ON FIXED SUBGROUPS OF MAXIMAL RANK

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Resum

We show that, in the free group F of rank n, n is the maximal length of strictly ascending chains of maximal rank fixed subgroups, that is, rank n subgroups of the form $\mathrm{Fix}\,\phi$ for some $\phi \in Aut(F)$. We further show that, in the rank two case, if the intersection of an arbitrary family of proper maximal rank fixed subgroups has rank two then all those subgroups are equal. In particular, every $G \leq Aut(F)$ with $r(\mathrm{Fix}\,G) = 2$ is either trivial or infinite cyclic.

1. The fringe of a subgroup

Throughout this section let I be an arbitrary non-empty set, and let $F_I = \langle I \mid \ \rangle$ denote the free group on I.

1.1 Definitions. A *graph* $X = (V, E, \iota, \tau)$ consists of two disjoint sets V, E (usually denoted VX and EX) and two maps $\iota, \tau : EX \to VX$. The elements of VX and EX are called the *vertices* and *edges* of X, respectively. The maps ι and τ are the *incident maps* of X. We consider ι and τ extended to the disjoint union $EX \vee (EX)^{-1}$ by setting $\iota e^{-1} = \tau e$ and $\tau e^{-1} = \iota e$, $e \in EX$.

The basic example is the *I-bouquet*, $R_I = (\{*\}, I, \iota, \tau)$ where ι and τ are each (necessarily) the constant map.

The concepts of *path*, *trivial path*, *fundamental group* at a vertex, *connected* graph, *connected components* of a graph, *subgraph*, *graph morphism* and *graph isomorphism* are the standard ones.

By the *link* of a vertex $v \in VX$, we mean $\mathrm{lk}(v) = \{e \in EX \vee (EX)^{-1} \mid \iota e = v\}$. The *valency* of v is $|\mathrm{lk}(v)|$. We say that X is a *finite core graph* if X is finite (i.e., VX and EX are both finite) and X has no vertices of valency 0 or 1. The *core* of a graph X, denoted $\mathrm{core}(X)$, is the union of the finite core subgraphs of X. If $\mathrm{core}(X) = X$ then we say that X is a *core graph.* A *forest* is a graph whose core is empty.

A *tree* is a connected forest. It is easy to see that, given a vertex v of X not in a tree component of X, the component of $X - E(\mathrm{core}(X))$ containing v contains exactly one vertex belonging to $\mathrm{core}(X)$; this vertex is called the *root* of v, denoted $\hat{v}$; note that $\hat{v} = v$ if and only if $v \in \mathrm{core}(X)$.

Let X be a connected graph and v a vertex of X. The fundamental group of X at v, denoted $\pi(X, v)$, is a free group and its isomorphism class (i.e. its rank) does not depend on v. This rank (possibly infinite) is the *rank* of X, denoted $r(X)$. For example, $\pi(R_I, *) = F_I$ and $r(R_I) = |I|$. It is easy to see that $r(X) = r(\mathrm{core}(X))$; in fact, for $v \in \mathrm{core}(X)$, $\pi(X, v) = \pi(\mathrm{core}(X), v)$. It is well known that if Y is a subgraph of X and $u \in VY \subseteq VX$ then $\pi(Y, u)$ is a free factor of $\pi(X, u)$.

Let $f : X \to Y$ be a graph morphism and $v \in VX$. We can define the group morphism $\pi(X, v) \to \pi(Y, vf)$, $p \mapsto pf$, and the map $\mathrm{lk}(v) \to \mathrm{lk}(vf)$, $e \mapsto ef$, $e^{-1} \mapsto (ef)^{-1}$, $e \in EX$, both denoted also by f. As in [6], when the latter map is injective (resp. surjective, resp. bijective) for every $v \in VX$, we say that f is an *immersion* (resp. *locally surjective*, resp. a *covering*).

Let X be a graph and let $\sim$ be an equivalence relation on the set of vertices VX. By the *quotient* graph $X/\!\sim$ we mean the graph $((VX)/\!\sim, EX, \bar{\iota}, \bar{\tau})$ where the incident functions are those of X composed with the projection onto $(VX)/\!\sim$. Let $^{-} : X \to X/\!\sim$ be the natural projection, a graph morphism. For example, if $\sim$ is the trivial equivalence relation on VX then $X/\!\sim = R_{EX}$. Suppose now that $f : X \to Y$ is a graph morphism and $\sim$ is an equivalence relation on VX such that $u \sim v$ implies $uf = vf$. In this case, f factors through the natural graph morphism $f_\sim : X/\!\sim \to Y$, that is, $f = {}^{-}f_\sim$. Furthermore, if f is locally surjective then so is $f_\sim$. $\square$

1.2 Covering theory. Covering theory for graphs is a special case of the topological theory of covering spaces. To fix notation we collect together some well-known results (see, for example, [6]) that we will use later.

(a) *(Path-lifting)* Let $f : X \to Y$ be a covering. For any vertex $u \in VX$ and any path p in Y with $\iota p = uf$ there exists a unique path $\tilde{p}$ in X such that $\iota\tilde{p} = u$ and $\tilde{p}f = p$. Such a $\tilde{p}$ is called the *lifting* of p with initial vertex u.

(b) *(Lifting Lemma)* Let $f : Y \to Z$ be a covering, let $g : X \to Z$ be a graph morphism with X a connected graph, and let $u \in VX$ and $v \in VY$ such that $ug = vf$. There exists a graph morphism $\tilde{g} : X \to Y$ satisfying $g = \tilde{g}f$ and $u\tilde{g} = v$ if and only if $(\pi(X, u))g \subseteq (\pi(Y, v))f$; in this case, $\tilde{g}$ is unique, and will be called the *lifting* of g sending u to v. Furthermore, if g is an immersion, or is locally surjective, or is a covering, then so is $\tilde{g}$.

(c) *(Existence of coverings)* Let Y be a connected graph, $v \in VY$ and $H \leq \pi(Y, v)$. There exists a connected covering, $f : X \to Y$, and a vertex $u \in VX$ such that $uf = v$ and $(\pi(X, u))f = H$; moreover, f is unique in the sense that for any such $f' : X' \to Y$ and $u' \in VX'$ there exist a graph isomorphism $g : X \to X'$ satisfying $ug = u'$ and $f = gf'$.

(d) *(π-injectivity)* If $f : X \to Y$ is an immersion then, for any vertex $v \in VX$, the corresponding group morphism $\pi(X, v) \to \pi(Y, vf)$ is injective. □

The Folding Lemma due to J. Stallings (see 3.3 in [6]) folds pairs of edges one by one, so works for finite graphs. We need here an extended version suitable for arbitrary graphs.

