibor Gallai, *Transitiv orienterbare Graphen*, Acta Math. Acad. Sci. Hungar 18, 25-66, (1967).

[8] Elisabeth Ruth Green, *Graph Product of Groups*, The University of Leeds, Department of pure mathematics, (1990).

[9] Mauricio Gutierrez and Adam Piggott, *Rigidity of Graph Products of Abelian Groups*, (2008).

[10] Ronald Hirshon, *Some New Groups Admitting Essentially Unique Directly Indecomposable Decompositions*, Math. Ann. 194, 123 - 125, (1971).

[11] Ronald Hirshon, *Cancellation and Hopficity in Direct Products*, Journal of Algebra 50, 26-33, (1978).

[12] Ronald Hirshon, *On Cancellation in Groups*, The American Mathematical Monthly, Vol. 76, No. 9, 1037-1039, (1969).

[13] Aleksandr G. Kurosh, *The Theory of Groups*, Chelsea Publishing Company, (1960).

[14] Michael R. Laurence, *Automorphisms of graph products of groups*, Ph.D. thesis, Queen Mary College, University of London, (1993).

[15] Ross M. McConnell, Jeremy P. Spinrad, *Modular decomposition and transitive orientation*, Discrete Mathematics 201, 189-241, (1999).

[16] David G. Radcliffe, *Rigidity of Graph Products of Groups*, Algebraic and Geometric Topology 3, 1079-1088, (2003).

[17] Joseph J. Rotman, *An Introduction to the Theory of Groups.*, Springer-Verlag, (1991).

[18] J.P Serre, *Trees*, Springer-Verlag, (1980).

[19] Y. Watatani, *Property T of Kazhdan implies property FA of Serre*, Math. Japon. 27: 97-103, (1981).