1.3 Folding Lemma. *Let $f : X \to Y$ be a graph morphism. There exists a graph X', an immersion $f' : X' \to Y$ and a graph morphism $\kappa : X \to X'$ such that $f = \kappa f'$ and, for every vertex $u \in VX$, $(\pi(X, u))f = (\pi(X', u\kappa))f'$. Furthermore, if f is locally surjective then so is f'.*

Proof. Let $\sim$ be the equivalence relation on the set of vertices of X such that $u \sim v$ if and only if there exist a path p in X from u to v such that pf cancels to a trivial path in Y. In particular, if $u \sim v$ then u and v belong to the same component of X, and $uf = vf$.

Consider now the quotient graph $X/\!\sim$, and the graph morphisms $^- : X \to X/\!\sim$ and $f_\sim : X/\!\sim \to Y$. Let X' denote the graph obtained from $X/\!\sim$ by identifying edges which have the same initial vertex, and the same terminal vertex, and the same image under $f_\sim$; so, $VX' = V(X/\!\sim) = (VX)/\!\sim$. Let $\kappa' : X/\!\sim \to X'$ and $f' : X' \to Y$ be the natural graph morphisms, and let $\kappa = {}^-\kappa'$.

By construction, it is clear that $f = \kappa f'$ and that f' is locally surjective if f is. Note that if two edges $e_1, e_2 \in E(X/\!\sim)$ satisfy $\overline{\iota} e_1^\epsilon = \overline{\iota} e_2^\epsilon$ and $e_1 f_\sim = e_2 f_\sim$ for some $\epsilon = \pm 1$, then $\overline{\tau} e_1^\epsilon = \overline{\tau} e_2^\epsilon$, so they are identified in X'. Thus, f' is an immersion. Now consider a vertex $u \in VX$. It is clear that $(\pi(X', u\kappa))f' = (\pi(X/\!\sim, \overline{u}))f_\sim$, and that $(\pi(X, u))f \subseteq (\pi(X/\!\sim, \overline{u}))f_\sim$. Thus, it remains to prove the inclusion $(\pi(X/\!\sim, \overline{u}))f_\sim \subseteq (\pi(X, u))f$. Let $p = e_1^{\epsilon_1} \cdots e_n^{\epsilon_n} \in \pi(X/\!\sim, \overline{u})$ be a closed path at $\overline{u}$. The edges $e_1, \cdots, e_n \in EX$ need not form a path in X but, by definition of $\sim$, there exist paths in X, p_i, $i = 0, \cdots, n$, such that $\iota p_0 = u$, $\iota p_i = \tau e_i^{\epsilon_i}$ for $i = 1, \cdots, n$, $\tau p_i = \iota e_{i+1}^{\epsilon_{i+1}}$ for $i = 0, \cdots, n-1$, $\tau p_n = u$, and $p_i f$ is trivial for $i = 0, \cdots, n$. Now $q = p_0 e_1^{\epsilon_1} p_1 \cdots p_{n-1} e_n^{\epsilon_n} p_n \in \pi(X, u)$ is a path in X closed at u and satisfying $qf = pf_\sim$. This completes the proof of the lemma. □

1.4 Definitions. In the group F_I, we will use the exponential notation to denote right conjugation, that is, $x^y = y^{-1}xy$, $x, y \in F_I$.

An *I-labelled graph* (X, ρ) is a graph X together with a map $\rho : EX \to I$. This can be thought of as a graph X with a specified graph morphism $\rho : X \to R_I$. An *I-labelled graph morphism* (resp. *I-labelled graph isomorphism*), $f : (X, \rho) \to (Y, \sigma)$, is a graph morphism (resp. graph isomorphism) between the corresponding graphs, $f : X \to Y$, respecting the labels, that is, satisfying $\rho = f\sigma$. By an *I-labelled* *graph immersion*, (resp. a *locally surjective I-labelled graph*, resp. an *I-labelled graph covering*) we mean an I-labelled graph (X, ρ) such that ρ is an immersion (resp. a locally surjective graph morphism, resp. a covering). Note that when $f : (X, \rho) \to (Y, \sigma)$ is an I-labelled graph morphism, if ρ is an immersion then so is f; and if ρ and σ are both coverings then so is f.

Applying 1.2(c) to R_I, we can canonically associate to each subgroup $H \leq F_I$, a connected I-labelled graph covering; it will be denoted (T_H, ρ_H). A vertex $v \in VT_H$ as in 1.2(c) (not necessarily unique) is called a *base point.* By construction, $(\pi(T_H, v))\rho_H = H$. The core of T_H will be denoted X_H. It is easy to see that $H \leq F_I$ is finitely generated if and only if X_H is a finite graph. □

Let $H \leq F_I$. Consider (T_H, ρ_H), the corresponding I-labelled graph covering with base point u. Take now an equivalence relation, $\sim$, on the set of vertices of T_H and consider the quotient I-labelled graph $(T_H/\!\sim, (\rho_H)_\sim)$; this is locally surjective but not necessarily an immersion. It is clear that the subgroup of F_I, $H' = (\pi(T_H/\!\sim, \overline{u}))(\rho_H)_\sim$, contains $(\pi(T_H, u))\rho_H = H$. Now, applying the Folding Lemma to the locally surjective I-labelled graph $(T_H/\!\sim, (\rho_H)_\sim)$ (and the uniqueness in 1.2(c)) we will obtain the I-labelled graph covering associated to H', $(T_{H'}, \rho_{H'})$, with base point v, the image of $\overline{u}$. Thus, the unique I-labelled graph morphism from (T_H, ρ_H) to $(T_{H'}, \rho_{H'})$ sending u to v (which exists by the Lifting Lemma and the fact $H \leq H'$) factors through $(T_H/\!\sim, (\rho_H)_\sim)$. The following lemma states that this is the situation for every subgroup $H' \leq F_I$ containing H.

1.5 Lemma. *Let $H \leq H' \leq F_I$ and let u and v be base points for the corresponding coverings (T_H, ρ_H) and $(T_{H'}, \rho_{H'})$, respectively. The unique I-labelled graph morphism $f : (T_H, \rho_H) \to (T_{H'}, \rho_{H'})$ sending u to v factors through $(T_H/\!\sim, (\rho_H)_\sim)$, for some equivalence relation $\sim$ on the set VT_H.*

Proof. Let $\sim$ denote the equivalence relation on VT_H such that $x \sim y$ if and only if $xf = yf$. Consider the quotient I-labelled graph $(T_H/\!\sim, (\rho_H)_\sim)$ (which is locally surjective) and apply the Folding Lemma to get a new I-labelled graph covering (Y, σ), and an I-labelled graph morphism $\kappa : (T_H/\!\sim, (\rho_H)_\sim) \to (Y, \sigma)$. Note that f and $^-\kappa$ are both coverings.

Suppose now that e_1 and e_2 are two edges in $T_H/\!\sim$ such that $\overline{\iota} e_1^\epsilon = \overline{\iota} e_2^\epsilon$ and $e_1(\rho_H)_\sim = e_2(\rho_H)_\sim$ for some $\epsilon = \pm 1$. In T_H we have two edges, $e_1, e_2 \in ET_H$, with $(\iota e_1^\epsilon)f = (\iota e_2^\epsilon)f$ and with the same label. But this implies $e_1 f = e_2 f$ and so, in $T_H/\!\sim$, $\overline{\tau} e_1^\epsilon = \overline{\tau} e_2^\epsilon$. This argument shows that κ collapses together only edges with the same initial vertex and the same terminal vertex. Consequently, $V(T_H/\!\sim) = VY$ and κ is one to one on vertices.

Let $p \in \pi(Y, \overline{u}\kappa)$ be a closed path at $\overline{u}\kappa$ and let $\tilde{p}$ be its lifting to T_H with initial vertex u. We have $\overline{\tau\tilde{p}} = \overline{u}$, that is, $(\tau\tilde{p})f = uf = v$. In other words, $\tilde{p}f$ is a closed path in $T_{H'}$ based at v and satisfying $(\tilde{p}f)\rho_{H'} = \tilde{p}\rho_H = p\sigma$. This shows that $(\pi(Y, \overline{u}\kappa))\sigma \leq (\pi(T_{H'}, v))\rho_{H'} = H'$. Similarly, if $q \in \pi(T_{H'}, v)$ is a closed path at v, let $\tilde{q}$ be its lifting to T_H with initial vertex u; then $\overline{\tilde{q}}\kappa$ is a closed path at $\overline{u}\kappa$ satisfying $(\overline{\tilde{q}}\kappa)\sigma = \tilde{q}\rho_H = q\rho_{H'}$. Thus, $(\pi(Y, \overline{u}\kappa))\sigma = H'$. Now, using twice the uniqueness in 1.2(c), $(Y,\,\sigma)$ and $(T_{H'},\,\rho_{H'})$ are isomorphic and $f = {}^{-}\kappa g$ where g is an I-labelled graph isomorphism $g:(Y,\,\sigma)\to(T_{H'},\,\rho_{H'})$ sending $\overline{u}\kappa$ to v. □

1.6 Definition. Let $H \le F_I$. Consider the corresponding I-labelled graph covering $(T_H,\,\rho_H)$ with base point u. Restricting ρ_H to the core of T_H, we have the I-labelled graph immersion $(X_H,\,\rho_H)$ where VX_H contains the distinguished vertex $\hat{u}$. We have

$$(\pi(X_H,\,\hat{u}))\rho_H = (\pi(T_H,\,\hat{u}))\rho_H = ((\pi(T_H,\,u))\rho_H)^{x(H,u)} = H^{x(H,u)},$$

where $x(H,u) = p\rho_H \in F_I$, and p is the unique reduced path from u to $\hat{u}$ in $T_H - EX_H$. Note that if the initial base point u belongs to the core then $\hat{u} = u$ and $x(H,u) = 1$.

In this situation, given an equivalence relation $\sim$ on the vertex set of the core graph X_H, we can consider the quotient graph $(X_H/\!\sim,\,(\rho_H)_\sim)$, and the subgroup $H_\sim = (\pi(X_H/\!\sim,\,\overline{\hat{u}}))(\rho_H)_\sim$ of F_I which clearly contains $H^{x(H,u)}$. We write

$$\mathcal{O}(H;F_I) = \{H_\sim^{x(H,u)^{-1}} \mid \sim \text{ eq. rel. on } VX_H\},$$

and call this set the *fringe* of H with respect to F_I. Observe that this set does not depend on the base point u. In fact, if v is another base point of $(T_H,\,\rho_H)$ then, by 1.2(b), there exist an I-labelled graph isomorphism $f:(T_H,\,\rho_H)\to(T_H,\,\rho_H)$ sending u to v, and so sending $\hat{u}$ to $\hat{v}$. Furthermore, $x(H,\,u) = x(H,\,v)$ and f restricts to an I-labelled graph isomorphism $(X_H,\,\rho_H)\to(X_H,\,\rho_H)$. And this restriction gives us a permutation, $\sim\to\sim\!f$, of the set of equivalence relations on VX_H such that $(\pi(X_H/\sim,\,\overline{\hat{u}}))(\rho_H)_\sim = (\pi(X_H/\sim\!f,\,\overline{\hat{v}}))(\rho_H)_{\sim f}$.

Note that $\mathcal{O}(H;F_I)$ contains H and a free factor of F_I, corresponding to equality and the trivial equivalence relation on VX_H, respectively. Note also that $\mathcal{O}(H;F_I)$ is finite when X_H is finite, that is, when H is finitely generated. □

The finitely generated case of the following theorem is essentially due to Takahasi (see Theorem 2 of [8]). The result will be proved here using the graph-theoretic tecniques developed above, and generalizing the construction in Definition 3 of [9].

1.7 Theorem. *(Takahasi) Let I be a set and let $H \le F_I$. For every $K \le F_I$ containing H there exists $H' \in \mathcal{O}(H;F_I)$ which is a free factor of K, i.e. such that $H \le H' \le K = H' * L$ for some $L \le F_I$.*

Proof. Consider the I-labelled graph covering $(T_H,\,\rho_H)$ with base point u; we have $(\pi(T_H,\,\hat{u}))\rho_H = H^{x(H,u)} \le K^{x(H,u)}$. Consider also the I-labelled graph covering $(T_{K^{x(H,u)}},\,\rho_{K^{x(H,u)}})$ with base point v chosen in such a way that ${}^{-}\kappa$ is the lifting of ρ_H sending $\hat{u}$ to v, where ${}^{-}: T_H\to T_H/\!\sim$ and $\kappa: T_H/\!\sim\,\to T_{K^{x(H,u)}}$ are as in Lemma 1.5, for some equivalence relation $\sim$ on the vertex set VT_H.

Now restrict $\sim$ to VX_H (again denoted $\sim$) and consider the quotient I-labelled graph $(X_H/\!\sim,\,(\rho_H)_\sim)$. Clearly the group $H' = H_\sim^{x(H,u)^{-1}}$ is an element of $\mathcal{O}(H;F_I)$, where $H_\sim = (\pi(X_H/\!\sim,\,\overline{\hat{u}}))(\rho_H)_\sim$. Moreover, $X_H/\!\sim$ is a subgraph of $T_H/\!\sim$. So $H_\sim$ is a free factor of $(\pi(T_H/\!\sim,\,\overline{\hat{u}}))(\rho_H)_\sim = (\pi(T_{K^{x(H,u)}},\,v))\rho_{K^{x(H,u)}} = K^{x(H,u)}$. Thus,

$H^{x(H,u)} \leq H_\sim \leq K^{x(H,u)} = H_\sim * L^{x(H,u)}$ for some $L \leq F_I$. Conjugating by $x(H,u)^{-1}$ we get $H \leq H' \leq K = H' * L$ which completes the proof. □

The fringe of a subgroup $H \leq F_I$ contains enought information to decide weather H itself has or has not some properties. As examples, we can state the following two corollaries which make use of concepts from [3] and [9], respectively.

1.8 Corollary. *Let $H \leq F_I$. Then, H is compressed if and only if $r(H) \leq r(H')$ for every $H' \in \mathcal{O}(H; F_I)$.* □

1.9 Corollary. *Let $H \leq F_I$. Then, H is contained in a proper retract of F_I if and only if H' is a proper retract of F_I, for some $H' \in \mathcal{O}(H; F_I)$.* □

2. Maximal rank fixed subgroups

Throughout this section, let $I = \{a_1, \cdots, a_n\}$, $n \geq 1$, and let F_n (resp. R_n) denote F_I (resp. R_I).

2.1 Definitions. A *maximal rank fixed subgroup* of F_n is a subgroup H of F_n of rank n which is of the form $H = \mathrm{Fix}\,\phi$ for some $\phi \in Aut(F_n)$; that n is the largest possible rank was proved by Bestvina-Handel [1].

For a subgroup $H \leq F_n$, we define the *abelian rank of H with respect to F_n*, denoted $r^{ab}(H; F_n)$, to be the rank of the free abelian group HF'_n/F'_n, that is, the rank as an abelian group of the image of H under the abelianization $F_n \to F_n^{ab}$. □

If $H = \mathrm{Fix}\,\phi \leq F_n$ is a maximal rank fixed subgroup, Collins-Turner [2] showed that there exist a free basis $\{x_1, \cdots, x_n\}$ for F_n, and an integer $1 \leq r \leq n$ such that $\mathrm{Fix}\phi$ is freely generated by $\{x_1, \cdots, x_r, x_{r+1}w_{r+1}x_{r+1}^{-1}, \cdots, x_n w_n x_n^{-1}\}$ where, for $i = r+1, \cdots, n$,

$$\begin{aligned} w_i \in \ & \langle x_1, \cdots, x_r, x_{r+1}w_{r+1}x_{r+1}^{-1}, \cdots, x_{i-1}w_{i-1}x_{i-1}^{-1} \rangle \\ & = \mathrm{Fix}\,\phi \cap \langle x_1, \cdots, x_{i-1} \rangle \end{aligned}$$

is not a proper power. Note that if $r = n$ then $H = F_n$ and ϕ is the identity.

For a given maximal rank fixed subgroup $H \leq F_n$, the above expression need not be unique, but the number r is uniquely determined by H since it is the abelian rank of H with respect to F_n; in fact, HF'_n/F'_n is the abelian subgroup of F_n^{ab} freely generated by $\{x_1F'_n, \cdots, x_rF'_n\}$.

The main result in this section is that a strictly ascending chain of such subgroups in F_n has length at most n. This is an obvious consequence of the following theorem.

2.2 Theorem. *If $H < K \leq F_n$ with H a maximal rank fixed subgroup then either $r(H) < r(K)$ or $r^{ab}(H; F_n) < r^{ab}(K; F_n)$.*

Proof. We may assume that $x_i = a_i$, $i = 1, \cdots, n$ is the basis of F_n in which H has an expression in the above form given by Collins-Turner. The core X_H of (T_H, ρ_H) with base point $u = \hat{u}$ is depicted in Fig. 1 with each edge and circle

FIG. 1

labelled with its I-label. Let $E' = \{e \in EX_H \mid \iota e = u \neq \tau e\}$; for $i = r+1, \cdots, n$, denote by e_i the edge in E' with label a_i. Observe that, since $w_i \in \langle a_1, \cdots, a_{i-1}\rangle$, none of the edges in the component of $X_H - E'$ containing τe_i has label a_i.

Clearly, by Theorem 1.7, we may restrict our atention to the case $K \in \mathcal{O}(H; F_n)$, that is, $K = H_\sim$ for some equivalence relation $\sim$ in VX_H.

Consider first the case where there exist two vertices v, w in two different components of $X_H - E'$ such that $v \sim w$. Then there exist $i \in \{r+1, \cdots, n\}$ and a closed path in $X_H/\!\sim$ based at $\overline{u}$ which crosses no edge twice, and which crosses exactly one edge labelled a_i. This closed path determines an element in KF'_n/F'_n not contained in HF'_n/F'_n. Thus, $\langle a_1F'_n, \cdots, a_rF'_n\rangle = HF'_n/F'_n < KF'_n/F'_n$ and so, $r^{ab}(H; F_n) < r^{ab}(K; F_n)$.

This leaves the case where $v \sim w$ implies that v and w belong to the same component of $X_H - E'$. Here the quotient $(X_H)/\!\sim$, and the I-labelled graph immersion obtained from it by applying the Folding Lemma, are both I-labelled graphs formed by taking X_H, and, for each $i = r+1, \cdots, n$, taking the component of $X_H - E'$ containing τe_i, and either leaving it unchanged, or replacing it with an I-labelled graph X_i such that $\langle w_i\rangle < \pi(X_i, \tau e_i)$, so X_i has rank at least two since w_i is not a proper power. Consequently, either $H = K$ or $r(H) < r(K)$. This completes the proof. □

2.3 Theorem. *Among the strictly ascending chains of maximal rank fixed subgroups of F_n, the maximum length is exactly n.*

Proof. It follows from Theorem 2.2 that any strictly ascending chain of maximal rank fixed subgroups of F_n has length at most n. It is straightforward to construct a chain of length n using the above Collins-Turner description. □

2.4 Remark Ted Turner has pointed out to me that the poof of Theorem 2.2 does not use the fact $w_i \in \operatorname{Fix}\phi$. Thus, Theorems 2.2 and 2.3 extend to the (rank n) subgroups of F_n of the form $\langle x_1, \cdots, x_r, x_{r+1}w_{r+1}x_{r+1}^{-1}, \cdots, x_nw_nx_n^{-1}\rangle$ for some free basis $\{x_1, \cdots, x_n\}$ of F_n and elements $w_i \in \langle x_1, \cdots, x_{i-1}\rangle$, $i = r+1, \cdots, n$, which are not proper powers. It is not difficult to show that not all subgroups of this form are maximal rank fixed subgroups of F_n. □

3. The rank two case

The main result of this section, Theorem 3.4, is that, in the rank two case, if the intersection of an arbitrary family of maximal rank fixed subgroups has maximal rank (that is, two) then it is again a maximal rank fixed subgroup. Theorem 2.3 for the special case of rank two states that if H, K are two different maximal rank fixed subgroups and $H < K$, then K is the whole group. These two facts will allow us to deduce certain consequences for the rank two case.

Throughout this section, let $I = \{a, b\}$ and let F_2 (resp. R_2) denote F_I (resp. R_I).

3.1 Notation. Let (X, ρ) and (Y, σ) be two finite I-labeled graphs.

A *branch point* of X is a vertex in X with valence greater than two. An *arc* in X is a reduced path $e_1^{\epsilon_1} \cdots e_n^{\epsilon_n}$ such that $\iota e_1^{\epsilon_1}$ and $\tau e_n^{\epsilon_n}$ are branch points of X while $\iota e_i^{\epsilon_i}$ are valence two vertices, for $i = 2, \cdots, n$. A *b-arc* in X is an arc with at least one of the e_i labelled b. An *a-arc* in X is an arc which is not a b-arc, that is, with all the edges labelled a.

Let $f : (X, \rho) \to (Y, \sigma)$ be an I-labelled graph immersion. Here f carries branch points of X to branch points of Y. Also f carries an arc, p, in X to a reduced path in Y which decomposes in a unique way as a reduced product of arcs in Y, $pf = p_1 \cdots p_m$; in this situation we say that each p_i and p_i^{-1} *occur* in pf (and thus in $p^{-1}f$). Clearly p is a b-arc if and only if one of the arcs occuring in pf is a b-arc. Let p (resp. q) be an arc in X (resp. Y). We denote by $o_p(q)$ the number of occurences of q or q^{-1} in the above reduced expression for pf. Let $o(q)$ denote $\frac{1}{2}\sum_p o_p(q)$, where the sum is over the set of all the arcs p in X. Thus $o_{p^\epsilon}(q^\delta)$ is a nonnegative integer independent of ϵ, $\delta = \pm 1$, and is positive if and only if q occurs in pf. Further, $o(q)$ is the total number of times that q and q^{-1} occur in the image of an arc in X, up to orientation. □

Let $H = \langle a, bab^{-1} \rangle < F_2$ and let $K \leq F_2$ with $r(K) = 2$. The immersions $\rho_H : X_H \to R_2$ and $\rho_K : X_K \to R_2$ have a pullback, which we will denote by $(\sigma_H, \sigma_K) : W \to X_H \times X_K$; both σ_H and σ_K are immersions. See [6] for the construction and basic properties. Let σ denote the diagonal map $\sigma_H \rho_H = \sigma_K \rho_K$.

Observe that the I-labelled graph immersion (X_H, ρ_H) is that depicted in Fig. 2 where each edge is labelled with its I-label. Note that the base point u of (T_H, ρ_H) belongs to X_H and let v denote the other vertex of X_H. The rank two core graph X_K has one of the three topology types depicted in Fig. 3. In Fig. 3(c), if the two branch points have links with labels $\{a, a^{-1}, b\}$ and $\{a, a^{-1}, b^{-1}\}$, let x denote the former branch point, and let y denote the latter branch point; in all other cases, let x denote one of the branch points, and if there is another branch point, let y denote

FIG. 2

it.

3.2 Lemma. *In the above situation, if one of the components of W has rank two, and W_0 is its core, then one of the following holds:*

(a) $\sigma_K : W_0 \to X_K$ *is an I-labelled graph isomorphism,*

(b) $(u,\, x) \in VW_0$ *and for some* $h \in H$ *and* $n \geq 1$*,* $(\pi(X_K,\, x))\rho_K = \langle a^n,\, hb\rangle$ *and* $(\pi(W_0,\, (u,\, x)))\sigma = \langle a^n,\, hba^nb^{-1}h^{-1}\rangle$*.*

Proof. In the case where X_K has the topology type of Fig. 3(a), the two branch points of W are $(u,\, x)$ and $(v,\, x)$, and they belong to W_0, and have valence three in W_0.

In the case where X_K has the topology type of Fig. 3(b) or 3(c), the only possible branch points of W are $(u,\, x)$, $(v,\, x)$, $(u,\, y)$ and $(v,\, y)$. But if both $(u,\, x)$ and $(v,\, x)$ occur, then $(\mathrm{lk}(u))\rho_H = (\mathrm{lk}(u,\, x))\sigma = (\mathrm{lk}(x))\rho_K = (\mathrm{lk}(v,\, x))\sigma = (\mathrm{lk}(v))\rho_H$, which is a contradiction. Thus, at most one of $(u,\, x)$ and $(v,\, x)$ has valence three. Now $r(W_0) = 2$ implies that exactly one of $(u,\, x)$ and $(v,\, x)$ belongs to W_0 and has valence three in W_0. A similar argument holds with y in place of x. Furthermore, W has a unique rank two component and its core is W_0, and has the topology type of Fig. 3(b) or 3(c).

There is exactly one edge in X_H labelled b, so the graph morphism σ_K is injective on b-labelled edges. The following is another way of saying the same fact and it is the fundamental point of the proof:

(1) *every arc q of X_K with $o(q) \geq 2$ is an a-arc.*

Let us now consider three cases depending on the topology type of X_K.

Case 1: X_K has the topology type of Fig. 3(a). Here W_0 has exactly three distinct arcs, up to orientation, and X_K has exactly two. So, by (1), one of the two arcs in X_K is an a-arc. Moreover, one of the arcs is a b-arc whose first and last edges have label b. Let q denote the b-arc with first (and last) edge labelled b (and not b^{-1}). Then $(\pi(X_K,\, x))\rho_K = \langle a^n,\, k\rangle$, where $n \geq 1$ and $k = q\rho_K \in F_2$.

FIG. 3

But, by the pullback construction, there exist two closed a-arcs of length n in W with initial (and terminal) vertices $(u,\, x)$ and $(v,\, x)$; furthermore, they lie in W_0. To complete the description of W_0, we add a b-arc, denoted p, from $(u,\, x)$ to $(v,\, x)$, which is mapped under σ_K to q, and so has label k. Thus, $(u,\, x) \in VW_0$ and $(\pi(W_0,\, (u,\, x)))\sigma = \langle a^n,\, ka^nk^{-1}\rangle$. Finally, by considering the image of p under σ_H, we see that $k = hb$ for some $h \in H$. Thus, (b) holds.

Case 2: X_K has the topology type of Fig. 3(b). Let q_1 denote the unique non-closed arc in X_K joining x to y, and let q_2 and q_3 denote the two closed arcs in X_K with clockwise orientation, based at x and y respectively. Let $\mathrm{lk}(x) = \{e_1,\, e_2,\, e_3\} \subset (EX_K)^{\pm 1}$ and assume that e_1 is the first term of q_1.

It is clear that if W_0 has the topology type of Fig. 3(c) then $o(q_1) \geq 3$, $o(q_2) \geq 2$ and $o(q_3) \geq 2$ so, by (1), q_1, q_2 and q_3 are a-arcs, which is a contradiction. Thus W_0 has the topology type of Fig. 3(b). Its two branch points are $(w_x,\, x)$ and $(w_y,\, y)$ where $w_x,\, w_y \in \{u, v\}$. Let p_1 denote the arc in W_0 joining $(w_x,\, x)$ to $(w_y,\, y)$, and let e'_1 denote the first term of p_1. Let p_2 be a closed arc in W_0 at $(w_x,\, x)$, and let e'_2 denote the first term of p_2. Let e'_3 denote the first term of p_2^{-1}, so $\mathrm{lk}(w_x,\, x) = \{e'_1,\, e'_2,\, e'_3\}$. If $e'_1\sigma_K \neq e_1$ then $o(q_1) \geq 2$ and $o(q_2) \geq 2$ which contradicts the fact that q_1 and q_2 cannot both be a-arcs. Thus, $e'_1\sigma_K = e_1$. A similar result holds, with y in place of x. Now q_i occurs in $p_i\sigma_K$, $i = 1,\, 2,\, 3$. If q_2 or q_3 occur in $p_1\sigma_K$ then $o(q_1) \geq 3$, $o(q_2) \geq 2$ and $o(q_3) \geq 2$ which is impossible. So $p_1\sigma_K = q_1$. Now suppose that $p_2\sigma_K = q_2^{\epsilon} \cdot w$ for some $\epsilon = \pm 1$ and some nontrivial path w. Then $o(q_2) \geq 2$, so q_2 is an a-arc and there must exist a closed a-arc in W_0 with the same length as q_2 and based at the branch point $(w_x,\, x)$ which contradicts the non-triviality of w. Thus $p_2\sigma_K = q_2^{\pm 1}$, and similarly $p_3\sigma_K = q_3^{\pm 1}$. This completes the proof that (a) holds.

Case 3: X_K has the topology type of Fig. 3(c). Denote by q_1, q_2 and q_3 the three arcs in X_K joining x to y. It is easy to see that if at least two of them are b-arcs then W_0 has the topology type of Fig. 3(c), and (a) holds. So we may assume that, q_1 and q_2 are a-arcs and q_3 is a b-arc. There exist positive integers r, s such that $q_1\rho_K = a^r$ and $q_2\rho_K = a^{-s}$. Thus, in W there are two paths joining $(u,\, x)$ to $(u,\, y)$ whose last coordinates are respectively q_1 and q_2; and similarly with v in place of u. But $r(W_0) = 2$, so W also contains an arc, p_3, whose last coordinate is q_3, and joining $(u,\, x)$ to $(u,\, y)$, or $(v,\, x)$ to $(v,\, y)$, or $(u,\, x)$ to $(v,\, y)$ (but not

$(v,\, x)$ to $(u,\, y)$, by the choice of x and y). In the first two cases, (a) holds. It remains to consider the third case. Let $n = r + s$, and $k = q_3\rho_K = p_3\sigma \in F_2$. Consider the image of p_3 under σ_H, and observe that $k = h'b$ for some $h' \in H$. Now, $(u,\, x) \in VW_0$, $(\pi(X_K,\, x))\rho_K = \langle a^n,\, ka^s\rangle = \langle a^n,\, hb\rangle$ and $(\pi(W_0,\, (u,\, x)))\sigma = \langle a^n,\, ka^nk^{-1}\rangle = \langle a^n,\, hba^nb^{-1}h^{-1}\rangle$, where $h = h'(ba^sb^{-1}) \in H$. Thus (b) holds in this case. □

3.3 Theorem. *Let F_2 be the free group on $\{a,b\}$, and let $H = \langle a,\, bab^{-1}\rangle$. If K is a subgroup of F_2 with $r(K) = r(H \cap K) = 2$, then one of the following holds:*

(a) $K \subseteq H$, so $H \cap K = K$;

(b) $K = \langle a^n,\, h_1b\rangle^{h_2}$, for some elements $h_1,\, h_2 \in H$ and $n \geq 1$, so $H \cap K = \langle a^n,\, h_1ba^nb^{-1}h_1^{-1}\rangle^{h_2}$.

Proof. Consider the I-labelled graph coverings $(T_H,\, \rho_H)$ and $(T_K,\, \rho_K)$ with base points u and w, respectively. Note that the base point u of $(T_H,\, \rho_H)$ belongs to X_H (and so, $\hat{u} = u$), but w does not necessarily belong to X_K. We know that the fundamental group at $(u,\, w)$ of the pullback of these two coverings is (isomorphic to) $H \cap K$. So, this pullback has a component of rank two. The core of this component coincides with the core of the unique rank two component of the pullback of the two immersions $\rho_H : X_H \to R_2$ and $\rho_K : X_K \to R_2$; that is, it coincides with W_0 using the above notation. So Lemma 3.2 applies.

Suppose first that (a) of Lemma 3.2 holds. There exists a vertex in W_0 of the form $(u,\, z)$, for some $z \in VX_K$. We have $H \cap (\pi(X_K,\, z))\rho_K = (\pi(X_H,\, u))\rho_H \cap (\pi(X_K,\, z))\rho_K = (\pi(W_0,\, (u,\, z)))\sigma = (\pi(X_K,\, z))\rho_K$. We also have $(\pi(X_K,\, z))\rho_K = (\pi(X_K,\, w))^p\rho_K = K^g$, where p is an arbitrary path in T_K from w to z and $g = p\rho_K \in F_2$. So $H \cap K^g = K^g$, that is, $K^g \leq H$.

Suppose next that (b) of Lemma 3.2 holds. Choose an arbitrary path p in T_K from w to x and let $g = p\rho_K \in F_2$. We have $K^g = (\pi(X_K,\, x))\rho_K = \langle a^n,\, hb\rangle$ and $H \cap K^g = (\pi(W_0,\, (u,\, x)))\sigma = \langle a^n,\, hba^nb^{-1}h^{-1}\rangle$, for some $h \in H$ and $n \geq 1$.

In summary, there exists $g \in F_2$ such that one of the following holds:

(a′) $K^g \leq H$ and $H \cap K^g = K^g$;

(b′) $K^g = \langle a^n,\, hb\rangle$ and $H \cap K^g = \langle a^n,\, hba^nb^{-1}h^{-1}\rangle$ for some $h \in H$ and $n \geq 1$.

Now $r(H \cap K) = r(H \cap K^g) = 2$. But H is the fixed subgroup of the automorphism of F_2 given by $a \mapsto a,\, b \mapsto ba$ so, by Theorem IV.5.5 of [3], g and 1 belong to the same double coset in $K\backslash F/H$. That is, $g = kh_2^{-1}$ for some $k \in K$ and some $h_2 \in H$. Thus $K^g = K^{h_2^{-1}}$, $H \cap K^g = H \cap K^{h_2^{-1}} = (H \cap K)^{h_2^{-1}}$. Now, taking $h_1 = h$, the theorem is proved. □

Using Theorems 2.2 and 3.3 we obtain the main result of this section:

3.4 Theorem. *Let F be a free group of rank two, and S a non-empty set of non-identity endomorphisms of F such that $r(Fix\, S) = 2$. Then $S \subseteq Aut(F)$, and $Fix\, S = \langle a,\, bab^{-1}\rangle$ for some basis $\{a,\, b\}$ of F, and $Fix\, S = Fix\,\phi$ for each $\phi \in S$.*

Proof. Let $\phi \in S$, and let $H = \operatorname{Fix}\phi$. By the Corollary in [5], $r(H) \leq 2$. But H is non-abelian, since it contains $\operatorname{Fix} S$ which has rank two. Hence $r(H) = 2$. By Corollary 2 of [9], ϕ is an automorphism. By Theorem A of [2], we can assume that $F = \langle a, b\rangle$ and $H = \langle a,\, bab^{-1}\rangle$.

Hence $S \subseteq \operatorname{Aut}(F)$.

Let $\psi \in S$, and let $K = \operatorname{Fix}\psi$. By the foregoing, $r(K) = 2$. By Theorem IV.5.7 of [3], or Theorem 1 of [7], $r(H \cap K) \leq 2$. But $H \cap K$ is not abelian since it contains $\operatorname{Fix} S$ which has rank two. So, $r(H \cap K) = 2$. Now, by Theorem 3.3, either $K \leq H$ or $K = \langle a^n,\, h_1 b\rangle^{h_2}$ for some $h_1,\, h_2 \in H$ and $n \geq 1$. But in the latter case, we would have $r^{ab}(K; F_2) = 2$, which contradicts Theorem 2.2 applied to $K < F \leq F$. Thus, $K \leq H < F$ and, by Theorem 2.3, $K = H$. The result is now clear. □

3.5 Corollary. *Let F be a free group of rank two, and S a subset of $End(F)$. If $r(Fix\, S) = 2$, then S fixes some primitive element of F.* □

There is a simple description of the sets of endomorphisms of a free group of rank two with fixed group of rank two.

3.6 Proposition. *Let S be a subset of $End(F_2)$, and let ψ be the automorphism of F_2 given by $a\psi = a$, $b\psi = ba$. Then $Fix\, S = \langle a,\, bab^{-1}\rangle$ if and only if every element of S is a power of ψ, and some non-identity power of ψ belongs to S.*

Proof. We may assume that S is a non-empty set of non-identity endomorphisms of F_2.

Suppose $\operatorname{Fix} S = \langle a,\, bab^{-1}\rangle$. By Theorem 3.4, $S \subseteq \operatorname{Aut}(F_2)$, and, for every $\phi \in S$, $\operatorname{Fix}\phi = \langle a,\, bab^{-1}\rangle$. Now, using Theorem A in [2] (or directly working with the fact that a and bab^{-1} are fixed by the non-identity automorphism ϕ), we obtain $a\phi = a$ and $b\phi = ba^r$ for some $r \neq 0$. That is, $\phi = \psi^r$.

The result is now clear. □

Combining Theorem 3.4 and Proposition 3.6 we have the following.

3.7 Theorem. *Let F be a free group of rank two. If G is a subset of $End(F)$ which is maximal with the property that $r(Fix\, G) = 2$, then G is an infinite cyclic group. Hence each subgroup of $Aut(F)$ with rank two fixed subgroup is either trivial or infinite cyclic.* □

Observe that the infiniteness of G (which here comes from the fact that ψ is not periodic) agrees with Theorem 2 of [4].

Let us observe the following.

3.8 Corollary. *Let F be a free group of rank two and let H be a proper maximal rank fixed subgroup of F. Then no non-identity endomorphism of H with rank two fixed subgroup can be extended to an endomorphism of F.*

Proof. Let $\phi : H \to H$ be an endomorphism with $r(\operatorname{Fix}\phi) = 2$ and suppose ϕ can be extended to an endomorphism $\tilde{\phi}$ of F. Then $\operatorname{Fix}\tilde{\phi} \cap H = \operatorname{Fix}\phi$ has rank two, so, by Theorem 3.4, $\operatorname{Fix}\tilde{\phi} = H$. Thus, ϕ is the identity. □

Let us conclude by describing the subgroups of the free group of rank two which are the fixed subgroups for some set of endomorphisms. As a Corollary, we can answer a question of G. Levitt in the rank two case.

3.9 THEOREM. *Let F be a free group of rank two, and let H be a subgroup of F. The following are equivalent:*

(a) $H = \mathrm{Fix}\,\phi$ *for some* $\phi \in Aut(F)$*,*

(b) $H = \mathrm{Fix}\,\phi$ *for some* $\phi \in End(F)$*,*

(c) $H = \mathrm{Fix}\,S$ *for some* $S \subseteq End(F)$*,*

(d) H *is* $\{1\}$*, or* $\langle w\rangle$ *for some* $w \in F$ *which is not a proper power, or* $\langle a,\, bab^{-1}\rangle$ *for some basis* $\{a,\, b\}$ *of* F*, or* F*,*

(e) $H = Fix\,G$ *for some* $G \subseteq Aut(F)$*.*

Proof. Obviously, (a) implies (b) and (b) implies (c).

Suppose that $H = \mathrm{Fix}\,S$ for some $S \subseteq \mathrm{End}\,(F)$. By Theorem IV.5.7 in [3], H has rank at most two. If $r(H) = 0$ then $H = \{1\}$. If $r(H) = 1$ then $H = \langle w\rangle$ for some $w \in F$. But if a power of an element of a free group is fixed by an endomorphism then the element itself is also fixed. So, w is not a proper power. Finally, if $r(H) = 2$ then, by Theorem 3.4, $H = \langle a,\, bab^{-1}\rangle$ for some basis $\{a,\, b\}$ of F, or $H = F$. Thus, (c) implies (d).

The trivial subgroup is the fixed subgroup of an automorphism of F permuting the two elements of some basis. If w is not a proper power, then $\langle w\rangle$ is the fixed subgroup of (right or left) conjugation by w. If $\{a,\, b\}$ is a basis of F, then $\langle a,\, bab^{-1}\rangle$ is the fixed subgroup of the automorphism given by $a \mapsto a$, $b \mapsto ba$. Obviously, F is the fixed subgroup of the identity. So, (d) implies (a).

Finally, it is obvious that (a) implies (e), and that (e) implies (c). □

3.10 COROLLARY. *Let F be a free group of rank two, and let S be a subset of $End(F)$. Then there exists a finite subset $S_0 \subseteq S$ such that* $\mathrm{Fix}\,S_0 = \mathrm{Fix}\,S$.

Proof. Choose $\phi_1 \in S$. If $\mathrm{Fix}\,\phi_1 = \mathrm{Fix}\,S$ then the result is proven. Otherwise, choose $\phi_2 \in S$ such that $\mathrm{Fix}\,\{\phi_1,\, \phi_2\}$ is strictly contained in $\mathrm{Fix}\,\phi_1$. By Theorems 3.9 and 2.3 we can repeat this operation at most four times. Thus, there exist $S_0 \subseteq S$ such that $\mathrm{Fix}\,S_0 = \mathrm{Fix}\,S$ and $|S_0| \leq 3$. □

4. Comments about rank higher than two.

One might conjecture that Theorem 3.4 extends to higher finite ranks, in the form that the intersection of maximal rank fixed subgroups in F_n is again a maximal rank fixed subgroup, where now it is possible for the intersections to be proper, by Theorem 2.3. Here graphical arguments using coverings and immersions to represent subgroups of F_n, and pull-backs to control intersections, seem not to be sufficient to prove such a result. Any kind of argument distinguishing topology types of graphs with rank n and their pull-backs leads to a list of possibilities which grows very quickly with n. This growth makes it difficult to analize the general situation for $n \geq 3$.

For example, in the case where $F_3 = \langle a, b, c \mid \ \rangle$, and $H = \langle a, b, cwc^{-1} \rangle$, $w \in \langle a, b \rangle$, we have the following three types of subgroups K of F_3 with $r(K) = r(H \cap K) = 3$ (among possibly others):

- $K \leq H$, so $H \cap K = K$,
- $K = \langle w^m, h_1, h_2 c \rangle^{h_3}$, $h_1, h_2, h_3 \in H$, so $H \cap K = \langle w^m, h_1, h_2 cwc^{-1} h_2^{-1} \rangle^{h_3}$
- $K = \langle u_1, u_2, h_1 c \rangle^{h_2}$, $h_1, h_2 \in H$, $u_1, u_2 \in \langle a, b \rangle$ with $\langle u_1, u_2 \rangle$ having rank two, $w^m \in \langle u_1, u_2 \rangle$, so $H \cap K = \langle u_1, u_2, h_1 cw^m c^{-1} h_1^{-1} \rangle^{h_2}$.

And in the case where $F_3 = \langle a, b, c \mid \ \rangle$, and $H = \langle a, bab^{-1}, cwc^{-1} \rangle$, $w \in \langle a, bab^{-1} \rangle$, observe the following two examples:

- $K = \langle u_1, u_2, h_1 c \rangle^{h_2}$, $h_1, h_2 \in H$, $u_1, u_2 \in \langle a, bab^{-1} \rangle$ with $\langle u_1, u_2 \rangle$ having rank two, $w^m \in \langle u_1, u_2 \rangle$, so $H \cap K = \langle u_1, u_2, h_1 cw^m c^{-1} h_1^{-1} \rangle^{h_2}$.
- $K = \langle a^n, h_1 b, (h_2 c)^{h_3} \rangle^{h_4}$, $h_1, h_2, h_3, h_4 \in H$, $(w^m)^{h_3} \in \langle a^n, h_1 b \rangle$, so $H \cap K = \langle a^n, h_1 ba^n b^{-1} h_1^{-1}, (h_2 cw^m c^{-1} h_2^{-1})^{h_3} \rangle^{h_4}$.

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References

[1] M. Bestvina and M, Handel, *Train tracks and automorphisms of free groups*, Ann. of Math. (2) **135** (1992), 1–51.

[2] D.J. Collins and E.C. Turner, *All automorphisms of free groups with maximal rank fixed subgroups*, Math. Proc. Camb. Phil. Soc. **119** (1996), 615–630.

[3] W. Dicks and E. Ventura, *The group fixed by a family of injective endomorphisms of a free group*, Contemporary Mathematics **195** (1996), 1–81.

[4] J.L. Dyer and G.P. Scott, *Periodic automorphisms of free groups*, Comm. Algebra **3** (1975), 195–201.

[5] W. Imrich and E.C. Turner, *Endomorphisms of free groups and their fixed points*, Math. Proc. Cambridge Philos. Soc. **105** (1989), 421–422.

[6] J.R. Stallings, *Topology of finite graphs*, Invent. Math. **71** (1983), 551–565.

[7] G. Tardos, *On the intersection of subgroups of a free group*, Invent. Math. **108** (1992), 29–36.

[8] M. Takahasi, *Note on chain conditions in free groups*, Osaka Math. J. **3** (1951), 221–225.

[9] E.C. Turner, *Test words for automorphisms of free groups*, Bull. London Math. Soc. **28** (1996), 255–263.